Causal Effect Estimation using Proxy Variables

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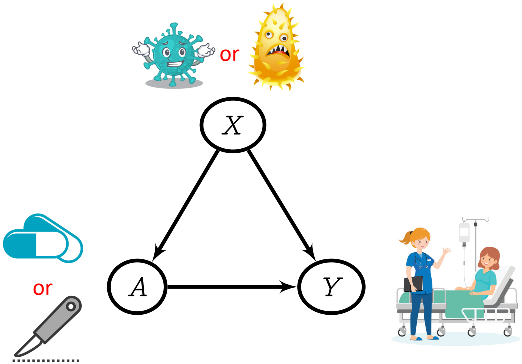
Observation vs intervention

Conditioning from observation: $\mathbb{E}[Y|A=a] = \sum_x \mathbb{E}[Y|a,x]p(x|a)$

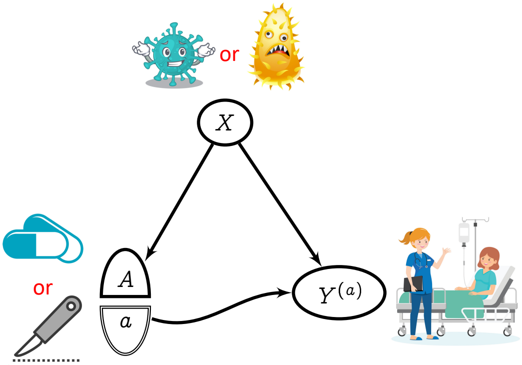


From our *observations* of historical hospital data:

- $P(Y = \text{cured}|A = \text{pills}) = 0.85$
- $P(Y = \text{cured}|A = \text{surgery}) = 0.72$

Observation vs intervention

Average causal effect (intervention): $\mathbb{E}[Y^{(a)}] = \sum_x \mathbb{E}[Y \mid a, x]p(x)$

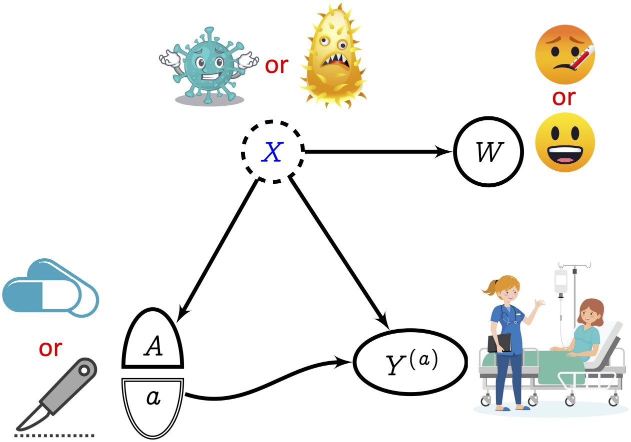


From our *intervention* (making all patients take a treatment):

- $P(Y^{(\text{pills})} = \text{cured}) = 0.64$
- $P(Y^{(\text{surgery})} = \text{cured}) = 0.75$

Richardson, Robins (2013), Single World Intervention Graphs (SWIGs): A Unification of the Counterfactual and Graphical Approaches to Causality

We record symptom W, not disease X



- $P(W = \text{fever} | X = \text{mild}) = 0.2$
- $P(W = \text{fever} | X = \text{severe}) = 0.8$

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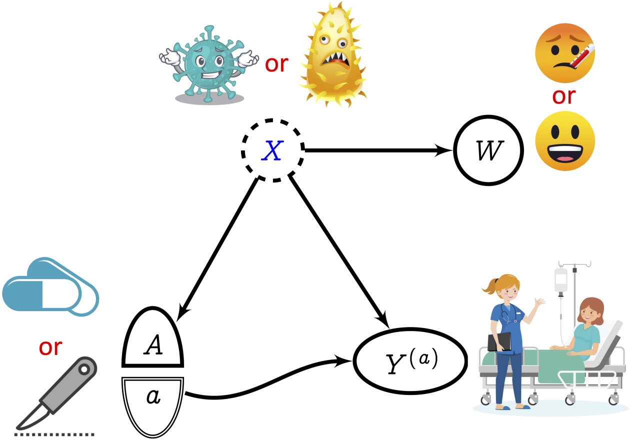

- $P(W = \text{fever} \mid X = \text{mild}) = 0.2$
- $P(W = \text{fever} \mid X = \text{severe}) = 0.8$

Could we just write: $P(Y^{(a)}) \overset{?}{=} \sum_{w \in \{0,1\}} \mathbb{E}[Y \mid a, w] p(w)$

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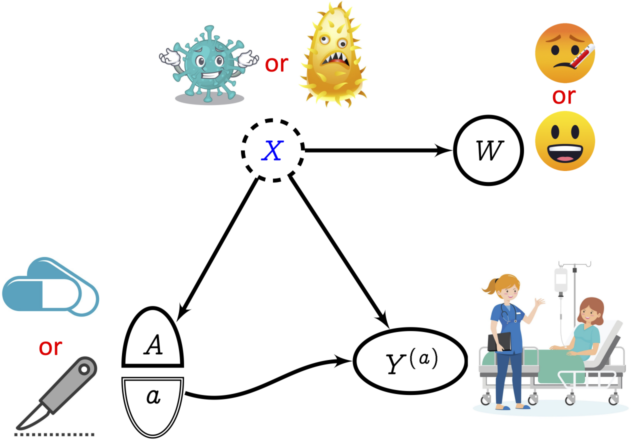

Wrong recommendation made:

- $\sum_{w\in\{0,1\}} \mathbb{E}[\text{cured}|\text{pills}, w]p(w) = 0.8 \quad (\neq 0.64)$
- $\sum_{w\in\{0,1\}} \mathbb{E}[\text{cured}|\text{surgery}, w]p(w) = 0.73 \quad (\neq 0.75)$

