Kernel Adaptive Metropolis-Hastings

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Metropolis-Hastings MCMC

- Unnormalized target $\pi(x) \propto p(x)$
- Generate Markov chain with invariant distribution p
  - Initialize $x_0 \sim p_0$
  - At iteration $t \geq 0$, propose to move to state $x' \sim q(\cdot|x_t)$
  - Accept/Reject proposals based on ratio

$$x_{t+1} = \begin{cases} x', & \text{w.p. } \min\left\{1, \frac{\pi(x')q(x_t|x')}{\pi(x_t)q(x'|x_t)}\right\}, \\ x_t, & \text{otherwise.} \end{cases}$$

- What proposal $q(\cdot|x_t)$?

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- What proposal $q(\cdot|x_t)$?
  - Too narrow or broad: $\rightarrow$ slow convergence
  - Does not conform to support of target $\rightarrow$ slow convergence

Adaptive MCMC

- **Adaptive Metropolis** (Haario, Saksman & Tamminen, 2001): Update proposal $q_t(\cdot|x_t) = \mathcal{N}(x_t, \nu^2\hat{\Sigma}_t)$, using estimates of the target covariance

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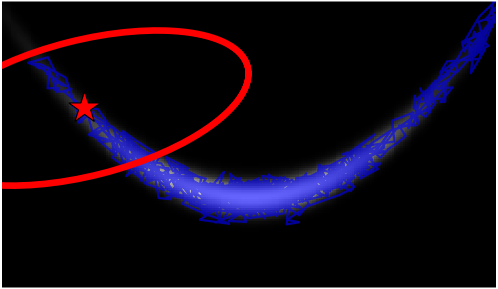

Locally miscalibrated for *strongly non-linear targets*: directions of large variance depend on the current location

Motivation: Intractable & Non-linear Targets

- Previous solutions for non-linear targets: Hamiltonian Monte Carlo (HMC) or Metropolis Adjusted Langevin Algorithms (MALA) (Roberts & Stramer, 2003; Girolami & Calderhead, 2011).
- Require target gradients and second order information

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Our case: not even target $\pi(\cdot)$ can be computed – **Pseudo-Marginal MCMC** (Beaumont, 2003; Andrieu & Roberts, 2009).

Bayesian Gaussian Process Classification

Example: when is target not computable?

- GPC model: latent process $\mathbf{f}$, labels $\mathbf{y}$, (with covariate matrix X), and hyperparameters θ:

$$p(\mathbf{f}, \mathbf{y}, \theta) = p(\theta) p(\mathbf{f}|\theta) p(\mathbf{y}|\mathbf{f})$$

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- Automatic Relevance Determination (ARD) covariance:

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... but cannot integrate out $\mathbf{f}$

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- MH ratio:

$$\alpha(\theta,\theta') = \min\left\{1, \frac{p(\theta')p(\mathbf{y}|\theta')q(\theta|\theta')}{p(\theta)p(\mathbf{y}|\theta)q(\theta'|\theta)}\right\}$$

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- **Filippone & Girolami, 2013** use Pseudo-Marginal MCMC: unbiased estimate of $p(\mathbf{y}|\theta)$ via importance sampling:

$$\hat{p}(\theta|\mathbf{y}) \propto p(\theta)\hat{p}(\mathbf{y}|\theta) \approx p(\theta)\frac{1}{n_{\text{imp}}}\sum_{i=1}^{n_{\text{imp}}} p(\mathbf{y}|\mathbf{f}^{(i)})\frac{p(\mathbf{f}^{(i)}|\theta)}{Q(\mathbf{f}^{(i)})}$$

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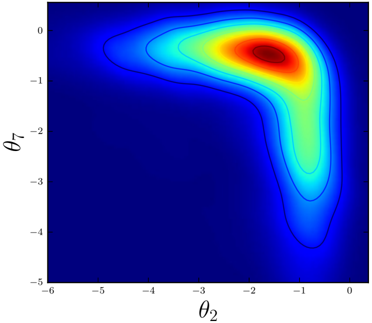
- Replacing marginal likelihood $p(\mathbf{y}|\theta)$ with *unbiased estimate* $\hat{p}(\mathbf{y}|\theta)$ still results in correct invariant distribution **[Beaumont, 2003; Andrieu & Roberts, 2009]**

