Gradient Flows on the Maximum Mean Discrepancy

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2nd RSS/Turing Workshop on Gradient Flows for Sampling, Inference, and Learning (2025)

Outline

MMD and MMD flow

- Introduction to MMD as an integral probability metric
- Connection with neural net training
- Wasserstein-2 Gradient Flow on the MMD
- Convergence: noise injection, adaptive kernel

Arbel, Korba, Salim, G., Maximum Mean Discrepancy Gradient Flow (NeurIPS 2019)

Galashov, De Bortoli, G., Deep MMD Gradient Flow without adversarial training (ICLR 2025)

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Main motivation: gradient flow when the target distribution represented by samples

- MMD (and related IPMs) are GAN critics
- Understand dynamics of GAN training
- Neural network training dynamics

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Galashov, De Bortoli, G., Deep MMD Gradient Flow without adversarial training (ICLR 2025)

The MMD, and MMD flow

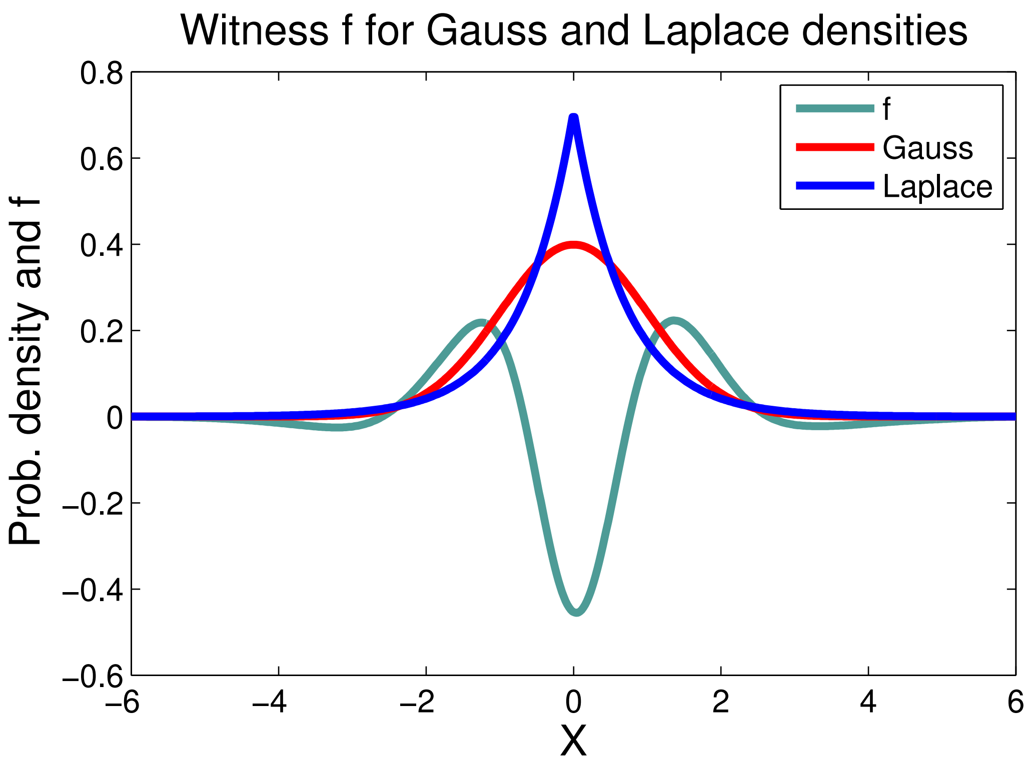
The MMD: an integral probability metric

Maximum mean discrepancy: smooth function for P vs Q

$$MMD(P, Q; F) := \sup_{\|f\| \leq 1} \left[\mathbf{E}_P f(X) - \mathbf{E}_Q f(Y)\right]$$

$$f(x) = \langle f, \varphi(x) \rangle_{\mathcal{F}}$$

$$\langle \varphi(x), \varphi(x') \rangle_{\mathcal{F}} = k(x, x')$$



The MMD: an integral probability metric

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$$\langle \varphi(x), \varphi(x') \rangle_{\mathcal{F}} = k(x, x')$$

For characteristic RKHS $\mathcal{F}$, $MMD(P, Q; F) = 0$ iff $P = Q$

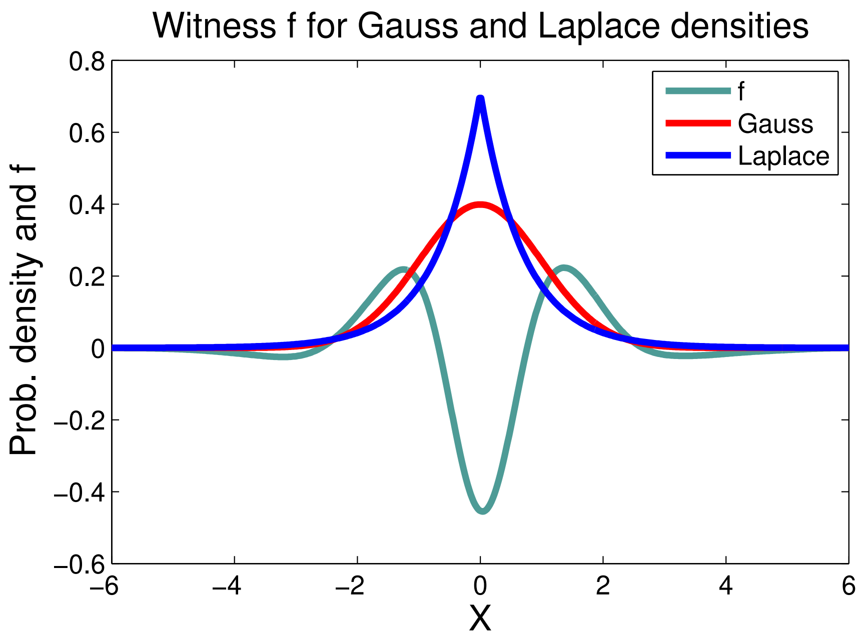
Other choices for witness function class:

- Bounded continuous [Dudley, 2002]
- Bounded varation 1 (Kolmogorov metric) [Müller, 1997]
- Bounded Lipschitz (Wasserstein distances) [Dudley, 2002]

The MMD and witness in closed form

The MMD:

$$
\begin{aligned}
&MMD(P, Q; F) \\
&= \sup_{||f||_{\mathcal{F}} \le 1} \left[\mathbf{E}_P f(X) - \mathbf{E}_Q f(Y) \right]
\end{aligned}
$$



The MMD and witness in closed form

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\begin{aligned}
& MMD(P, Q; F) \\
&= \sup_{\|f\|_{\mathcal{F}} \leq 1} \left[\mathrm{E}_P f(X) - \mathrm{E}_Q f(Y) \right] \\
&= \sup_{\|f\|_{\mathcal{F}} \leq 1} \langle f, \mu_P - \mu_Q \rangle_{\mathcal{F}}
\end{aligned}
$$

use

$$
\begin{aligned}
\mathrm{E}_P f(X) &= \mathrm{E}_P \langle \varphi(X), f \rangle_{\mathcal{F}} \\
&= \langle \mathrm{E}_P [\varphi(X)], f \rangle_{\mathcal{F}} \\
&= \langle \mu_P, f \rangle_{\mathcal{F}}
\end{aligned}
$$

The MMD and witness in closed form

The MMD:

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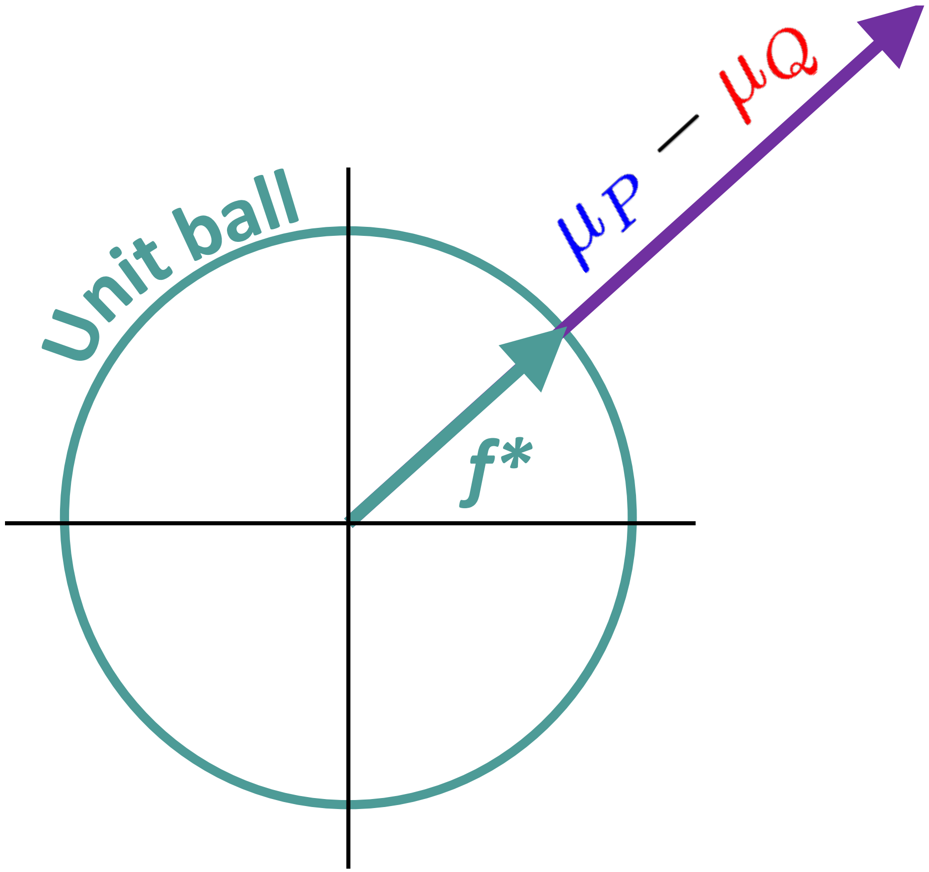
&= \|\mu_P - \mu_Q\|_{\mathcal{F}}
\end{aligned}
$$



$$
f^* = \frac{\mu_P - \mu_Q}{\|\mu_P - \mu_Q\|}
$$

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The MMD:

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&= \|\boldsymbol{\mu}_P - \boldsymbol{\mu}_Q\|_{\mathcal{F}}
\end{aligned}
$$

$$
\begin{aligned}
f^*(x) &\propto \langle \boldsymbol{\mu}_P - \boldsymbol{\mu}_Q, \varphi(x) \rangle_{\mathcal{F}} \\
&= \mathrm{E}_P k(X, x) - \mathrm{E}_Q k(Y, x)
\end{aligned}
$$

The MMD and witness in closed form

The MMD:

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&= \sup_{||f||_{\mathcal{F}} \leq 1} \langle f, \mu_P - \mu_Q \rangle_{\mathcal{F}} \\
&= ||\mu_P - \mu_Q||_{\mathcal{F}}
\end{aligned}
$$

In terms of kernels:

$$
\begin{aligned}
MMD^2(P, Q) &= ||\mu_P - \mu_Q||^2_{\mathcal{F}} \\
&= \underbrace{\mathrm{E}_P k(x, x')}_{\text{(a)}} + \underbrace{\mathrm{E}_Q k(y, y')}_{\text{(a)}} - 2\underbrace{\mathrm{E}_{P,Q} k(x, y)}_{\text{(b)}}
\end{aligned}
$$

(a)= within distrib. similarity, (b)= cross-distrib. similarity.