Correct answer impossible without observing X

Pearl (2010), On Measurement Bias in Causal Inference

Outline

Causal effect estimation, with hidden covariates X:

- Use proxy variables (negative controls)

Applications: effect of actions under

- privacy constraints (email, ads, DMA)
- data gathering constraints (edge computing)
- fundamental limitations (preferences, state of mind)

Related topic: domain shift (the blessings of multiple domains)

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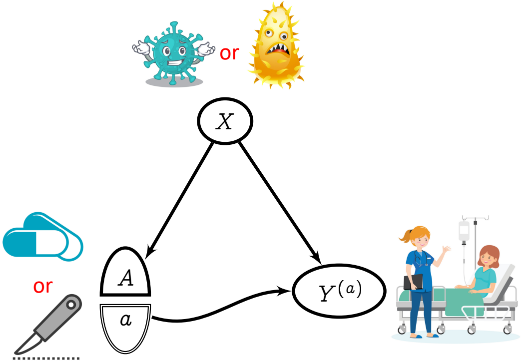
What's new and why?

- Treatment A, proxy variables, etc can be multivariate, complicated...
- ...by using kernel or neural net feature representations
- Don't ~~meet your heroes~~ model your hidden variables!

Some core assumptions



Assume:

- Stable Unit Treatment Value Assumption (aka "no interference"),
- Conditional exchangeability $Y^{(a)} \perp\!\!\!\perp A | X$.
- Overlap.

Proxy causal learning

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Identifying causal effects with proxy variables of an unmeasured confounder

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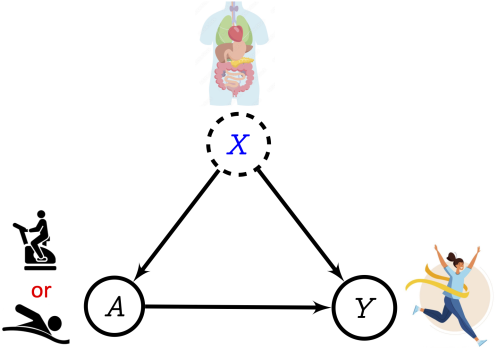
What are proxies, and when are they useful?

Unobserved X with (possibly) complex nonlinear effects on A, Y

In this example:

- X: true physical status
- A: exercise regimes
- Y: fitness goal



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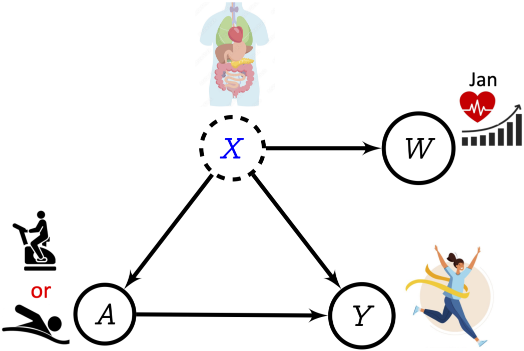

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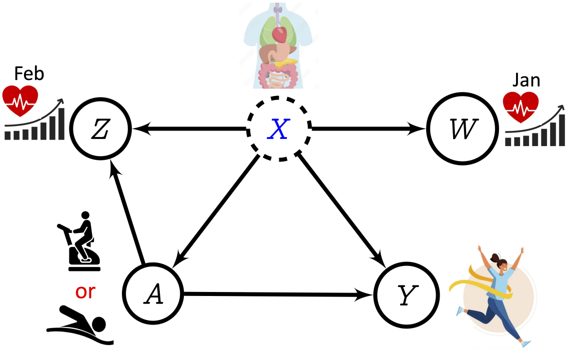

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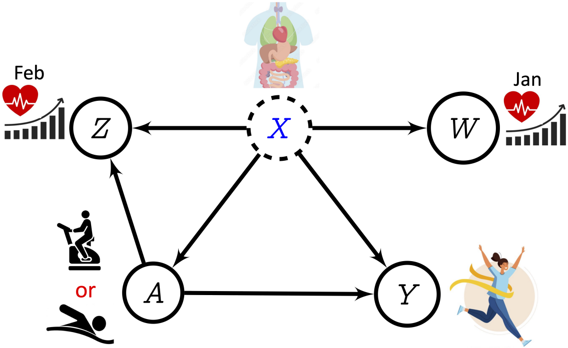


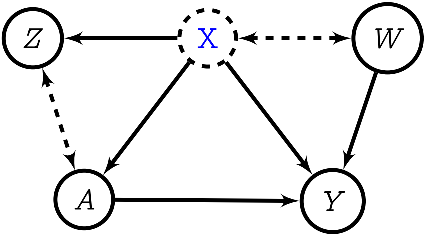
$\Longrightarrow$ Can recover $\mathbb{E}(Y^{(a)})$ from observational data

Proxy variables: general setting

Unobserved X with (possibly) complex nonlinear effects on A, Y

The definitions are:

- X: unobserved confounder.
- A: treatment
- Y: outcome
- Z: treatment proxy
- W outcome proxy



Miao, Geng, Tchetgen Tchetgen (2018): Identifying causal effects with proxy variables of an unmeasured confounder.