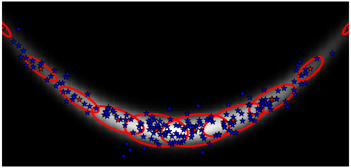
Intractable & Non-linear Target in GPC

- Sliced posterior over hyperparameters of a Gaussian Process classifier on UCI Glass dataset obtained using Pseudo-Marginal MCMC



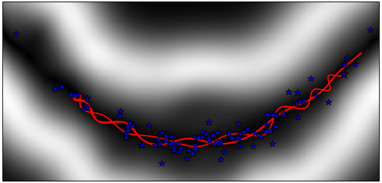
Adaptive sampler that learns the shape of non-linear targets without gradient information?

Two strategies for adaptive sampling



Kameleon (Sejdinovic et al. 2014)

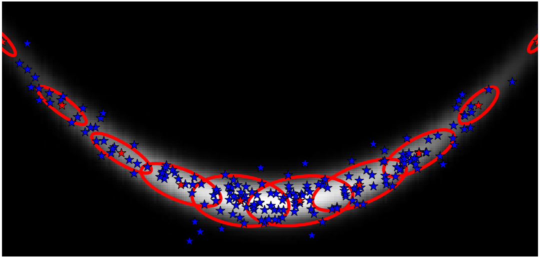
- Learns covariance in RKHS.
- *Locally* aligns to (non-linear) target covariance, gradient free.



Kernel Adaptive Hamiltonian Monte Carlo (Strathmann et al. 2015)

- Learns *global* estimate of gradient of log target density

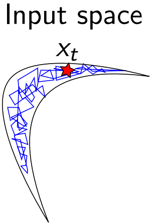
The Kameleon



D. Sejdinovic, H. Strathmann, M. Lomeli, C. Andrieu, and A. Gretton, ICML 2014

Use feature space covariance

- Capture non-linearities using linear covariance $C_{\mathbf{z}}$ in feature space $\mathcal{H}$



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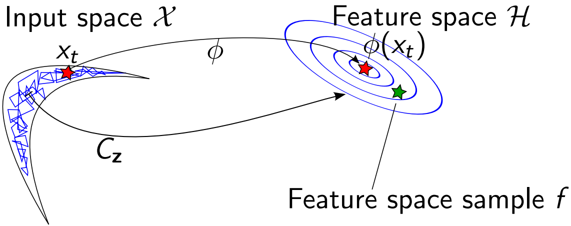

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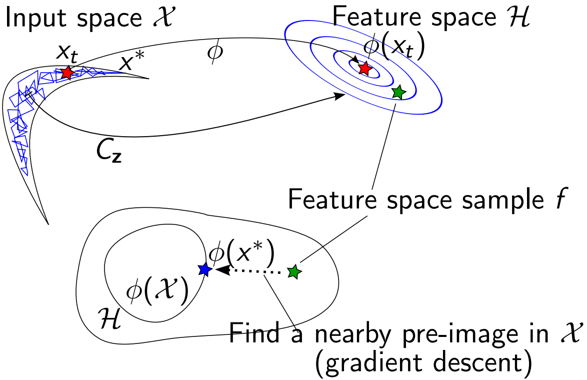

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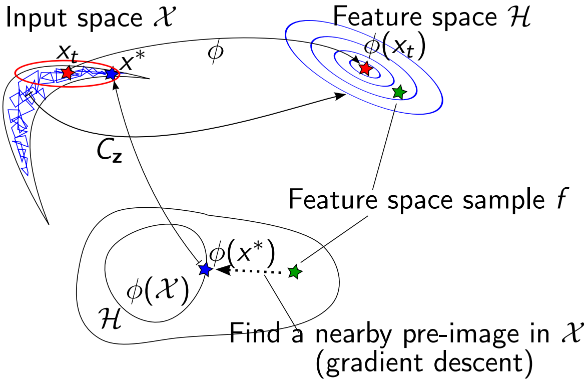

Proposal Construction Summary

1. Get a chain subsample $\mathbf{z} = \{z_i\}_{i=1}^n$
2. Construct an RKHS sample $f \sim \mathcal{N}(\phi(x_t), \nu^2 C_{\mathbf{z}})$
3. Propose x^* such that $\phi(x^*)$ is close to f (with an additional exploration term $\xi \sim \mathcal{N}\left(0, \gamma^2 I_d\right)$).