MMD Flow (NeurIPS 19)

arXiv > stat > arXiv:1906.04370

Statistics > Machine Learning

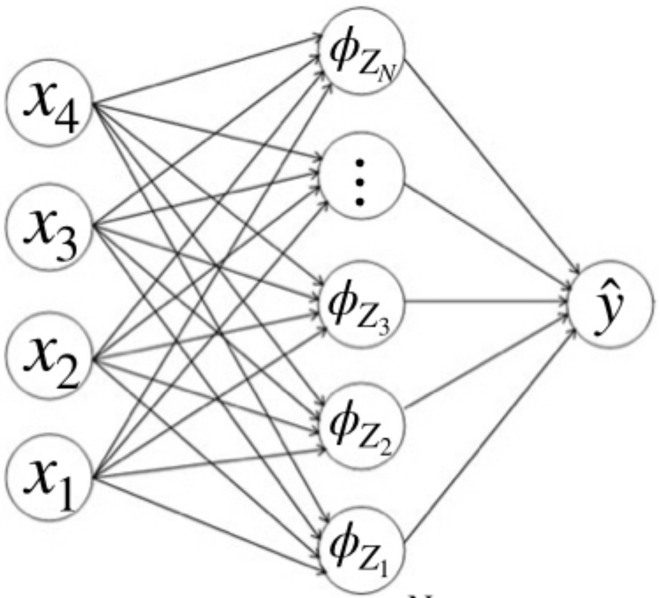
[Submitted on 11 Jun 2019 (v1), last revised 3 Dec 2019 (this version, v2)]

Maximum Mean Discrepancy Gradient Flow

Michael Arbel, Anna Korba, Adil Salim, Arthur Gretton

Motivation: Neural Net training

$(x, y) \sim data$



$$\min_{Z_1,\dots,Z_N} \mathbb{E}_{data}[\|y - \frac{1}{N}\sum_{i=1}^{N} \phi_{Z_i}(x)\|^2]$$

$$\min_{Z_1,\dots,Z_N \in \mathcal{Z}} \mathcal{L}\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{Z_i}\right)$$

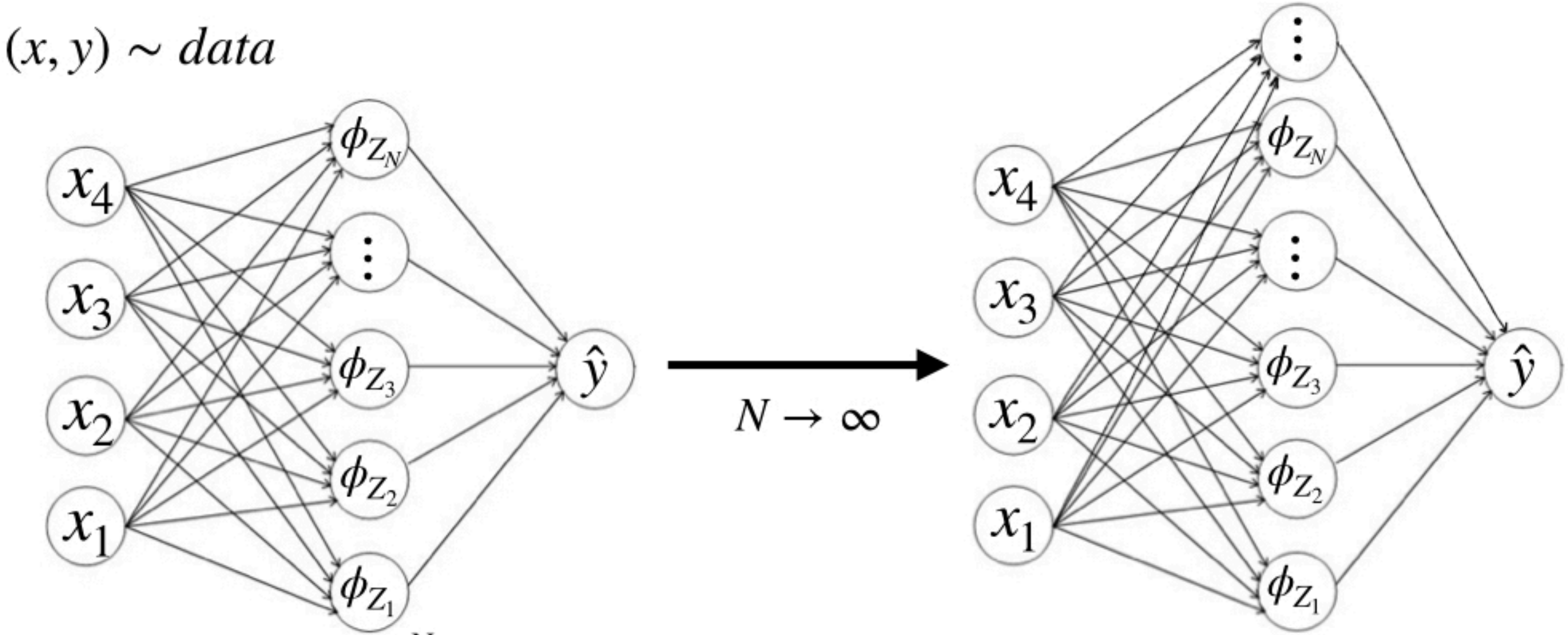
Optimization using gradient descent:

$$Z_i^{t+1} = Z_i^t - \gamma \nabla_{Z_i} \mathcal{L}\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{Z_i^t}\right)$$

Chizat, Bach. “On the global convergence of gradient descent for over-parameterized models using optimal transport”, NeurIPS (2018)

Motivation: Neural Net training

$$\min_{Z_1,\ldots,Z_n \in \mathcal{Z}} \mathcal{L}\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{Z_i}\right) \xrightarrow[n\to\infty]{} \min_{\nu \in \mathcal{P}} \mathcal{L}(\nu)$$



$$\min_{Z_1,\ldots,Z_N} \mathbb{E}_{data}[\|y - \frac{1}{N}\sum_{i=1}^{N} \phi_{Z_i}(x)\|^2] \xrightarrow[N \to \infty]{} \min_{\nu \in \mathscr{P}} \mathbb{E}_{data}[\|y - \mathbb{E}_{Z\sim\nu}[\phi_Z(x)]\|^2]$$

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Motivation: Neural Net training

From previous slide:

$$\min_{\nu \in \mathcal{P}} \mathcal{L}(\nu) := \mathbb{E}_{(x,y)}[\|y - \mathbb{E}_{Z\sim\nu}[\phi_Z(x)]\|^2]$$

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Motivation: Neural Net training

From previous slide:

$$\min_{\nu \in \mathcal{P}} \mathcal{L}(\nu) := \mathbb{E}_{(x,y)}[||y - \mathbb{E}_{Z\sim\nu}[\phi_Z(x)]||^2]$$

Connection to the MMD:

- Assume well-specified setting, $y(x) = \mathbb{E}_{U\sim\nu^\star}[\phi_U(x)]$
- Random feature formulation,

$$\mathcal{L}(\nu) = \mathbb{E}_x\left[||\mathbb{E}_{U\sim\nu^\star}[\phi_U(x)] - \mathbb{E}_{Z\sim\nu}[\phi_Z(x)]||^2\right] = MMD^2(\nu, \nu^\star)$$

- The kernel is: $k(U, Z) = \mathbb{E}_x[\phi_U(x)^\top \phi_Z(x)]$.

Chizat, Bach. "On the global convergence of gradient descent for over-parameterized models using optimal transport", NeurIPS (2018)

Intuition: MMD as "force field" on ν

Assume henceforth

$$\nu, \nu^* \in \mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) \; : \; \int \|x\|^2 d\mu(x) < \infty \right\}.$$

MMD as free energy: target ν^*, current distribution ν

$$\mathcal{F}(\nu) := \frac{1}{2} MMD^2(\nu^*, \nu) = \frac{1}{2} \underbrace{\mathrm{E}_{\nu} k(x, x')}_{\text{interaction}} + \frac{1}{2} \underbrace{\mathrm{E}_{\nu^*} k(y, y')}_{\text{constant}} - \underbrace{\mathrm{E}_{\nu, \nu^*} k(x, y)}_{\text{confinement}}$$

[A] Ambrosio, Gigli, and Savaré. Gradient flows: in metric spaces and in the space of probability measures. (2008)

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Consider $\{y_i\}_{i=1}^n \overset{\text{i.i.d.}}{\sim} \nu^*$ and $\{x_i\}_{i=1}^n \overset{\text{i.i.d.}}{\sim} \nu$.
Force on a particle z:

$$-\sum_j \nabla_z k(z, x_j) + \sum_j \nabla_z k(z, y_j) = -\nabla_z \hat{f}_{\nu^*, \nu_t}(z)$$

Can we formalize this?

[A] Ambrosio, Gigli, and Savaré. Gradient flows: in metric spaces and in the space of probability measures. (2008)

Wasserstein gradient flows

Tangent space of $\left(\mathcal{P}_2(\mathbb{R}^d), W_2\right)$ is $h \in L^2(\mu)$ where $h : \mathbb{R}^d \to \mathbb{R}^d$.