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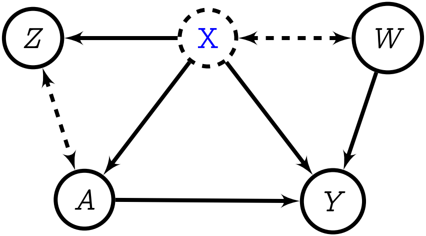

Structural assumptions:

$$W \perp\!\!\!\perp (Z, A) | X$$

$$Y \perp\!\!\!\perp Z | (A, X)$$

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Why proxy variables? A simple proof

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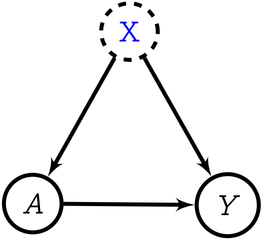

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$$\underbrace{P(Y^{(a)})}_{d_y \times 1} := \sum_{i=1}^{d_x} P(Y|x_i, a)P(x_i)$$

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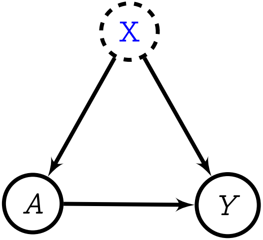

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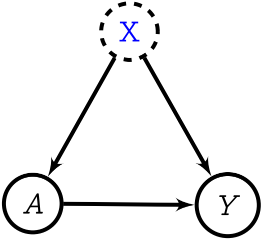

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Goal: “get rid of the blue” X

...add the outcome proxy W

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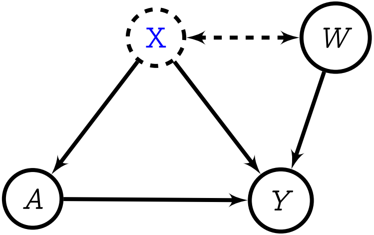

For each a, if we could solve:

$$\underbrace{P(Y|X, a)}_{d_y \times d_x} = \underbrace{H_{w,a}}_{d_y \times d_w} \underbrace{P(W|X)}_{d_w \times d_x}$$

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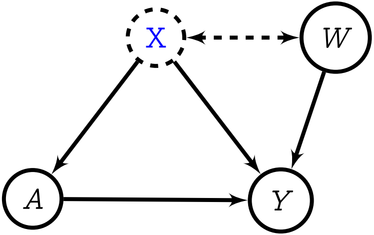

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.....then

$$P(Y^{(a)}) = P(Y|X,a)P(X)$$

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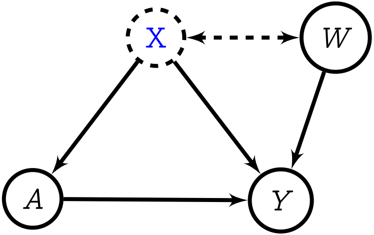

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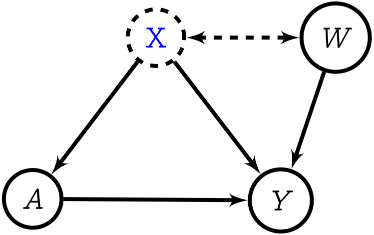

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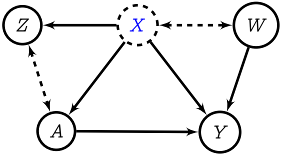
...now project onto $p(X|Z,a)$

From last slide,

$$P(Y|X,a) = H_{w,a} P(W|X)$$



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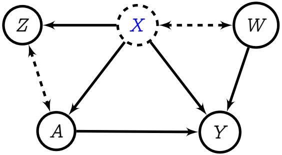

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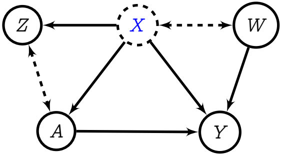

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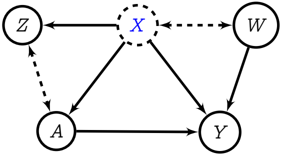

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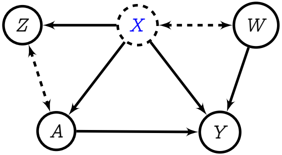

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Solve for $H_{w,a}$:

$$P(Y|Z,a) = H_{w,a} P(W|Z,a)$$

Everything observed!

Domain shift: the blessings of multiple domains

Outcome bridge for domain shift

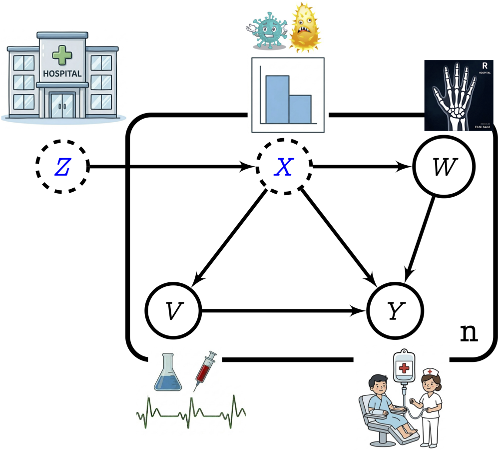
"The blessings of multiple domains"

Outcome bridge for domain shift

"The blessings of multiple domains"

In this example:

- X: which disease
- V: blood tests, ECG
- W: x-rays
- Y: diagnosis
- Z: domain (P_X param.)
- $n \in \{1 \ldots D+1\}$




arXiv > cs > arXiv:2403.07442

Computer Science > Machine Learning

[Submitted on 12 Mar 2024]

Proxy Methods for Domain Adaptation

Katherine Tsai, Stephen R. Pfohl, Olawale Salaudeen, Nicole Chiou, Matt J. Kusner, Alexander D'Amour, Sanmi Koyejo, Arthur Gretton

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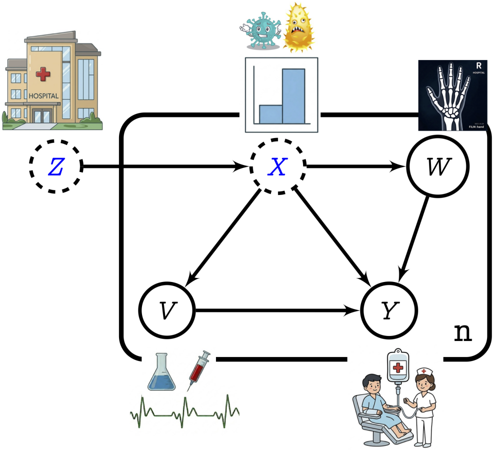