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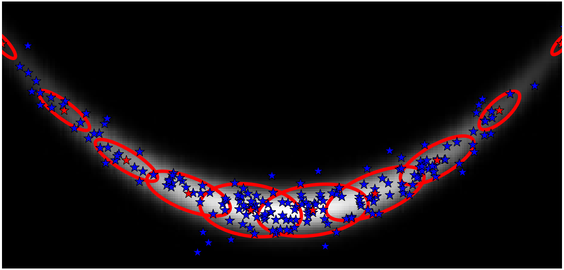
Integrate out RKHS samples f, gradient step, and ξ to obtain marginal Gaussian proposal on the input space:

$$q_{\mathbf{z}}(x^*|x_t) = \mathcal{N}(x_t, \gamma^2 I_d + \nu^2 M_{\mathbf{z},x_t} H M_{\mathbf{z},x_t}^\top)$$

$M_{\mathbf{z},x_t} = 2\left[\nabla_x k(x, z_1)|_{x=x_t}, \ldots, \nabla_x k(x, z_n)|_{x=x_t}\right]$,
$k(x, x') = \langle \phi(x), \phi(x') \rangle_{\mathcal{H}}$.

Examples of Covariance Structure for Standard Kernels

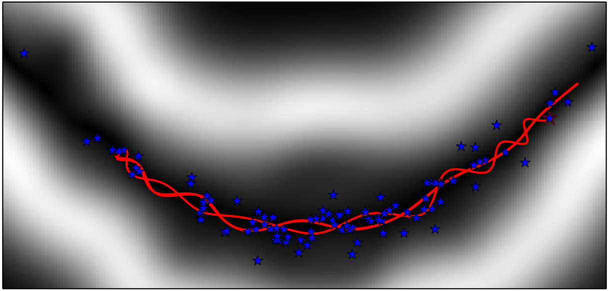


Kameleon proposals capture local covariance structure

Gaussian kernel: $k(x, x') = \exp\left(-\frac{1}{2}\sigma^{-2}\left\|x - x'\right\|_2^2\right)$

$$\left[\mathrm{cov}[q_{\mathbf{z}(\cdot|y)}]\right]_{ij} = \gamma^2\delta_{ij} + \frac{4\nu^2}{\sigma^4}\sum_{a=1}^{n}\left[k(y, z_a)\right]^2(z_{a,i} - y_i)(z_{a,j} - y_j) + \mathcal{O}\left(\frac{1}{n}\right).$$

Kernel Adaptive Hamiltonian Monte Carlo (KMC)



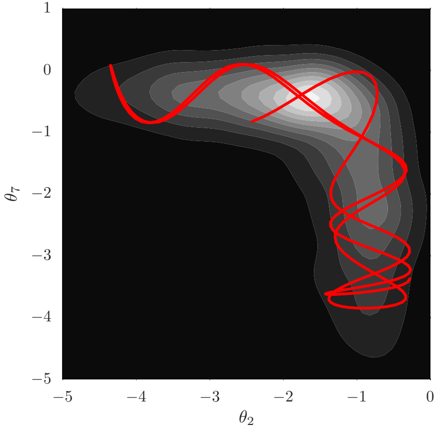
Heiko Strathmann, Dino Sejdinovic, Samuel Livingstone, Zoltan Szabo, and Arthur Gretton, NIPS 2015

Hamiltonian Monte Carlo

- HMC: distant moves, high acceptance probability.
- Potential energy $U(q) = -\log \pi(q)$, auxiliary momentum $p \sim \exp(-K(p))$, simulate for $t \in \mathbb{R}$ along Hamiltonian flow of $H(p,q) = K(p) + U(q)$, using operator

$$\frac{\partial K}{\partial p}\frac{\partial}{\partial q} - \frac{\partial U}{\partial q}\frac{\partial}{\partial p}$$

- Numerical simulation (i.e. leapfrog) depends on *gradient information*.