Define $\nabla_{W_2}\mathcal{F}(\mu)$ of $\mathcal{F}$ at μ using Taylor expansion

$$\mathcal{F}((\mathrm{Id} + \epsilon h)_{\#\mu}) = \mathcal{F}(\mu) + \epsilon \left\langle \nabla_{W_2}\mathcal{F}(\mu), h \right\rangle_\mu + o(\epsilon) \tag{1}$$

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Under reasonable assumptions [A. Theorem 10.4.13]

$$\nabla_{W_2}\mathcal{F}(\mu) = \nabla \mathcal{F}'(\mu).$$

where first variation in direction ξ:

$$\mathcal{F}(\mu + \epsilon\xi) = \mathcal{F}(\mu) + \epsilon \int \mathcal{F}'(\mu)(x)\, d\xi(x) + o(\epsilon) \qquad \mu + \epsilon\xi \in \mathcal{P}_2(\mathbb{R}^d) \tag{2}$$

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The gradient flow is then:

$$\partial_t \nu_t = \mathrm{div}(\nu_t \nabla_{W_2}\mathcal{F}(\nu_t))$$

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Wasserstein gradient flow on MMD

First variation of $\frac{1}{2}MMD^2(\nu^\star, \nu) =: \mathcal{F}(\nu)$

$$\mathcal{F}'(\nu)(z) := f_{\nu^\star,\nu}(z) = 2\left(\mathbb{E}_{U\sim\nu^\star}[k(U,z)] - \mathbb{E}_{U\sim\nu}[k(U,z)]\right)$$

The W_2 gradient flow of the MMD:

$$\partial_t \nu_t = \mathrm{div}(\nu_t \nabla_{W_2}\mathcal{F}(\nu_t)) = \mathrm{div}(\nu_t \nabla f_{\nu^\star,\nu_t})$$

Ambrosio, Gigli, and Savaré. Gradient flows: in metric spaces and in the space of probability measures. (2008, Ch. 10)

Mroueh. Sercu, and Raj. Sobolev Descent. (AISTATS, 2019)

Arbel, Korba, Salim, G. (NeurIPS 2019)

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McKean-Vlasov dynamics for particles (existence and uniqueness under Assumption A):

$$dZ_t = -\nabla_{Z_t} f_{\nu^\star,\nu_t}(Z_t)dt, \qquad Z_0 \sim \nu_0$$

Assumption A: $k(x,x) \leq K$, for all $x \in \mathbb{R}^d$, $\sum_{i=1}^d ||\partial_i k(x,\cdot)||^2 \leq K_{1d}$ and $\sum_{i,j=1}^d ||\partial_i \partial_j k(x,\cdot)||^2 \leq K_{2d}$, d indicates scaling with dimension.

Ambrosio, Gigli, and Savaré. Gradient flows: in metric spaces and in the space of probability measures. (2008, Ch. 10)

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Wasserstein gradient flow on the MMD

Forward Euler scheme [A, Section 2.2]:

$$\nu_{n+1} = (I - \gamma \nabla f_{\nu^\star, \nu_t})_{\#} \nu_n$$
$$Z_{n+1} = Z_n - \gamma \nabla_{Z_n} f_{\nu^\star, \nu_n}(Z_n), \qquad Z_0 \sim \nu_0, \; Z_n \sim \nu_n$$

Under Assumption A, ν_n approaches ν_t as $\gamma \to 0$

[A] Arbel, Korba, Salim, G. (NeurIPS 2019)

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Consistency? Does ν_t converge to $\nu^\star$ as $t \to \infty$?

[A] Arbel, Korba, Salim, G. (NeurIPS 2019)

Consistency

Can we use geodesic (displacement) convexity?

- A geodesic ρ_t between ν_1 and ν_2 is given by the transport map $T_{\nu_1}^{\nu_2} : \mathbb{R}^d \to \mathbb{R}^d$:

$$\rho_t = \left((1-t)\mathrm{Id} + tT_{\nu_1}^{\nu_2}\right)_{\#\nu_1}$$

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- A functional $\mathcal{F}$ is displacement convex if:

$$\mathcal{F}(\rho_t) \leq (1-t)\mathcal{F}(\nu_1) + t\mathcal{F}(\nu_2)$$

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MMD is not displacement convex in general (it is always mixture convex[1]).

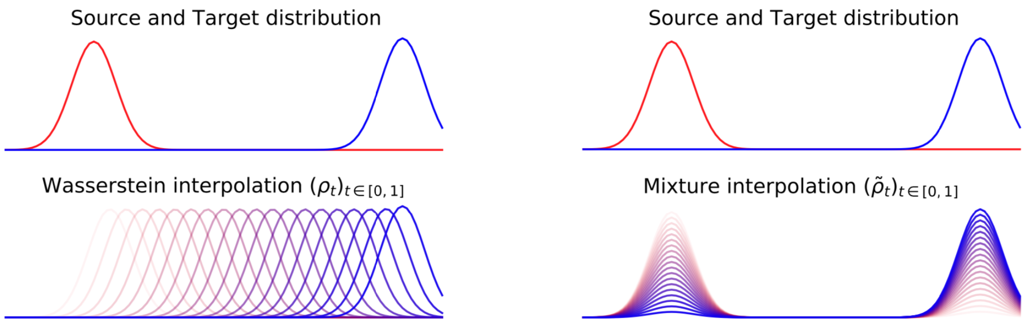

Figure from Korba, Salim, ICML 2022 Tutorial, "Sampling as First-Order Optimization over a space of probability measures"

1. $\mathcal{F}(t\nu_1 + (1-t)\nu_2) \leq t\mathcal{F}(\nu_1) + (1-t)\mathcal{F}(\nu_2) \qquad \forall t \in [0,1]$).

Noise injection for convergence

Noise injection: Evaluate $\nabla f_{\nu^\star,\nu_t}$ outside of the support of ν_t to get a better signal!

- Sample $u_t \sim \mathcal{N}(0,1)$ and β_t is the noise level:

$$Z_{t+1} = Z_t - \gamma \nabla f_{\nu^\star,\nu_t}(Z_t + \beta_t u_t); \qquad Z_t \sim \nu_t$$

- Similar to continuation methods,[1] but extended to interacting particles.
- Different from entropic regularization:

$$Z_{t+1} = Z_t - \gamma \nabla f_{\nu^\star,\nu_t}(Z_t) + \beta_t u_t$$

[1] Chaudhari, Oberman, Osher, Soatto, Carlier. Deep relaxation: partial differential equations for optimizing deep neural networks. Research in the Mathematical Sciences (2017)
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Noise injection: consistency

Recall: $$Z_{t+1} = Z_t - \gamma \nabla f_{\nu^\star,\nu_t}(Z_t + \beta_t u_t); \qquad Z_t \sim \nu_t$$

Tradeoff for β_t

- Large β_t: $\nu_{t+1} - \nu_t$ not a descent direction any more: $\mathcal{F}(\nu_{t+1}) > \mathcal{F}(\nu_t)$
- Small β_t: does not converge

[A] Arbel, Korba, Salim, G. (NeurIPS 2019)

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- Small β_t: does not converge

Need β_t such that:

$$\mathcal{F}(\nu_{t+1}) - \mathcal{F}(\nu_t) \leq -C\gamma \mathbb{E}_{\substack{X_t \sim \nu_t \\ u_t \sim \mathcal{N}(0,1)}} [\|\nabla f_{\nu^\star,\nu_t}(X_t + \beta_t u_t)\|^2]$$

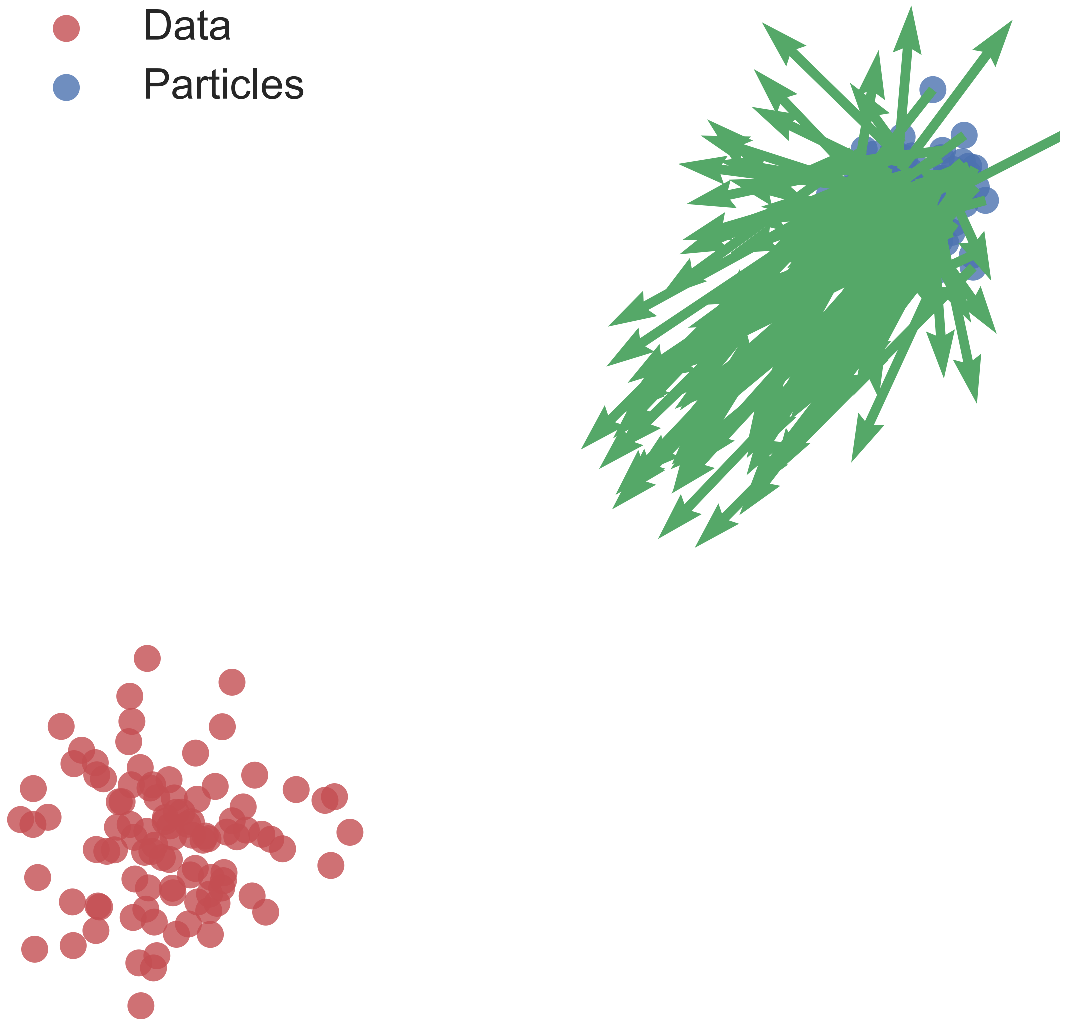
$$\sum_i^t \beta_i^2 \underset{t\to\infty}{\to} \infty$$