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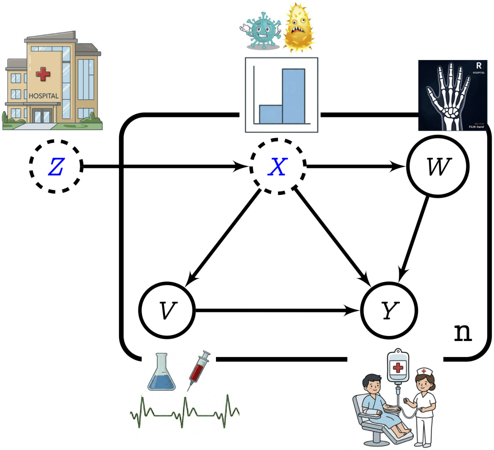
Goal: prediction on new (∗) domain D+1 via outcome bridge



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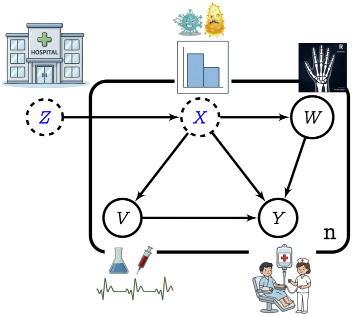
Proof (1)

- X: unobserved variable.
- V: "view 1"
- W: "view 2"
- Y: outcome
- n observations per domain



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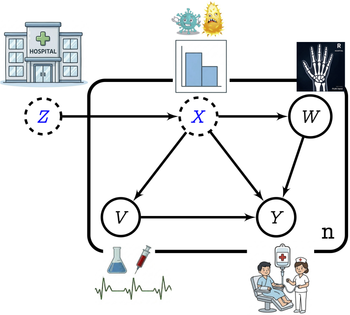

Core assumption: only $P(X|Z)$ changes with Z, all $P(Q|X)$ invariant for $Q \subseteq \{V, Y, W\}$.

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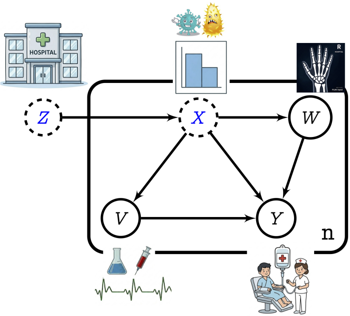

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$$P(Y|X, v) = M_{w,v} P(W|X)$$

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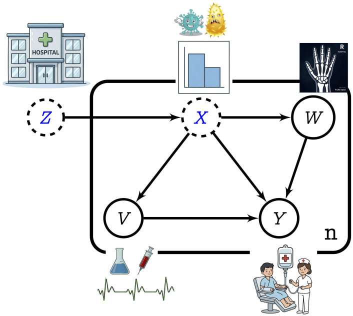

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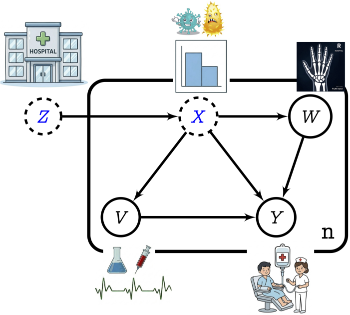

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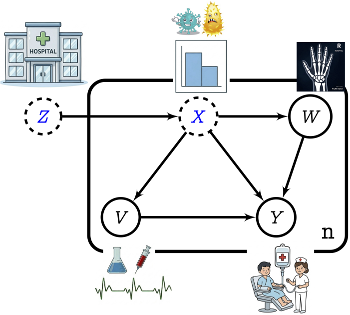

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Proof (2) - how to solve for outcome bridge?

Get rid of the blue!

For each v, define matrix of per-domain probabilities:

$$\underbrace{P^{(1..D)}(X|v)}_{d_x \times D} := \begin{bmatrix} P^{(1)}(X|v) & \dots & P^{(D)}(X|v) \end{bmatrix}$$

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$$\begin{aligned} P^{(1..D)}(W|v) &:= \begin{bmatrix} P^{(1)}(W|v) & \dots & P^{(D)}(W|v) \end{bmatrix} \\ &= P(W|X)P^{(1..D)}(X|v) \end{aligned}$$

Proof (2) - how to solve for outcome bridge?

Get rid of the blue!

For each v, define matrix of per-domain probabilities:

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Similarly

$$\begin{aligned} P^{(1..D)}(Y|v) &:= \begin{bmatrix} P^{(1)}(Y|v) & \dots & P^{(D)}(Y|v) \end{bmatrix} \\ &= P(Y|X,v)P^{(1..D)}(X|v) \end{aligned}$$

Then

$$P^{(1..D)}(Y|v) = M_{w,v} P^{(1..D)}(W|v)$$

Solve for $M_{w,v}$ when $\mathrm{rank}(P^{(1..D)}(W|v)) \geq d_x$.

Proxy/Negative Control Methods in the Real World

Outcome bridge and proxy variables

Kernel features (ICML 2021):


arXiv.org > cs > arXiv:2105.04544

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Computer Science > Machine Learning

[Submitted on 10 May 2021 (v1), last revised 9 Oct 2021 (this version, v4)]

Proximal Causal Learning with Kernels: Two-Stage Estimation and Moment Restriction

Afsaneh Mastouri, Yuchen Zhu, Limor Gultchin, Anna Korba, Ricardo Silva, Matt J. Kusner, Arthur Gretton, Krikamol Muandet







NN features (NeurIPS 2021):


arXiv.org > cs > arXiv:2106.03907

Search...