What if gradient *unavailable*, e.g. in Bayesian GP classification?

Infinite dimensional exponential families

Proposal is RKHS exponential family model [Fukumizu, 2009; Sriperumbudur et al. 2014], but accept using true Hamiltonian (to correct for both model and leapfrog)

$$\text{const} \times \pi(x) \approx \exp\left(\langle f, k(x,\cdot)\rangle_{\mathcal{H}} - A(f)\right)$$

- Sufficient statistics: feature map $k(\cdot, x) \in \mathcal{H}$, satisfies $f(x) = \langle f, k(x,\cdot)\rangle_{\mathcal{H}}$ for any $f \in \mathcal{H}$.
- Natural parameters: $f \in \mathcal{H}$.

The model is

- dense in continuous densities on compact domains (TV, KL, etc.),
- relatively robust to increasing dimensions, as opposed to e.g. KDE.

How to learn f from samples without access to $A(f)$?

Score matching

- Estimation of unnormalised density models from samples [Sriperumbudur et al. 2014]
- Minimises *Fisher divergence*

$$J(f) = \frac{1}{2} \int \pi(x) \left\| \nabla f(x) - \nabla \log \pi(x) \right\|_2^2 dx$$

- Possible *without* accessing $\nabla \log \pi(x)$ and accessing $\pi(x)$ only through samples $\mathbf{x} := \{x_i\}_{i=1}^t$

$$\hat{J}(f) = \widehat{\mathbb{E}}_{\mathbf{x}} \left\{ \sum_{\ell=1}^{d} \left[\frac{\partial^2 f(x)}{\partial x_\ell^2} + \frac{1}{2} \left(\frac{\partial f(x)}{\partial x_\ell} \right)^2 \right] \right\}$$

Expensive: full solution requires solving $(td + 1)$*-dimensional linear system.*

Approximate solution: KMC finite

$$f(x) = \theta^\top \phi_x$$

- *Random Fourier Features* $\phi_x^\top \phi_y \approx k(x,y)$
- $\theta \in \mathbb{R}^m$ can be computed from

$$\hat{\theta}_\lambda := (C + \lambda I)^{-1} b$$

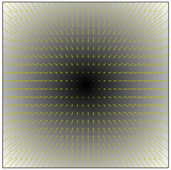
$$b := -\frac{1}{t}\sum_{i=1}^{t}\sum_{\ell=1}^{d}\ddot{\phi}_{x_i}^{\ell} \quad C := \frac{1}{t}\sum_{i=1}^{t}\sum_{\ell=1}^{d}\dot{\phi}_{x_i}^{\ell}\left(\dot{\phi}_{x_i}^{\ell}\right)^T$$

where $\dot{\phi}_x^\ell := \frac{\partial}{\partial x_\ell}\phi_x$ and $\ddot{\phi}_x^\ell := \frac{\partial^2}{\partial x_\ell^2}\phi_x$.

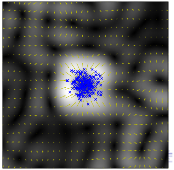
- *On-line updates* cost $\mathcal{O}(dm^2)$.

Updates fast, uses *all* Markov chain history. Caveat: need to initialise correctly.

Gradient norm: Gaussian



KMC Finite



Approximate solution: KMC lite

$$f(x) = \sum_{i=1}^{n} \alpha_i k(z_i, x)$$

- $\mathbf{z} \subseteq \mathbf{x}$ sub-sample.
- α from linear system

$$\hat{\alpha}_\lambda = -\frac{\sigma}{2}(C + \lambda I)^{-1} b$$

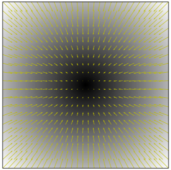
where $C \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ depend on kernel matrix

- Cost $\mathcal{O}(n^3 + n^2 d)$ (or cheaper with low-rank approx., conjugate gradient).

Geometrically ergodic on log-concave targets (fast convergence).