Then [A, Proposition 8]

$$\mathcal{F}(\nu_t) \leq \mathcal{F}(\nu_0) e^{-C\gamma \sum_i^t \beta_i^2}.$$

[A] Arbel, Korba, Salim, G. (NeurIPS 2019)

Noise injected MMD flow in practice



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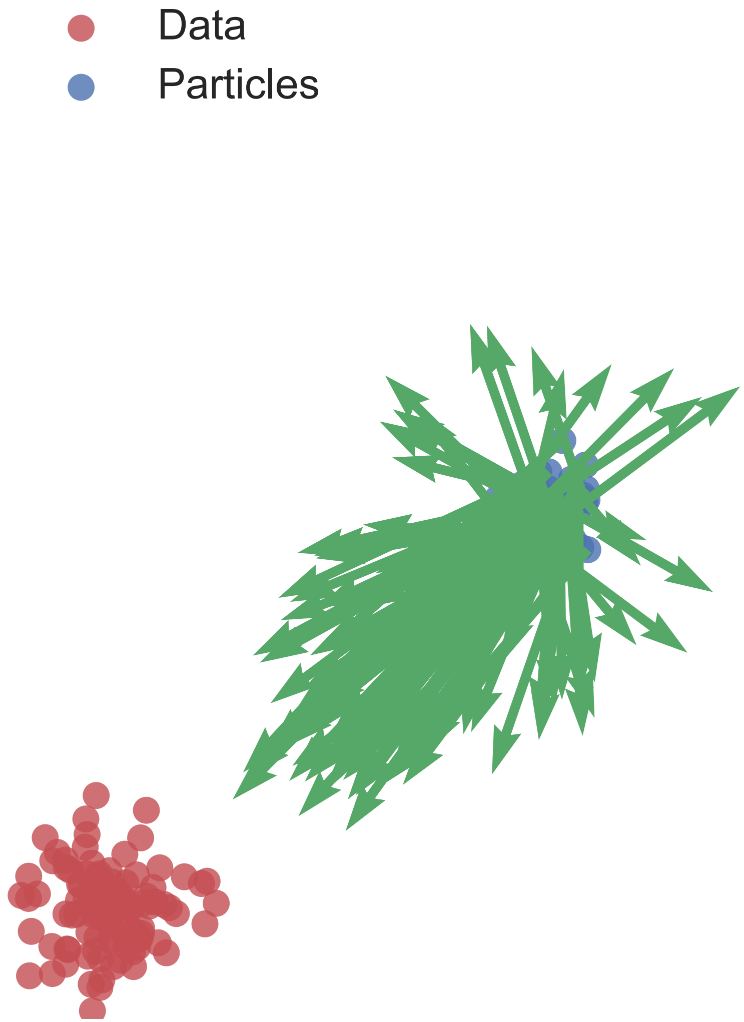

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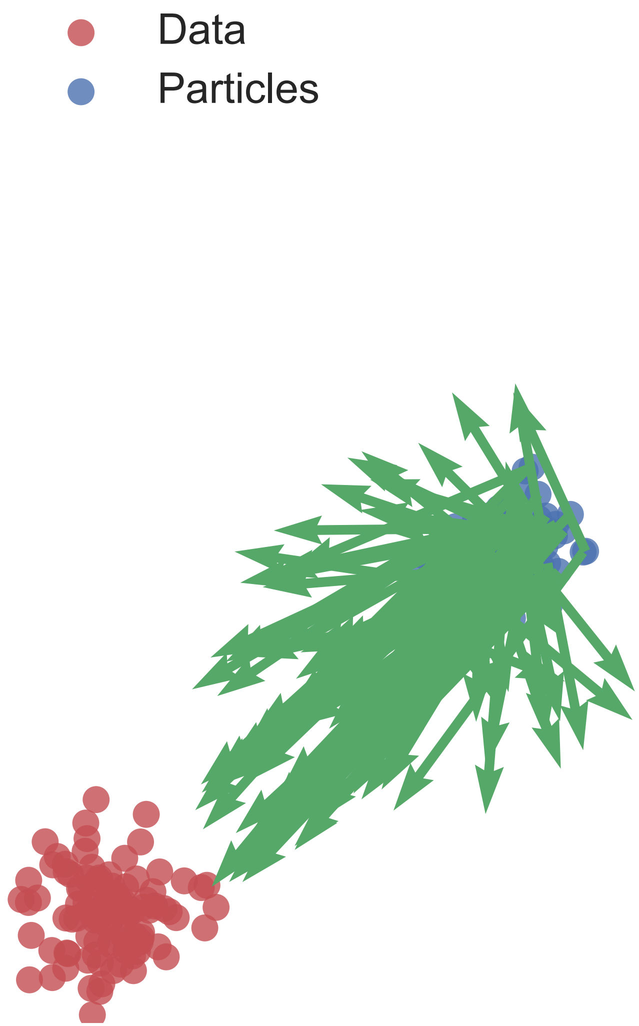

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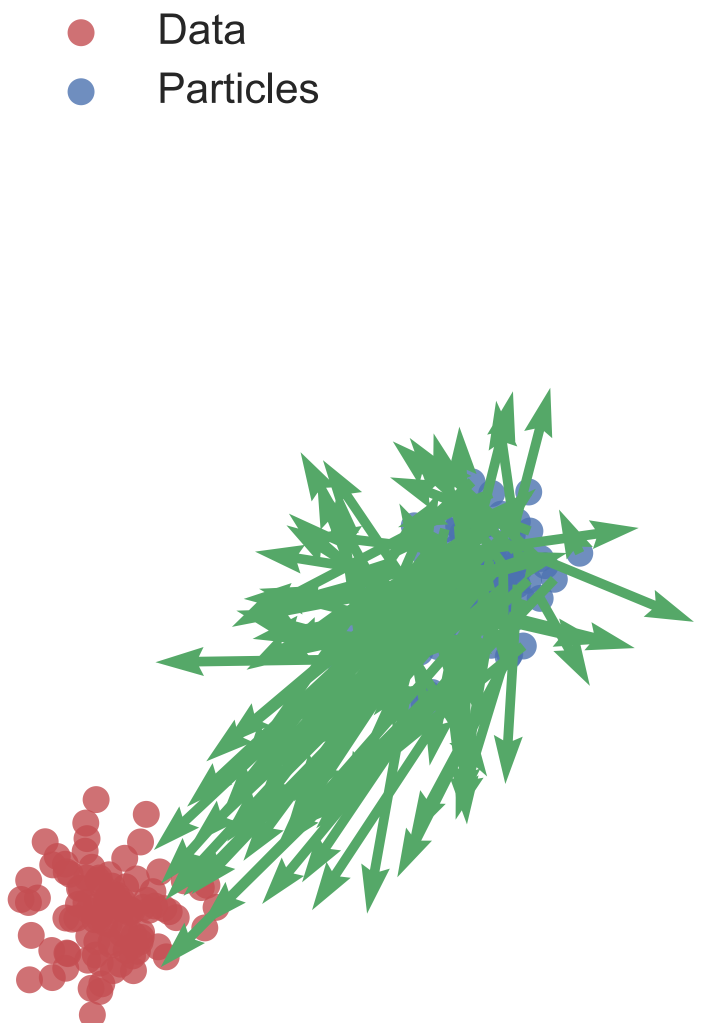

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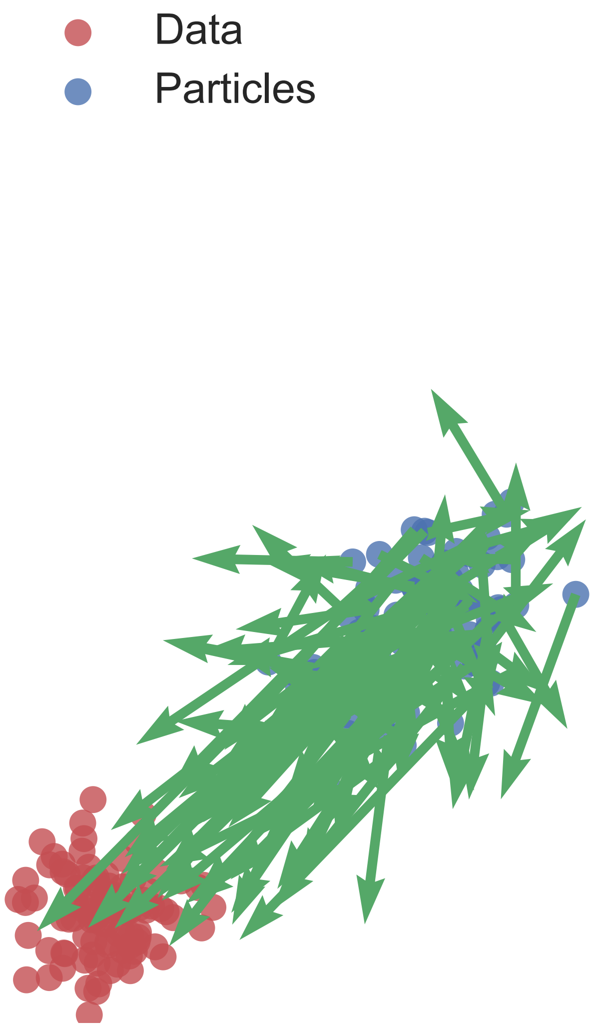
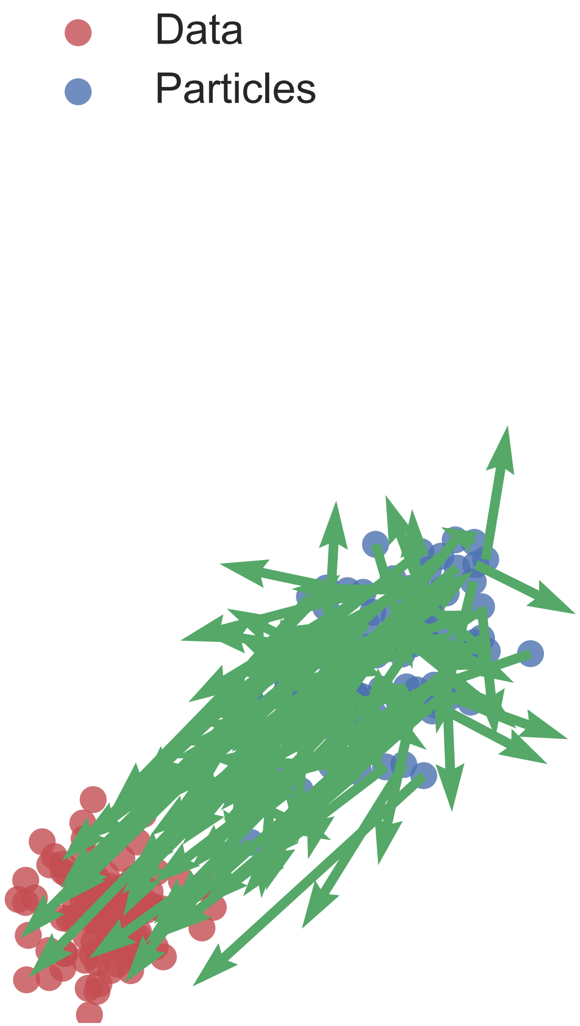