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Computer Science > Machine Learning

[Submitted on 7 Jun 2021 (v1), last revised 7 Dec 2021 (this version, v2)]

Deep Proxy Causal Learning and its Application to Confounded Bandit Policy Evaluation

Liyuan Xu, Heishiro Kanagawa, Arthur Gretton


Code for NN and kernel proxy methods:

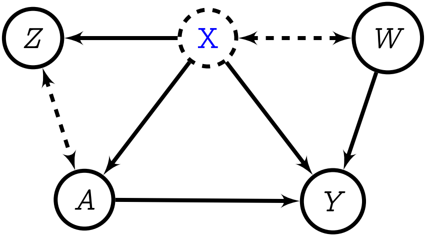
https://github.com/liyuan9988/DeepFeatureProxyVariable/

Reminder: proxy variables, finite setting

Unobserved X with (possibly) complex nonlinear effects on A, Y

The definitions are:

- X: unobserved confounder.
- A: treatment
- Y: outcome
- Z: treatment proxy
- W outcome proxy



We can obtain

$$P(Y^{(a)}) = P(Y|X,a)P(X)$$
$$= H_{w,a}P(W)$$

by solving

$$P(Y|Z,a) = H_{w,a}P(W|Z,a)$$

Proxy relation, outcome bridge

If X were observed, we would write (dose-response curve)

$$\mathbb{E}(Y^{(a)}) = \int_x \mathbb{E}(Y|a, x)p(x)dx.$$

....but we do not observe X.

Proxy relation, outcome bridge

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....but we do not observe X.

Main theorem: Assume we solved for outcome bridge:

$$\mathbb{E}(Y|a, z) = \mathbb{E}_{W|a,z} h_y(W, a)$$

- "Primary" $\mathbb{E}(Y|a, z)$, "secondary" $\mathbb{E}_{W|a,z}$ linked by h_y
- All variables observed, X not seen *or modeled.*

Fredholm equation of first kind. Bridge existence requires $\diamondsuit$, identification of DR requires $\triangle$ (and further technical assumptions) [XKG: Asspumption 2, Prop. 1,Corr. 1; Deaner]

$$\mathbb{E}[f(X)|A = a, Z = z] = 0, \ \forall(z, a) \iff f(X) = 0, \ \mathbb{P}_X \text{ a.s.} \quad \triangle$$

$$\mathbb{E}[f(X)|A = a, W = w] = 0, \ \forall(w, a) \iff f(X) = 0, \ \mathbb{P}_X \text{ a.s.} \quad \diamondsuit$$

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Dose-response curve via $p(w)$:

$$\mathbb{E}(Y^{(a)}) = DR^{(O)}(a; h_y) := \int_w h_y(a, w)p(w)dw$$

Proxy relation, outcome bridge

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Dose-response curve via $p(w)$:

$$\mathbb{E}(Y^{(a)}) = DR^{(O)}(a;h_y) := \int_w h_y(a,w)p(w)dw$$

Challenge: need a loss function for h_y

Primary loss function for $h_y(w, a)$

Goal:

$$\mathbb{E}(Y|a, z) = \mathbb{E}_{W|a,z} h_y(W, a)$$

Primary loss function:

$$h_y = \arg\min_{h \in \mathcal{H}} \mathbb{E}_{Y,A,Z} \left(Y - \mathbb{E}_{W|A,Z} h(W, A) \right)^2$$

Why?

Deaner (2021).
Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).
Xu, Kanagawa, G. (2021).

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Why?

$f^*(a, z) = \mathbb{E}(Y|a, z)$ solves

$$\underset{f}{\operatorname{argmin}} \, \mathbb{E}_{Y,A,Z} (Y - f(A, Z))^2$$

Deaner (2021).
Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).
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$f^*(a, z) = \mathbb{E}(Y|a, z)$ solves

$$\underset{f}{\operatorname{argmin}} \, \mathbb{E}_{Y,A,Z} \, (Y - f(A, Z))^2$$

...and by the proxy model above,

$$\mathbb{E}(Y|a, z) = \mathbb{E}_{W|a,z} h_y(W, a)$$

Deaner (2021).
Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).
Xu, Kanagawa, G. (2021).

Feature paramatrization of bridge $h_y(a, w)$

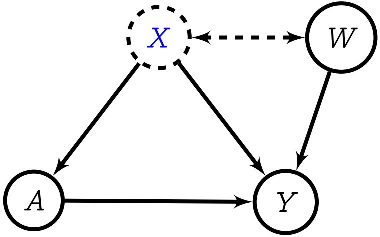
The outcome bridge function class $\mathcal{H}$ defined as:

$$h_y(a, w) = \gamma^\top [\varphi_\theta(w) \otimes \varphi_\xi(a)] = \gamma^\top \begin{bmatrix} \varphi_{\theta,1}(w)\varphi_{\xi,1}(a) \\ \varphi_{\theta,1}(w)\varphi_{\xi,2}(a) \\ \vdots \\ \varphi_{\theta,2}(w)\varphi_{\xi,1}(a) \\ \vdots \end{bmatrix}$$

Assume we have:

- output proxy kernel/NN features $\varphi_\theta(w)$
- treatment kernel/NN features $\varphi_\xi(a)$
- linear final layer γ

(argument of feature map indicates feature space)



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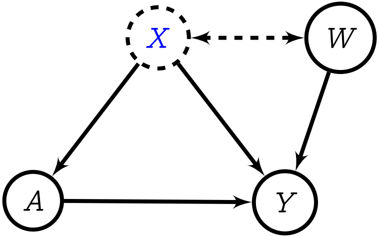

Questions:

- Why feature map $\varphi_\theta(w) \otimes \varphi_\xi(a)$?
- Why final linear layer γ?

Both are necessary (next slide)!

Regression for $h_y(w, a)$

Goal:

$$\mathbb{E}(Y|a, z) = \mathbb{E}_{W|a,z} h_y(W, a)$$

Primary regression:

$$h_y^{(\lambda)} = \arg\min_{h \in \mathcal{H}} \mathbb{E}_{Y,A,Z} \left(Y - \mathbb{E}_{W|A,Z} h(W, A) \right)^2 + \lambda \Omega(\|\gamma\|^2)$$

Deaner (2021).
Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).
Xu, Kanagawa, G. (2021).