Gradient norm:

Gaussian



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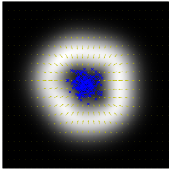
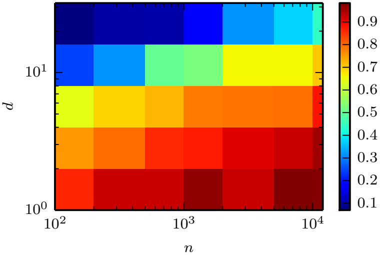


Does kernel HMC work in high dimensions?

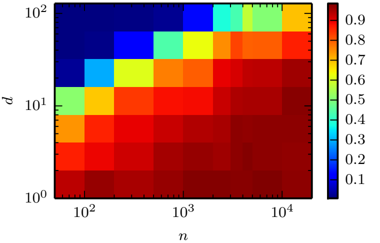
Challenging Gaussian target (**top**):

- Eigenvalues: $\lambda_i \sim \texttt{Exp}(1)$.
- Covariance: $\mathrm{diag}(\lambda_1, \ldots, \lambda_d)$, randomly rotate.
- Use Rational Quadratic kernel to account for resulting highly 'non-singular' length-scales.
- KMC scales up to $d \approx 30$.



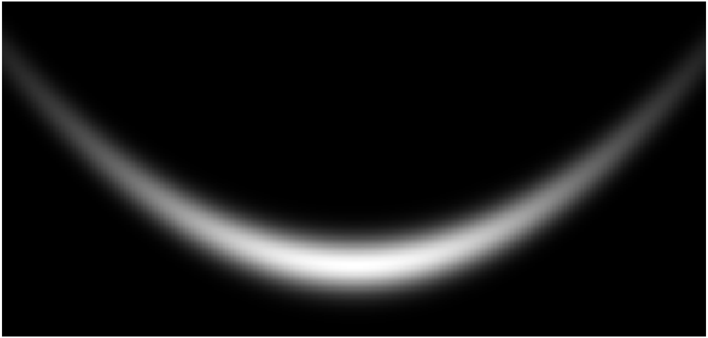
An easy, isotropic Gaussian target (**bottom**):

- More smoothness allows KMC to scale up to $d \approx 100$.



Synthetic targets: Banana

Banana: $\mathcal{B}(b, v)$: take $X \sim \mathcal{N}(0, \Sigma)$ with $\Sigma = \text{diag}(v, 1, \ldots, 1)$, and set $Y_2 = X_2 + b(X_1^2 - v)$, and $Y_i = X_i$ for $i \neq 2$. (Haario et al, 1999; 2001)



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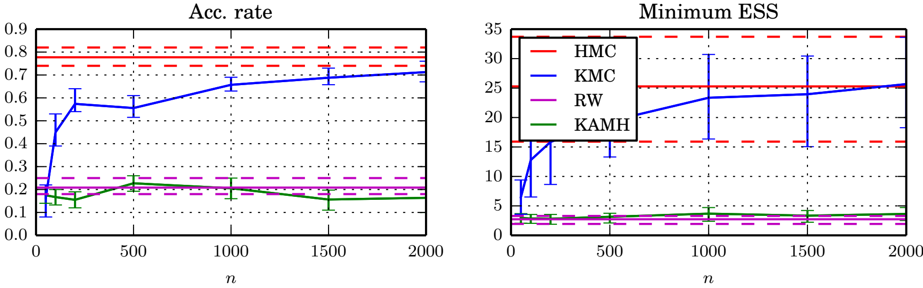

KMC behaves like HMC as number n of oracle samples increases.

Gaussian Process Classification on UCI data

- Standard GPC model

$$p(\mathbf{f}, \mathbf{y}, \theta) = p(\theta)p(\mathbf{f}|\theta)p(\mathbf{y}|\mathbf{f})$$

  where $p(\mathbf{f}|\theta)$ is a GP and with a sigmoidal likelihood $p(\mathbf{y}|\mathbf{f})$.
- Goal: sample from $p(\theta|\mathbf{y}) \propto p(\theta)p(\mathbf{y}|\theta)$.
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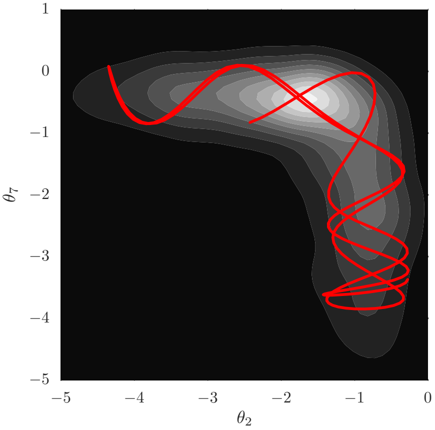