Noise injected MMD flow in practice



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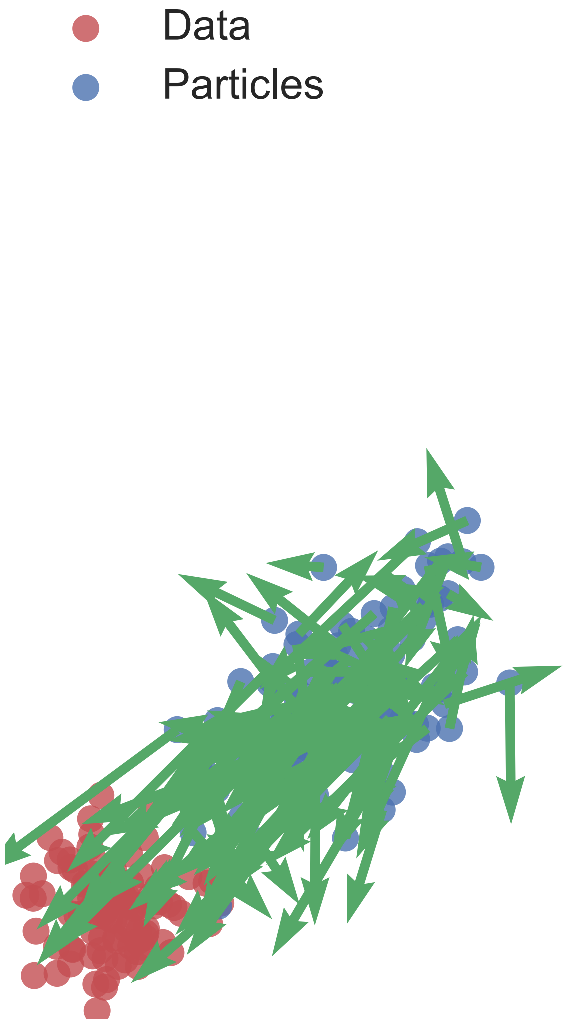

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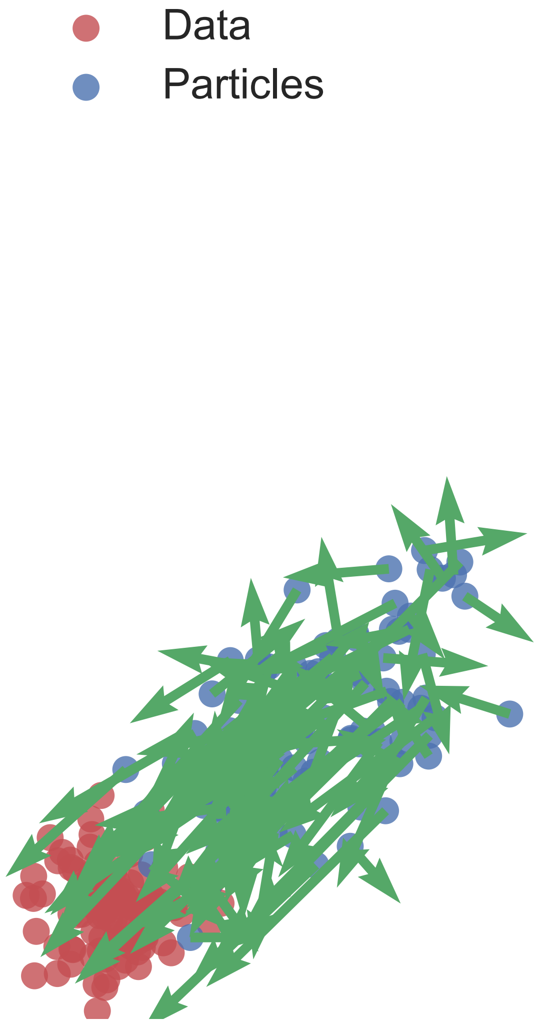

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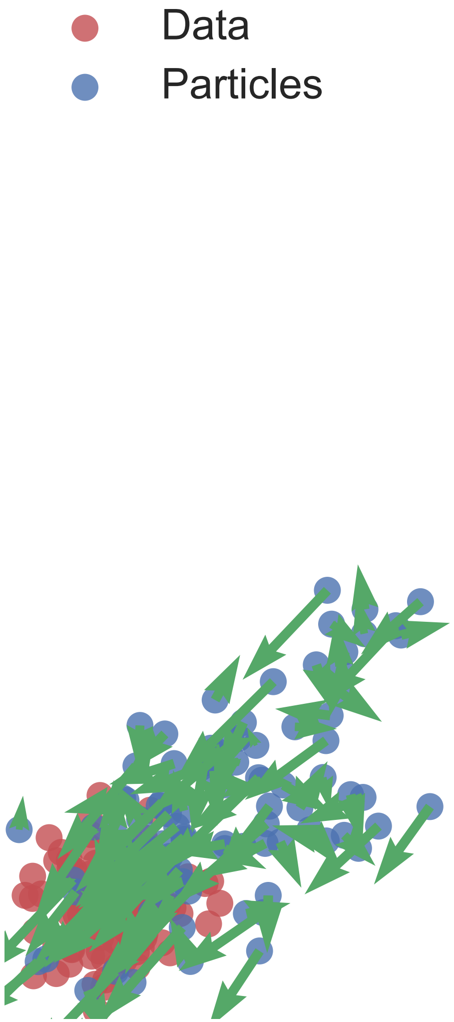
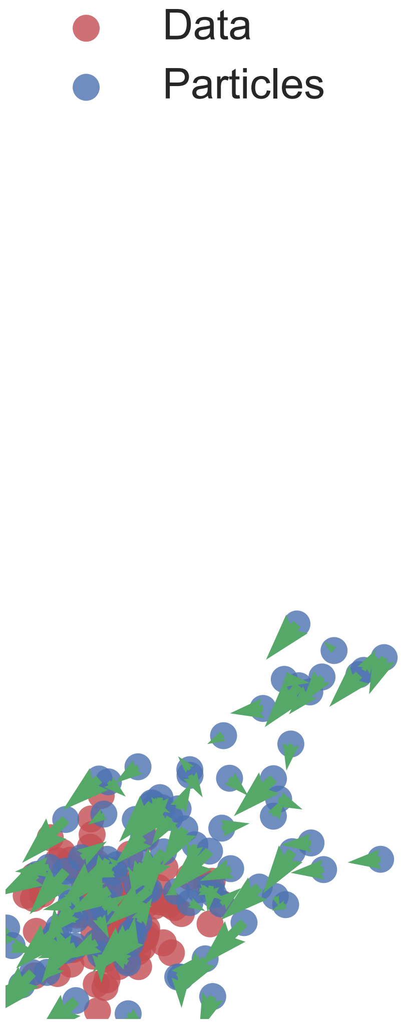


Noise injected MMD flow in practice



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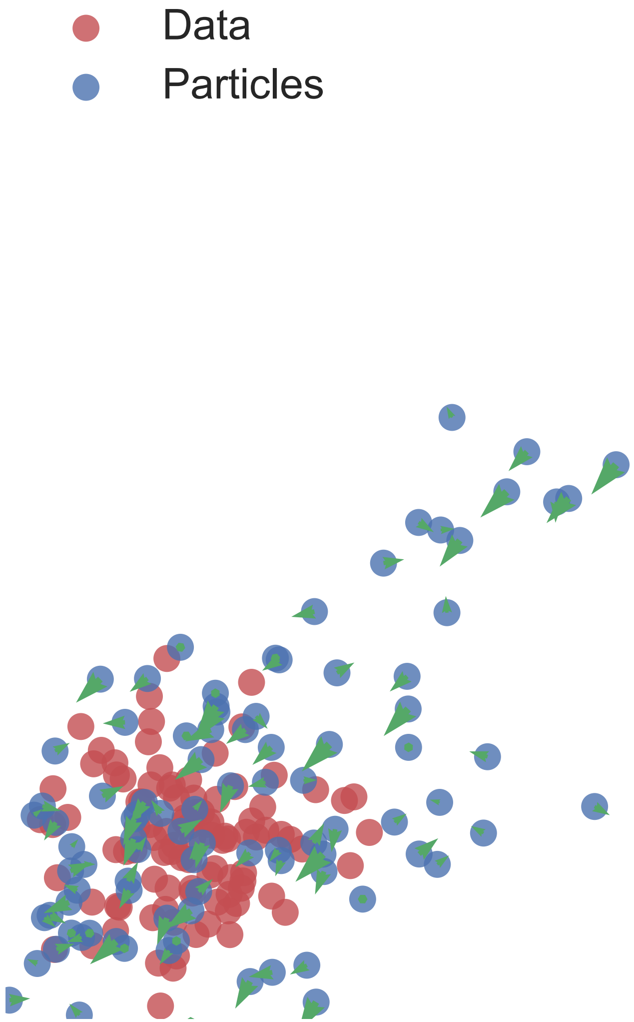
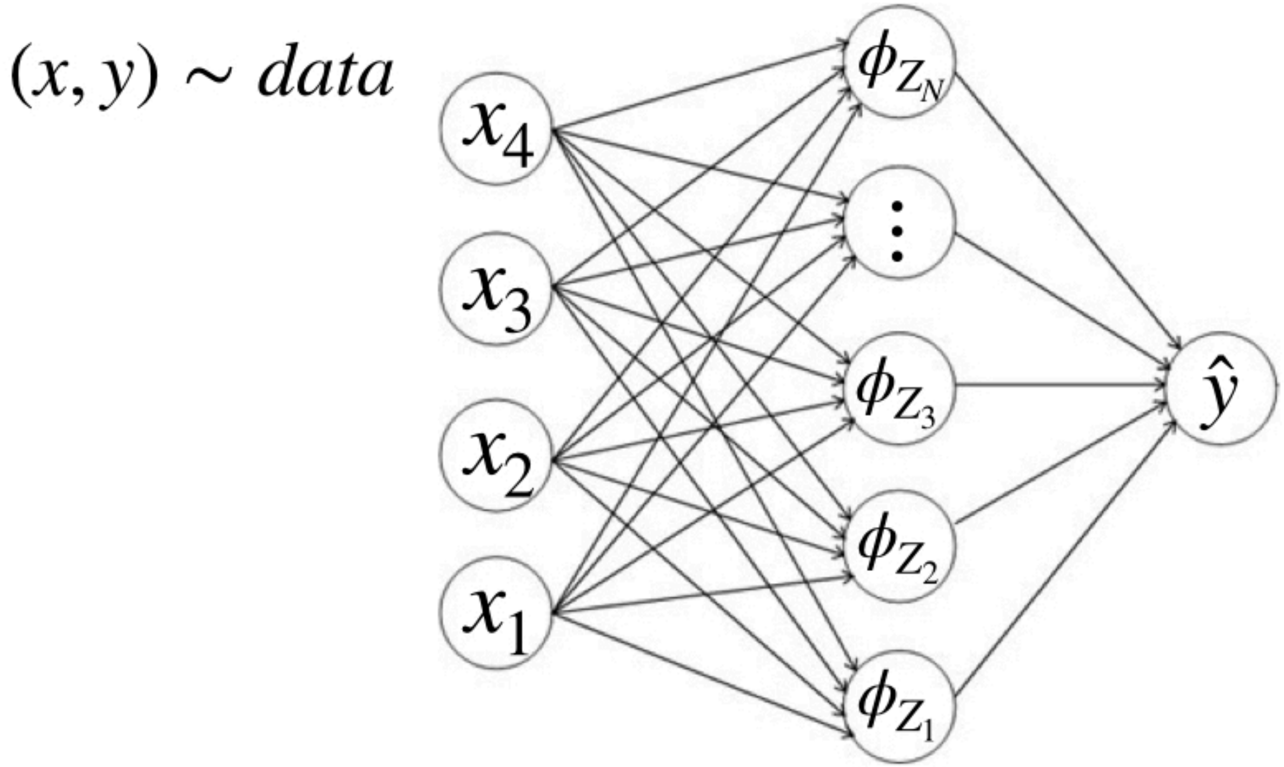