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How to get conditional expectation $\mathbb{E}_{W|a,z} h(W, a)$?

Density estimation for $p(W|a, z)$? Sample from $p(W|a, z)$?

Deaner (2021).

Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).

Xu, Kanagawa, G. (2021).

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Recall bridge function

$$h(W, a) = \qquad \left[\gamma^\top (\varphi_\theta(W) \otimes \varphi_\xi(a))\right]$$

Deaner (2021).

Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).

Xu, Kanagawa, G. (2021).

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Recall bridge function

$$\mathbb{E}_{W|a,z}\, h(W, a) = \mathbb{E}_{W|a,z} \left[\gamma^\top (\varphi_\theta(W) \otimes \varphi_\xi(a)) \right]$$

Deaner (2021).
Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).
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Recall bridge function

$$\begin{aligned} \mathbb{E}_{W|a,z}\, h(W, a) &= \mathbb{E}_{W|a,z} \left[\gamma^\top (\varphi_\theta(W) \otimes \varphi_\xi(a)) \right] \\ &= \gamma^\top \Big(\underbrace{\mathbb{E}_{W|a,z} [\varphi_\theta(W)]}_{\text{cond. feat. mean}} \otimes \varphi_\xi(a) \Big) \end{aligned}$$

(this is why linear γ and feature map $\varphi_\theta(w) \otimes \varphi_\xi(a)$)

Deaner (2021).
Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).
Xu, Kanagawa, G. (2021).

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Ridge regression (again!): 2SLS

$$\mathbb{E}_{W|a,z} \varphi_\theta(W) = F_{\theta,\zeta} \varphi_\zeta(a, z)$$

Deaner (2021).
Mastouri, Zhu, Gultchin, Korba, Silva, Kusner, G., Muandet (2021).
Xu, Kanagawa, G. (2021).

Treatment bridge and doubly robust proxy learning

AISTATS 2025

arXiv > cs > arXiv:2503.08371

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Computer Science > Machine Learning

[Submitted on 11 Mar 2025]

Density Ratio-based Proxy Causal Learning Without Density Ratios

Bariscan Bozkurt, Ben Deaner, Dimitri Meunier, Liyuan Xu, Arthur Gretton




NeurIPS 2025

arXiv > cs > arXiv:2505.19807

Computer Science > Machine Learning

[Submitted on 26 May 2025]

Density Ratio-Free Doubly Robust Proxy Causal Learning

Bariscan Bozkurt, Houssam Zenati, Dimitri Meunier, Liyuan Xu, Arthur Gretton




Code for treatment bridge and doubly robust:

https://github.com/BariscanBozkurt/Doubly-Robust-Kernel-Proxy-Variable-Algorithm

Other DR approaches:

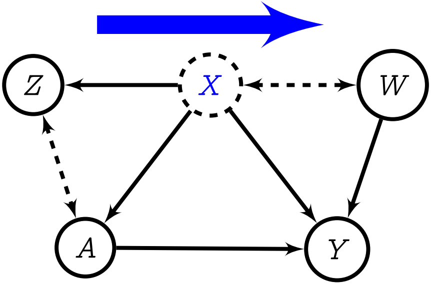
Cui, Pu, Shi, Miao, Tchetgen-Tchetgen. Semiparametric proximal causal inference. JASA (2024): binary treatment.

Wu, Fu, Wang, Sun. Doubly robust proximal causal learning for continuous treatments. ICLR (2024): density ratio + Parzen smoothed treatment

Treatment bridge: idea

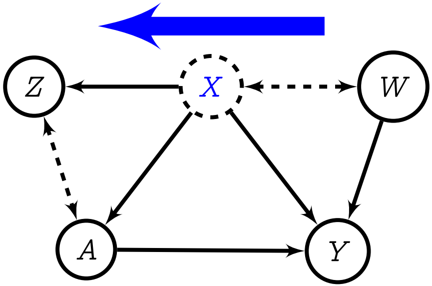
Outcome bridge

$$\mathbb{E}(Y \mid a, z) = \mathbb{E}_{W \mid a,z} h_y(W, a)$$



Treatment bridge

$$?? = \mathbb{E}_{Z|a,w}[g_y(Z, a)]$$



Treatment bridge dose-response

Main theorem: Assume we solved for treatment bridge:

$$\frac{p(a)p(w)}{p(a,w)} = \mathbb{E}_{Z|a,w}[g_y(Z,a)]$$

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Dose-response curve via $p(Z|A=a)$:

$$\mathbb{E}(Y^{(a)}) = DR^{(T)}(a;g_y) := \mathbb{E}_{Y,Z|a}[Y\,g_y(Z,a)]$$

Bridge existence requires $\triangle$, identification of DR requires $\diamondsuit$ [BDMXG25: Assumptions 3.2, 3.3, 3.7].

$$W \perp\!\!\!\perp (Z,A)|X$$

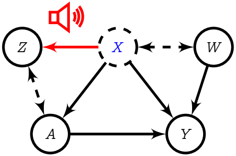
$$Y \perp\!\!\!\perp Z|(A,X)$$

$$\mathbb{E}[f(X)|A=a,Z=z]=0,\ \forall(z,a) \iff f(X)=0,\ \mathbb{P}_X \text{ a.s.} \quad \triangle$$

$$\mathbb{E}[f(X)|A=a,W=w]=0,\ \forall(w,a) \iff f(X)=0,\ \mathbb{P}_X \text{ a.s.} \quad \diamondsuit$$

Orange assumption is slightly stronger than the one we used, and implies our actual Assumption 3.2