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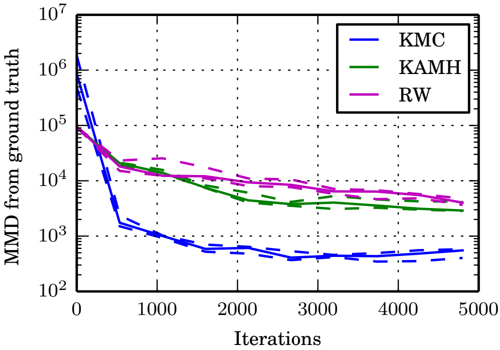

Significant mixing improvements over state-of-the-art.

Conclusions

- Simple, versatile, gradient-free adaptive MCMC samplers:
- Kameleon:
  - Uses local covariance structure of the target distribution at the current chain state
- Kernel HMC
  - Derivative of log density fit to samples, use this as proposal in HMC.
- Outperforms existing adaptive approaches on nonlinear target distributions
- Future work: For Kameleon, does feature space covariance track high density regions in original space? For kernel HMC, how does convergence rate degrade with increasing dimension?

- **Kameleon code: https://github.com/karlnapf/kameleon-mcmc**
- **Kernel HMC code: https://github.com/karlnapf/kernel hmc**

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- Unbiased estimate of $\hat{p}(\mathbf{y}|\theta)$ via importance sampling:

$$\hat{p}(\theta|\mathbf{y}) \propto p(\theta)\hat{p}(\mathbf{y}|\theta) \approx p(\theta) \frac{1}{n_{\text{imp}}} \sum_{i=1}^{n_{\text{imp}}} p(\mathbf{y}|\mathbf{f}^{(i)}) \frac{p(\mathbf{f}^{(i)}|\theta)}{Q(\mathbf{f}^{(i)})}$$

Bayesian Gaussian Process Classification (2)

- Fully Bayesian treatment: Interested in the posterior $p(\theta|y)$
- Cannot use a Gibbs sampler on $p(\theta, \mathbf{f}|y)$, which samples from $p(\mathbf{f}|\theta, y)$ and $p(\theta|\mathbf{f}, y)$ in turns, since $p(\theta|\mathbf{f}, y)$ is extremely sharp
- Filippone & Girolami, 2013 use Pseudo-Marginal MCMC to sample $p(\theta|y) = p(\theta)\int p(\theta, \mathbf{f}|y)p(\mathbf{f}|\theta)d\mathbf{f}$.
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- No access to likelihood, gradient, or Hessian of the target.

RKHS and Kernel Embedding

- For any positive semidefinite function k, there is a unique RKHS $\mathcal{H}_k$. Can consider $x \mapsto k(\cdot, x)$ as a feature map.

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Definition (Kernel embedding)

Let k be a kernel on $\mathcal{X}$, and P a probability measure on $\mathcal{X}$. The *kernel embedding* of P into the RKHS $\mathcal{H}_k$ is $\mu_k(P) \in \mathcal{H}_k$ such that $\mathbb{E}_P f(X) = \langle f, \mu_k(P) \rangle_{\mathcal{H}_k}$ for all $f \in \mathcal{H}_k$.

- Alternatively, can be defined by the Bochner integral $\mu_k(P) = \int k(\cdot, x)\, dP(x)$ (expected canonical feature)
- For many kernels k, including the Gaussian, Laplacian and inverse multi-quadratics, the kernel embedding $P \mapsto \mu_P$ is injective: characteristic (Sriperumbudur et al, 2010),
- captures all moments (similarly to the characteristic function).

Covariance operator

Definition

The covariance operator of P is $C_P : \mathcal{H}_k \to \mathcal{H}_k$ such that $\forall f, g \in \mathcal{H}_k$,
$\langle f, C_P g \rangle_{\mathcal{H}_k} = \mathrm{Cov}_P [f(X) g(X)]$.