Noise injected MMD flow in practice

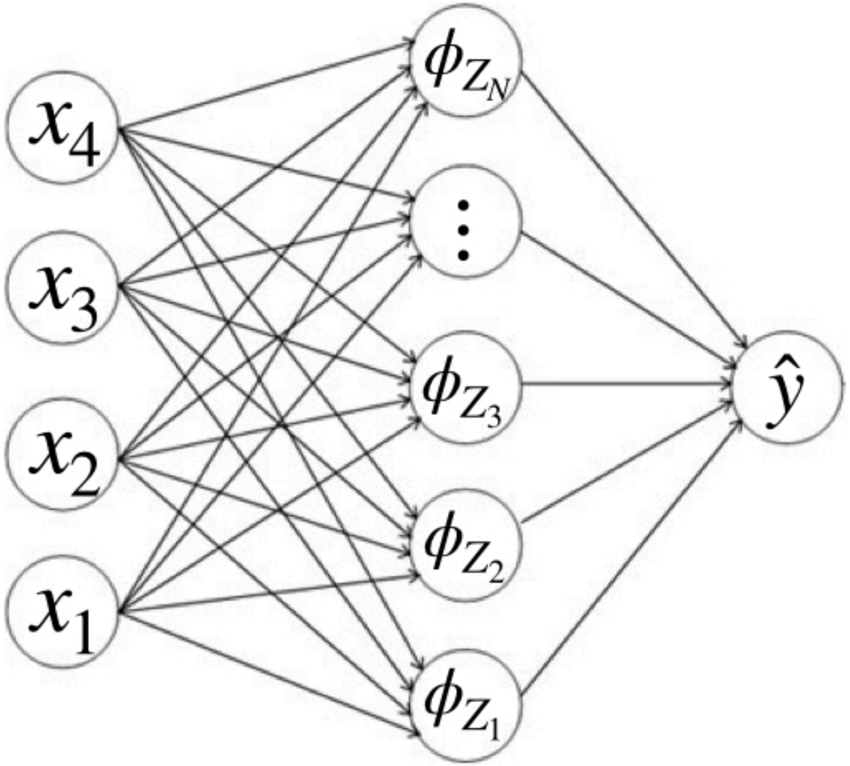
Noise injection: neural net setting



$$\min_{Z_1,\ldots,Z_N} \mathbb{E}_{data}[\| \frac{1}{M} \sum_{m}^{M} \phi_{U^m}(x) - \frac{1}{N} \sum_{n=1}^{N} \phi_{Z^n}(x) \|^2]$$

Noise injection: neural net setting

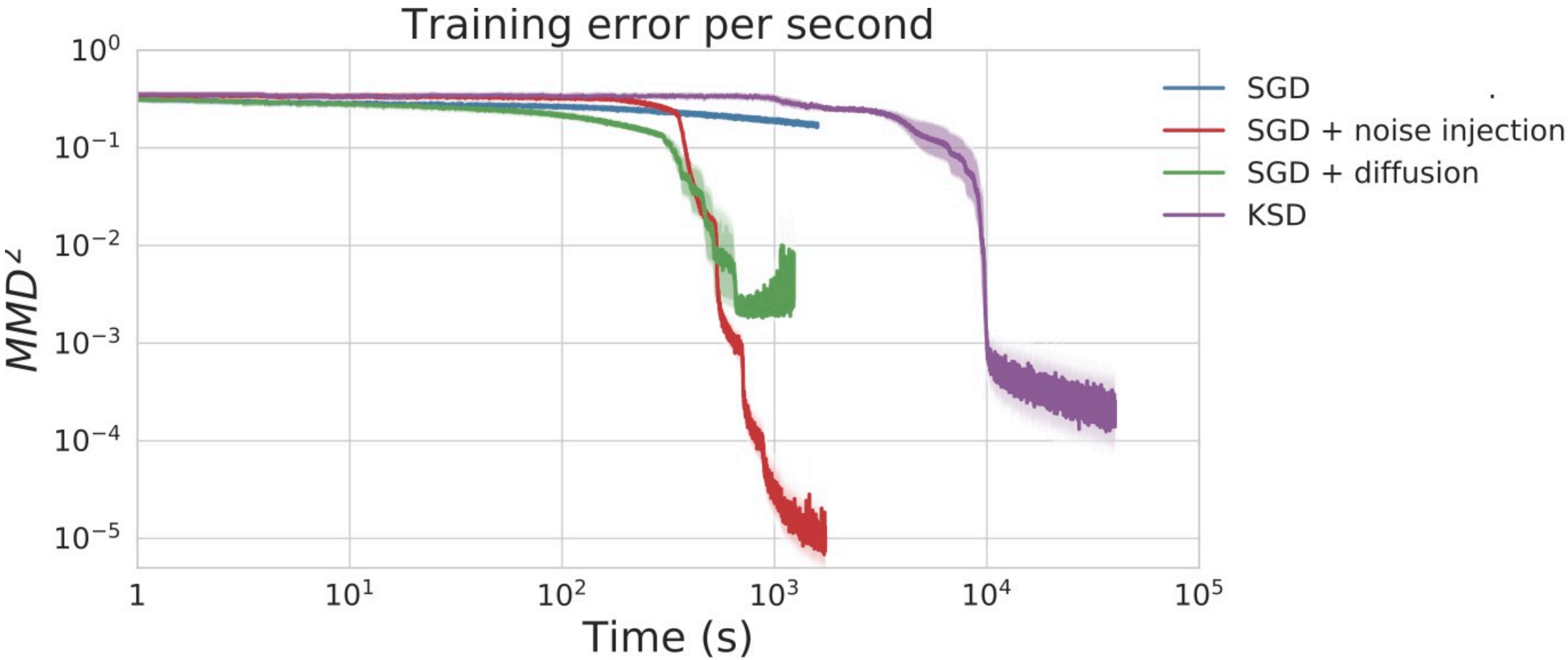
$(x, y) \sim data$



$$\min_{Z_1,\ldots,Z_N} MMD^2(\nu^*, \frac{1}{N}\sum_{n=1}^{N} \delta_{Z^n})$$

$$k(Z, Z') = \mathbb{E}_{data}[\phi_Z(x)\phi_{Z'}(x)]$$

Noise injection: neural net setting



KSD is Kernel Sobolev Discrepancy. Y. Mroueh, T. Sercu, and A. Raj. “Sobolev Descent.” In: AISTATS. 2019.

Adaptive MMD Flow (ICLR 25)

arXiv > cs > arXiv:2405.06780

Computer Science > Machine Learning

[Submitted on 10 May 2024]

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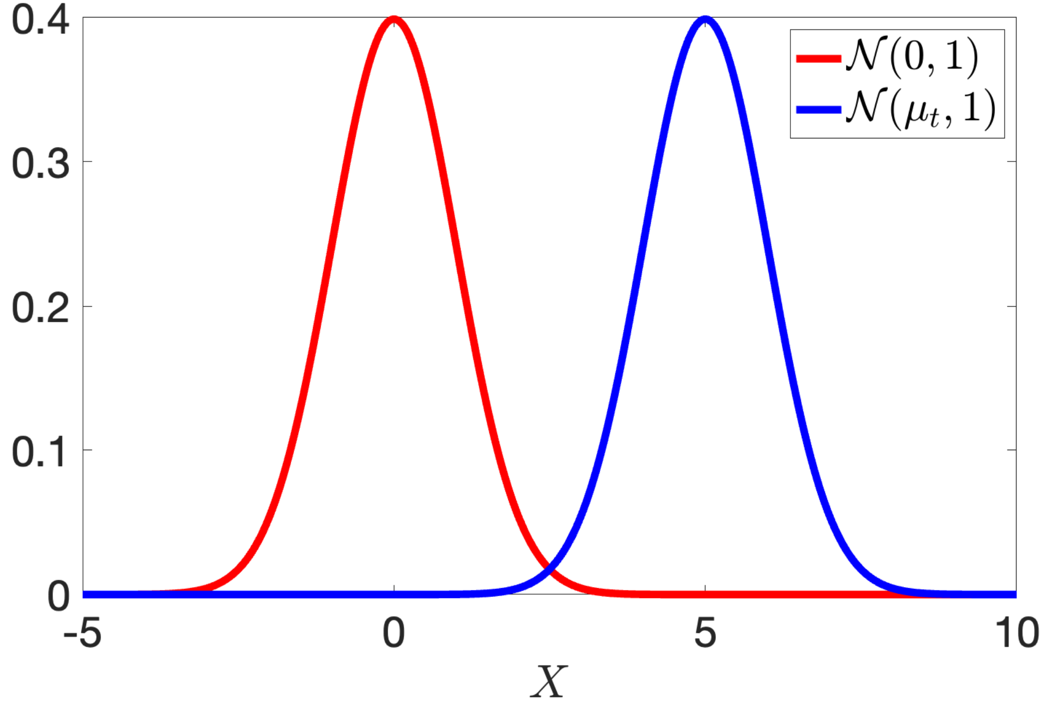
Will an adaptive kernel help?