Why treatment bridge? Synthetic demo



Setting	Outcome bridge	Treatment bridge
Set. 1	41.84 ± 26.61	5.53 ± 0.69

Bozkurt, Deaner, Meunier, Xu, G (2025, Section 13.5)
Tsai, Pfohl, Salaudeen, Chiou, Kusner, D'Amour, Koyejo, G. (2024, Appendix D.5)

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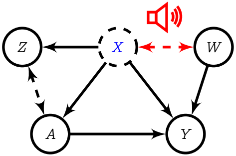

Setting	Outcome bridge	Treatment bridge
Set. 1	41.84 ± 26.61	5.53 ± 0.69
Set. 2	5.41 ± 2.17	9.32 ± 5.29

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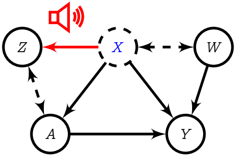

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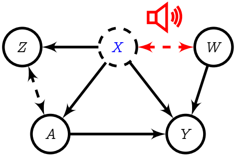

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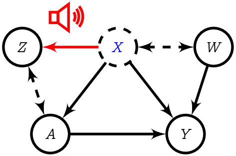

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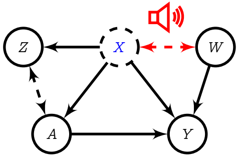

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Bozkurt, Deaner, Meunier, Xu, G (2025, Section 13.5)
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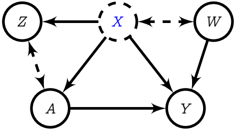

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Doubly robust proxy causal learning

$$
\begin{aligned}
& DR^{(DR)}(a; h_y, g_y) \\
& = \mathbb{E}_{Y,Z,W|a}[g_y(Z,a)(Y - h_y(W,a))] + \mathbb{E}_W[h_y(W,a)]
\end{aligned}
$$

Doubly robust proxy causal learning

$$
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&= \mathbb{E}_{Y,Z,W|a}[g_y(Z,a)(Y - h_y(W,a))] + \mathbb{E}_W[h_y(W,a)] \\
&= \underbrace{\mathbb{E}_{Y,Z|a}[Y\, g_y(Z,a)]}_{\text{3rd regression}} - \underbrace{\mathbb{E}_{Z,W|a}[g_y(Z,a) h_y(W,a)]}_{\text{4th regression}} + \mathbb{E}_W[h_y(W,a)]
\end{aligned}
$$

Doubly robust proxy causal learning

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\end{aligned}
$$

Guarantees?

Doubly robust proxy causal learning

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&= \underbrace{\mathbb{E}_{Y,Z|a}[Y\, g_y(Z,a)]}_{\text{3rd regression}} - \underbrace{\mathbb{E}_{Z,W|a}[g_y(Z,a) h_y(W,a)]}_{\text{4th regression}} + \mathbb{E}_W[h_y(W,a)]
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- If either of h_y, g_y is correct then $DR^{(DR)}(a; h_y, g_y) = DR(a)$ (true dose-response)

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- Convergence:

$$
\begin{aligned}
&|DR(a) - \widehat{DR}^{(DR)}(a; \hat{h}_y, \hat{g}_y)| \\
&\lesssim \|g_y - \hat{g}_y\| \|h_y - \hat{h}_y\| + \|C_{YZ|A} - \widehat{C}_{YZ|A}\|_{HS} + \|C_{WZ|A} - \widehat{C}_{WZ|A}\|_{HS}
\end{aligned}
$$

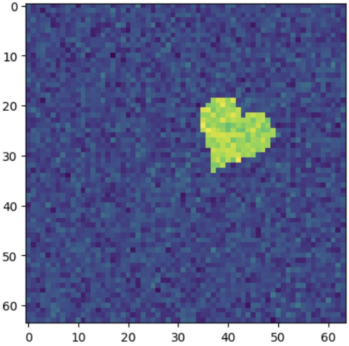
where

$$
C_{YZ|A}\varphi(a) = \mathbb{E}_{Y,Z|a}[Y\varphi(Z)] \qquad C_{WZ|A}\varphi(a) = \mathbb{E}_{W,Z|a}[\varphi(W) \otimes \varphi(Z)]
$$

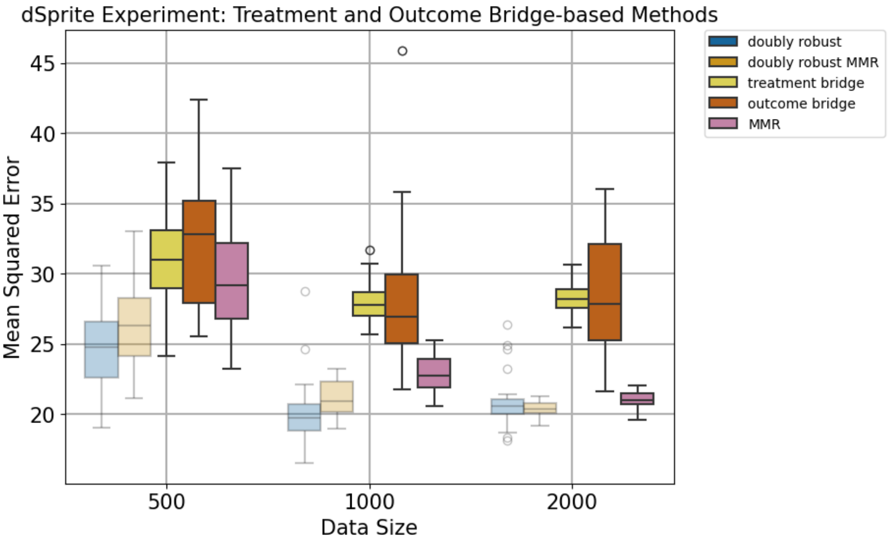
Synthetic experiment, kernel features

dSprite example:

- $X = \{\mathtt{scale}, \mathtt{rotation}, \mathtt{posX}, \mathtt{posY}\}$
- Treatment $A \in \mathbb{R}^{4096}$ is the image generated (with Gaussian noise)
- Outcome Y is linear function of A with multiplicative confounding by posY.
- $Z = \{\mathtt{scale}, \mathtt{rotation}, \mathtt{posX}\}$,
  W = noisy image sharing posY



Synthetic experiment, results



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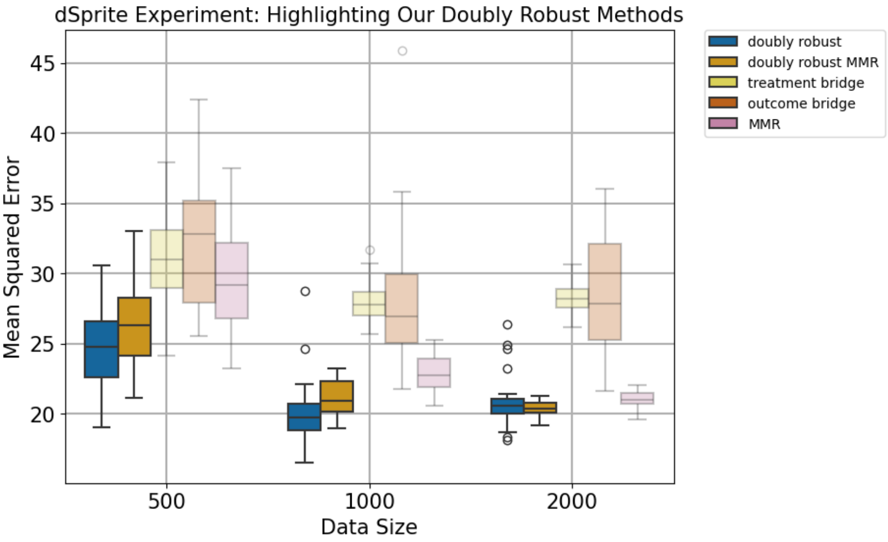
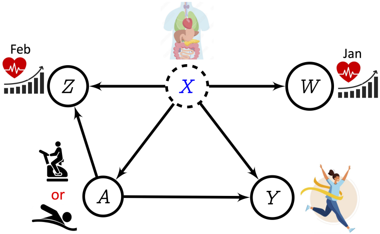


Conclusion

Causal effect estimation with unobserved X, (possibly) complex nonlinear effects on A, Y

We need to observe:

- Treatment proxy Z (interacts with A, but not directly with Y)
- Outcome proxy W (no direct interaction with A, can affect Y)



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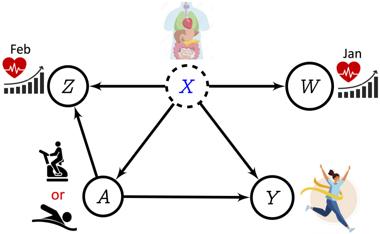

Key messages:

- Don't ~~meet your heroes~~ model/sample latents X
- Don't model all of W, only relevant features for Y
- "Ridge regression is all you need"

Code available:

https://github.com/liyuan9988/DeepFeatureProxyVariable/

https://github.com/BariscanBozkurt/Doubly-Robust-Kernel-Proxy-Variable-Algorithm

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Work supported by:

The Gatsby Charitable Foundation

Google DeepMind

Questions?


THE KERNEL
BREWERY
LONDON
BALTIC PORTER
THE KERNEL
EXPORT STOUT

Treatment bridge regression loss

Loss function

$$\mathcal{L}(g_y) = \mathbb{E}_{A,W}\left[\left(\frac{p(A)p(W)}{p(A,W)} - \mathbb{E}_{Z|A,W}[g_y(Z,A)]\right)^2\right]$$

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$$-2\int \left(\underbrace{\frac{p(a)p(w)}{p(a,w)}}_{\text{density ratio!}} \mathbb{E}_{Z|a,w}[g_y(Z,a)]\right) p(a,w)\, da dw$$

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Empirical estimates:

1. Squared mean
2. U-statistic

Feature paramatrization of bridge $g_y(z,a)$

The treatment bridge is a function of two arguments

$$g_y(z,a) = \eta^\top \left[\varphi_\theta(z) \otimes \varphi_\xi(a)\right]$$

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Need a third regression

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Failures of completeness assumptions (1)

Recall (one of the) completeness assumptions:

$$\mathbb{E}[f(X)|A=a,Z=z]=0,\forall(a,z) \iff f(X)=0,\ \mathbb{P}_X \text{ a.s.} \quad (\triangle)$$

For conciseness, assume conditioning on some a.

Failure 1: $Z \perp\!\!\!\perp X$ (no information about X in proxy)

$$g(X|) = \tilde{g}(X) - \mathbb{E}_X \tilde{g}(X)$$
$$\mathbb{E}(g(X)|Z,a) = \mathbb{E}g(X) = 0.$$

Failures of identifiability assumptions (2)

Failure 2: "exploitable invariance" of $p(X|z)$

$$X \sim \mathcal{N}(0,1),$$
$$Z = |X| + \mathcal{N}(0,1),$$

where $p(X|z) \propto p(z|X)p(X)$ symmetric in X. Consider square integrable *antisymmetric* function $g(X) = -g(-X) \neq 0$. Then

$$\begin{aligned}
\mathbb{E}[g(X)|Z=z] &= \int_{-\infty}^{\infty} g(X)p(X|z)dX \\
&= \int_{-\infty}^{0} g(X)p(X|z)dX + \int_{0}^{\infty} g(X)p(X|z)dX \\
&= 0.
\end{aligned}$$

If distribution of $X|Z$ retains the same "symmetry class" over a set of Z with nonzero measure, then the assumption is violated by $g(X)$ with zero mean on this class.