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- Covariance operator: $C_P : \mathcal{H}_k \to \mathcal{H}_k$ is given by $C_P = \int k(\cdot, x) \otimes k(\cdot, x)\, dP(x) - \mu_P \otimes \mu_P$ (covariance of canonical features)
- Empirical versions of embedding and the covariance operator:

$$\mu_{\mathbf{z}} = \frac{1}{n} \sum_{i=1}^{n} k(\cdot, z_i) \qquad C_{\mathbf{z}} = \frac{1}{n} \sum_{i=1}^{n} k(\cdot, z_i) \otimes k(\cdot, z_i) - \mu_{\mathbf{z}} \otimes \mu_{\mathbf{z}}$$

The empirical covariance captures **non-linear** features of the underlying distribution, e.g. Kernel PCA

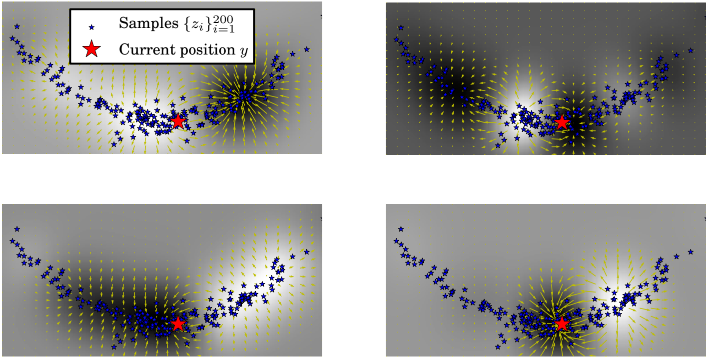
Kernel distance gradient

$$g(x) = k(x,x) - 2k(x,y) - 2\sum_{i=1}^{n} \beta_i \left[k(x,z_i) - \mu_{\mathbf{z}}(x)\right]$$

$$\nabla_x g(x)|_{x=y} = \underbrace{\nabla_x k(x,x)|_{x=y} - 2\nabla_x k(x,y)|_{x=y}}_{=0} - M_{\mathbf{z},y} H\beta$$

where $M_{\mathbf{z},y} = 2\left[\nabla_x k(x,z_1)|_{x=y}, \ldots, \nabla_x k(x,z_n)|_{x=y}\right]$ and $H = I_n - \frac{1}{n}\mathbf{1}_{n\times n}$

Cost function g

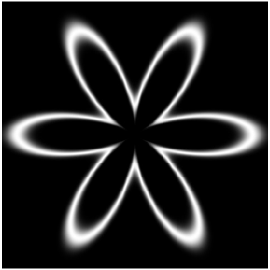


g varies most along the high density regions of the target

Synthetic targets: Flower

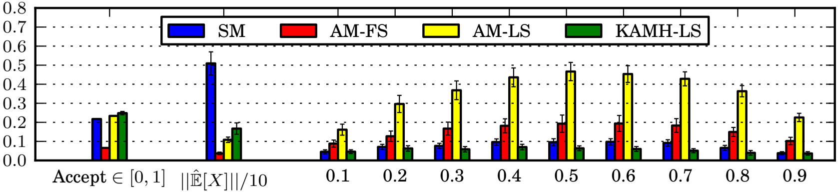
Flower: $\mathcal{F}(r_0, A, \omega, \sigma)$, a d-dimensional target with:

$$\mathcal{F}(x; r_0, A, \omega, \sigma) \propto \exp\left(-\frac{\sqrt{x_1^2 + x_2^2} - r_0 - A\cos\left(\omega \mathrm{atan2}\left(x_2, x_1\right)\right)}{2\sigma^2}\right) \times \prod_{j=3}^{d} \mathcal{N}(x_j; 0, 1).$$



Concentrates on r_0-circle with a periodic perturbation (with amplitude A and frequency ω) in the first two dimensions.

Synthetic targets: convergence statistics



8-dimensional $\mathcal{F}(10, 6, 6, 1)$ target;
iterations: 120000, burn-in: 60000