Define the two measures:

$$\nu^\star := \mathcal{N}(0, \sigma^2 \mathrm{Id}) \qquad \nu_t := \mathcal{N}(\mu_t, \sigma^2 \mathrm{Id}).$$

Consider the family of MMDs:

$$\mathrm{MMD}^2_\alpha(\nu^\star, \nu_t) \qquad \text{with} \qquad k_\alpha(x, y) = \alpha^{-d} \exp[-||x - y||^2/(2\alpha^2)]$$



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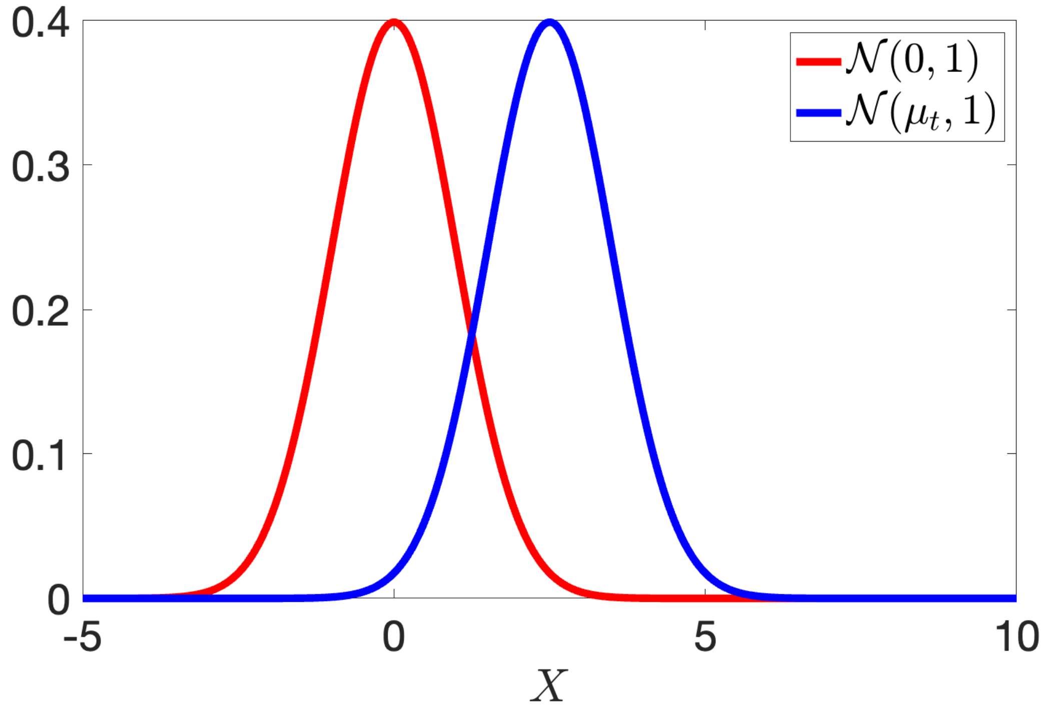


Will an adaptive kernel help?

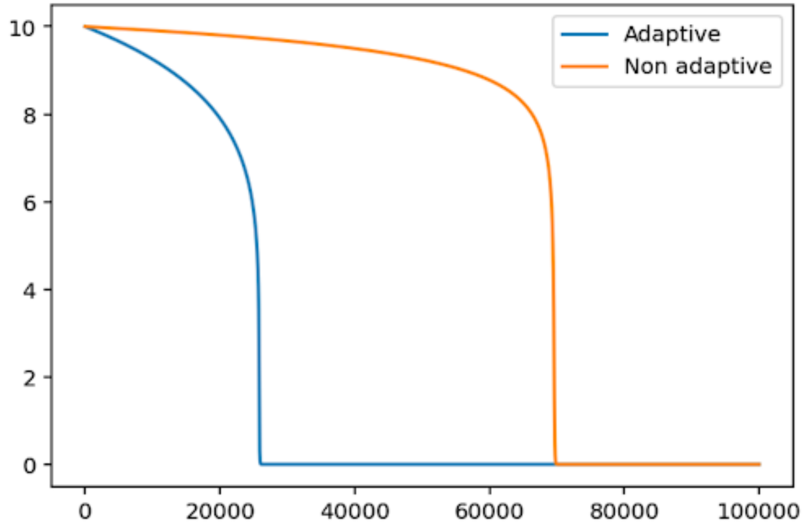
Choose kernel such that:

$$\alpha^\star = \text{argmax}_{\alpha \geq 0} \|\nabla_{\mu_t} \text{MMD}^2_\alpha(\nu^\star, \nu_t)\|.$$

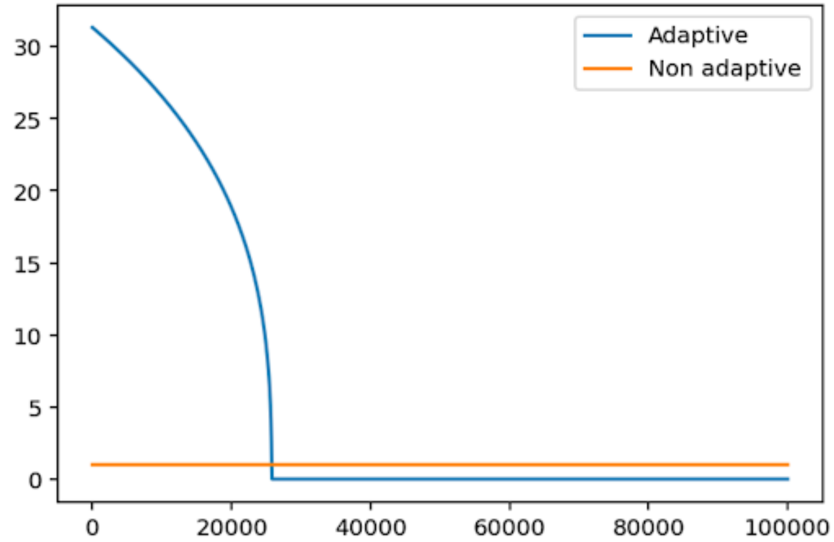
Then

$$\alpha^\star = \text{ReLU}(\|\mu_t\|^2/(d+2) - 2\sigma^2)^{1/2}.$$

Value of mean μ_t

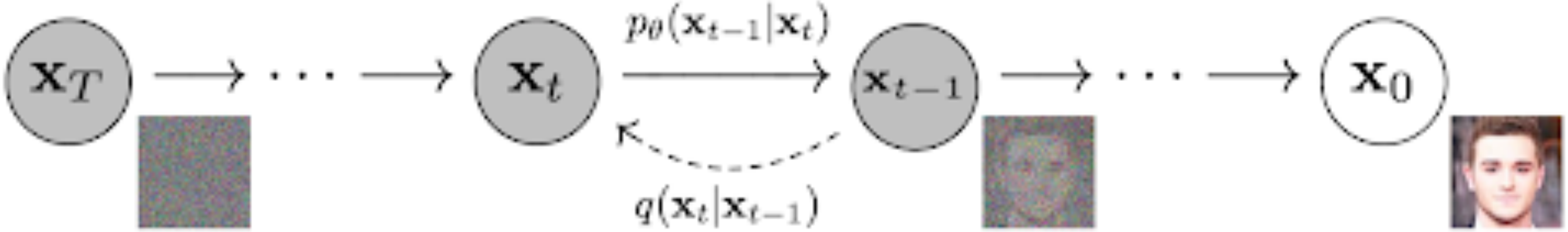


Kernel width α



How to train an adaptive MMD (1)

Diffusion:



Generate forward path $\tilde{\nu}_t$, $t \in [0, 1]$, such that $\tilde{\nu}_0 = \nu^\star$, and $\tilde{\nu}_1 = \mathrm{N}(0, \mathrm{Id})$ is a Gaussian noise.

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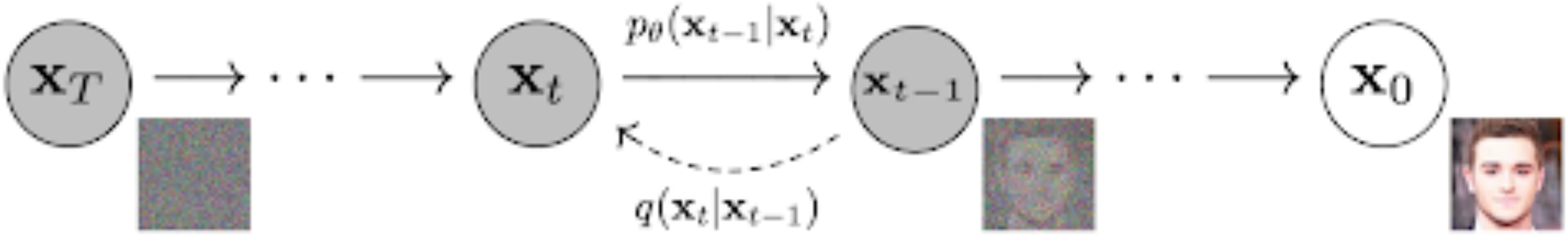

Generate forward path $\tilde{\nu}_t$, $t \in [0,1]$, such that $\tilde{\nu}_0 = \nu^\star$, and $\tilde{\nu}_1 = \mathrm{N}(0, \mathrm{Id})$ is a Gaussian noise.

Given samples $\tilde{x}_0 \sim \tilde{\nu}_0$, the samples $\tilde{x}_t | \tilde{x}_0$ are given by

$$\tilde{x}_t = \alpha_t \tilde{x}_0 + \beta_t \epsilon, \quad \epsilon \in \mathrm{N}(0, \mathrm{Id}),$$

with $\alpha_0 = \beta_1 = 1$ and $\alpha_1 = \beta_0 = 0$.

- low t: $\tilde{x}_t$ close to the original data $\tilde{x}_0$,
- high t: $\tilde{x}_t$ close to a unit Gaussian

Schedule (α_t, β_t) is the variance-preserving one of Song, Sohl-Dickstein, Kingma, Kumar, Ermon, Poole. Score-based generative modeling through stochastic differential equations (ICLR 2021)

How to train an adaptive MMD (2)

Time-dependent MMD training loss:

$$\mathcal{F}(\theta, t) := \frac{1}{2}\mathrm{E}_{\tilde{\nu}_t} k_{\theta,t}(\tilde{x}_t, \tilde{x}_t') + \mathrm{E}_{\tilde{\nu}_t,\nu^*} k_{\theta,t}(\tilde{x}_t, y)$$

with kernel

$$k_{\theta,t}(x, y) = \phi(x; t, \theta)^\top \phi(y; t, \theta)$$

and witness $f^{(\theta,t)}_{\nu^\star,\tilde{\nu}_t}$.

Gulrajani, Ahmed, Arjovsky, Dumoulin, Courville, Improved Training of Wasserstein GANs (NeurIPS 2017)

Binkowski, Sutherland, Arbel, G. (NeurIPS 2018)

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and witness $f^{(\theta,t)}_{\nu^\star, \tilde{\nu}_t}$.

Train θ by minimizing noise-conditional loss on forward path:

$$\mathcal{F}_{\mathrm{tot}}(\theta, t) = \mathcal{F}(\theta, t) + \lambda_{\ell_2}\mathcal{F}_{\ell_2}(\theta, t) + \lambda_\nabla \mathcal{F}_\nabla(\theta, t),$$

$$\mathcal{F}_{\mathrm{tot}}(\theta) = \mathbb{E}_{t \sim U[0,1]}\left[\mathcal{F}_{\mathrm{tot}}(\theta, t)\right]$$

where

- $\mathcal{F}_{\ell_2}(\theta, t)$ is a "variance"-style penalty
- $\mathcal{F}_\nabla(\theta, t) = \frac{1}{N}\sum_{i=1}^N (\|\nabla f^{(\theta,t)}_{\nu^\star, \tilde{\nu}_t}(\tilde{x}_{t,i})\|_2 - 1)^2$, is a gradient penalty

Gulrajani, Ahmed, Arjovsky, Dumoulin, Courville, Improved Training of Wasserstein GANs (NeurIPS 2017)

Binkowski, Sutherland, Arbel, G. (NeurIPS 2018)

Sample generation

Algorithm Noise-adaptive MMD gradient flow

Sample initial particles $Z \sim \mathrm{N}(0, \mathrm{Id})$
Set $\Delta t = (t_{\mathrm{max}} - t_{\mathrm{min}})/T$
for $i = T$ to 0 do
  Set the noise level $t = i\Delta t$
  Set $Z_t^0 = Z$
  for $n = 0$ to $N_\mathrm{s} - 1$ do
    $Z_t^{n+1} = Z_t^n - \eta \nabla f_{\nu^\star, \nu_t}^{(\theta^\star, t)}(Z_t^n)$
  end for
  Set $Z = Z_t^N$
end for
Output Z

Results

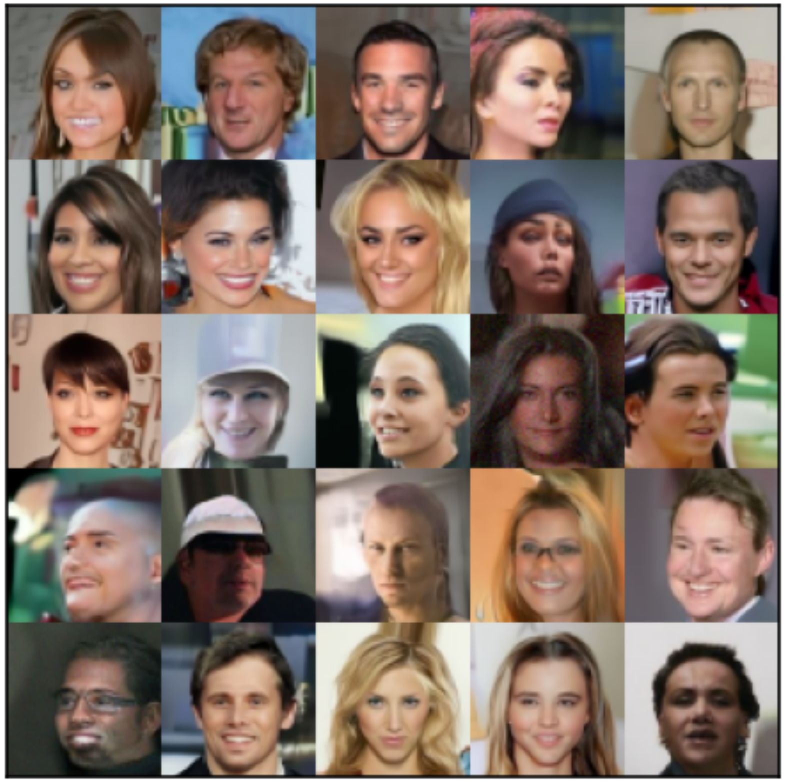
Table: Unconditional generation, CIFAR-10. MMD GAN (orig.), used mixed-RQ kernel. "Orig." – original paper, "impl." – our implementation.

Method	FID	IS	NFE
MMD GAN (orig.)	39.90	6.51	-
MMD GAN (impl.)	13.62	8.93	-
DDPM (orig.)	3.17	9.46	1000
DDPM (impl.)	5.19	8.90	100
Discriminator flows			
DGGF-KL	28.80	-	110
JKO-Flow	23.10	7.48	$\sim$ 150
GS-MMD-RK	55.00	-	86
DMMD (ours)	8.31	9.09	100
DMMD (ours)	7.74	9.12	250

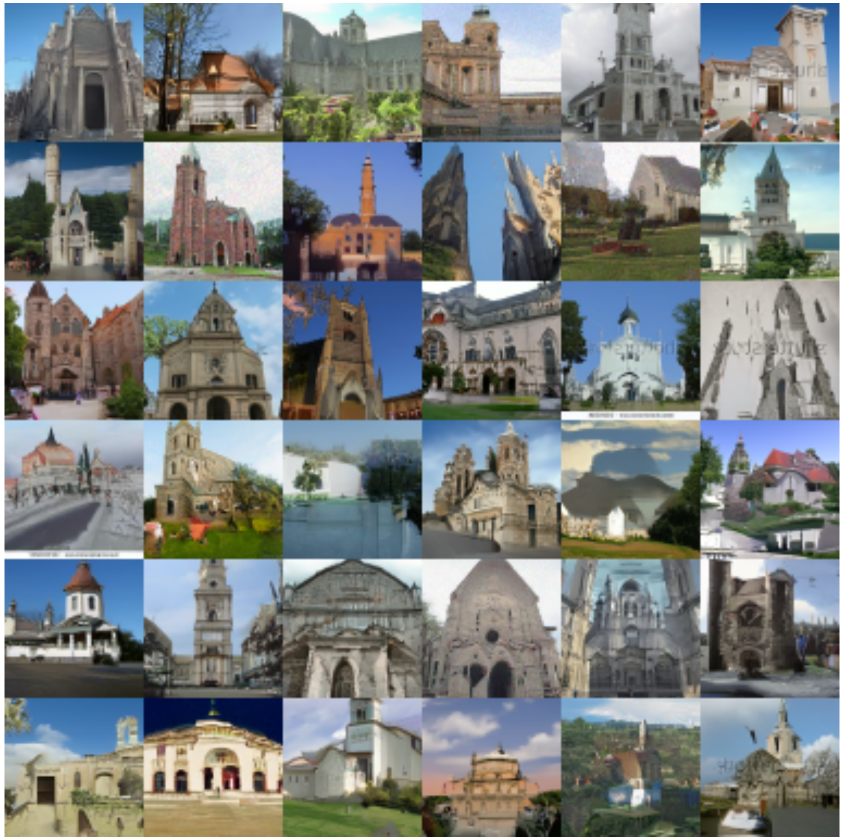
DDPM from (Ho et al., 2020). Discriminator flows include two KL gradient flows trained adversarially: JKO-Flow (Fan et al., 2022) and Deep Generative Wasserstein Gradient Flows (DGGF-KL) (Heng et al., 2023). GS-MMD-RK is Generative Sliced MMD Flows with Riesz Kernels (Hertrich et al., 2024)

Images

CELEB-A (64x64)



LSUN Church (64x64)



Summary

- Gradient flows based on kernel dependence measures:
  - MMD flow is simpler, KALE flow is mode-seeking
  - Noise injection can improve convergence
- NeurIPS 2019, ICLR 2025

NeurIPS 2019:

arXiv > stat > arXiv:1906.04370

Statistics > Machine Learning

[Submitted on 11 Jun 2019 (v1), last revised 3 Dec 2019 (this version, v2)]

Maximum Mean Discrepancy Gradient Flow

Michael Arbel, Anna Korba, Adil Salim, Arthur Gretton

Adaptive MMD (ICLR 25):

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NeurIPS 2021:

arXiv > stat > arXiv:2106.08929

Statistics > Machine Learning

[Submitted on 16 Jun 2021 (v1), last revised 29 Oct 2021 (this version, v2)]

KALE Flow: A Relaxed KL Gradient Flow for Probabilities with Disjoint Support

Pierre Glaser, Michael Arbel, Arthur Gretton

Adaptive MMD (ICLR 25):

arXiv > cs > arXiv:2405.06780

Computer Science > Machine Learning

[Submitted on 10 May 2024]

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Alexandre Galashov, Valentin de Bortoli, Arthur Gretton

(De)regularized MMD (JMLR, submitted):

arXiv > stat > arXiv:2409.14980

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[Submitted on 23 Sep 2024]

(De)-regularized Maximum Mean Discrepancy Gradient Flow

Zonghao Chen, Aratrika Mustafi, Pierre Glaser, Anna Korba, Arthur Gretton, Bharath K. Sriperumbudur

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The Gatsby Charitable Foundation

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