Alternatives to the DFT

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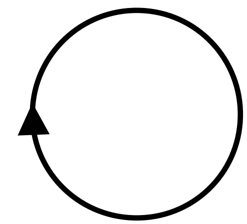
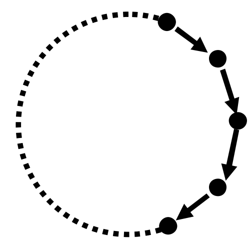
Introduction

- Discrete time signal with finite support $\mathbf{s} = (s_0, s_1, \ldots, s_{n-1})$

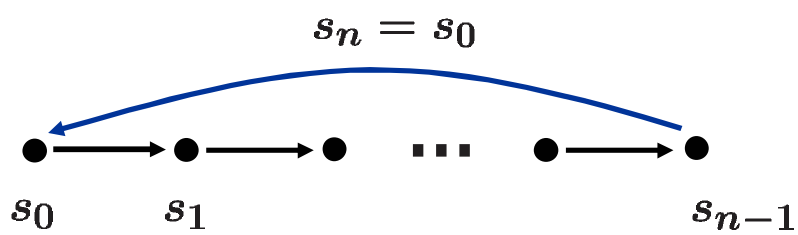
- DTFT $y(\theta) = \sum_{0 \le \ell < n} s_\ell e^{-j\theta\ell}, \quad \theta \in [-\pi, \pi).$
  - evaluate polynomial $s(x) = \sum_{0 \le \ell < n} s_\ell x^\ell$ on unit circle



- DFT$_n$ $y_k = \sum_{0 \le \ell < n} s_\ell e^{-j\frac{2\pi k}{n}\ell}, \quad 0 \le k < n.$
  - evaluate polynomial $s(x)$ at the n^{th} roots of unity
  - DFT$_n$ samples DTFT at equidistant points on the unit circle
  - As $n \to \infty$, DFT$_n$ approaches DTFT

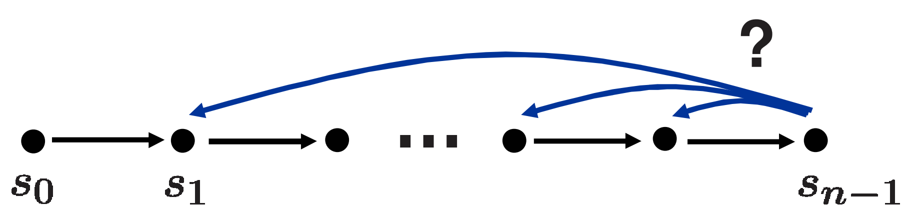


- Furthermore:



periodic signal extension $\Longleftrightarrow$ cyclic boundary condition

- ***Question:*** Can we design transforms that imply more general, other than periodic, signal extensions or boundary conditions?



- ***Answer:*** Yes!

Alternatives to DFT

- Large class of transforms associated with non-periodic signal extensions
- Consequently, their corresponding notion of convolution is not circular
- As $n \to \infty$, each of these transforms approaches DTFT
- Afford fast implementation: $O(n \log^2 n)$
- How?
  - Framework given by the **algebraic theory of signal processing**

[Püschel&Moura'06]

Algebraic Signal Processing Theory

- **Central concepts:**

  set of filters/linear systems = an algebra $\mathcal{A}$

  set of signals = an $\mathcal{A}$-module $\mathcal{M}$

CORRESPONDENCE BETWEEN DISCRETE SIGNAL PROCESSING CONCEPTS AND ALGEBRAIC CONCEPTS.

signal processing concept	algebraic concept (coordinate free)	in coordinates
filter	$h \in \mathcal{A}$ (algebra)	$\phi(h) \in \mathbb{C}^{I \times I}$
signal	$s = \sum s_i b_i \in \mathcal{M}$ ($\mathcal{A}$-module)	$\mathbf{s} = (s_i)_{i \in I} \in \mathbb{C}^I$
filtering	$h \cdot s$	$\phi(h) \cdot \mathbf{s}$
impulse	base vector $b_i \in \mathcal{M}$	$\mathbf{b}_i = (\dots, 0, 1, 0, \dots)^T \in \mathbb{C}^I$
impulse response of $h \in \mathcal{A}$	$h \cdot b_i \in \mathcal{M}$	$\phi(h) \cdot \mathbf{b}_i \in \mathbb{C}^I$
Fourier transform	$\Delta : \mathcal{M} \to \bigoplus_{\omega \in W} \mathcal{M}_\omega$	$\mathcal{F} : \mathbb{C}^I \to \bigoplus_{\omega \in W} \mathbb{C}^{d_\omega} \Leftrightarrow \phi \to \bigoplus_{\omega \in W} \phi_\omega$
spectrum of signal	$\Delta(s) = (s_\omega)_{\omega \in W} = \omega \mapsto s_\omega$	$\mathcal{F}(\mathbf{s}) = (\mathbf{s}_\omega)_{\omega \in W} = \omega \mapsto \mathbf{s}_\omega$
frequency response of $h \in \mathcal{A}$	n.a.	$(\phi_\omega(h))_{\omega \in W} = \omega \mapsto \phi_\omega(h)$

Algebraic Signal Processing Theory

	Definition	Example
Algebra	Vector space + Ring	$\mathbb{C},\ \mathbb{C}[x],\dots$
Polynomial Algebra	$p_n(x) = x^n + \sum_{0\le i<n} \beta_i x^i,\ \ \deg(p_n) = n$	$p_n(x) = x^n - 1$
	$\mathbb{C}[x]/p_n(x) = \{s(x) = \sum_{0\le \ell<n} s_\ell x^\ell \mid \deg(s) < n\}$	$C[x]/(x^n-1)$
	$\dim(\mathbb{C}[x]/p_n(x)) = n$	basis $b = (1, x, \dots, x^{n-1})$
Boundary condition	x^n mod $p_n(x) = -\sum_{0\le i<n} \beta_i x^i$	x^n mod $(x^n - 1) = 1$ cyclic b.c.
Signal extension	x^m mod $p_n(x)$ for $m \ge n$	x^m mod $(x^n-1) = x^{m \bmod n}$ periodic s.e.
Convolution	$h(x)s(x)$ mod $p_n(x)$	circular convolution
Spectrum	$\alpha = (\alpha_0, \dots, \alpha_{n-1})$ zeros/roots of $p_n(x)$ (assumed distinct)	$(\omega_n^0, \dots, \omega_n^{n-1})$ $\omega_n = e^{-\frac{2\pi j}{n}}$

Algebraic Signal Processing Theory (cont.)

	Definition	Example
Spectrum	$\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ zeros/roots of $p_n(x)$ (assumed distinct)	$(\omega_n^0, \ldots, \omega_n^{n-1})$ $\omega_n = e^{-\frac{2\pi j}{n}}$
Fourier Transform	$\mathcal{F}: \mathbb{C}[x]/p_n(x) \to \bigoplus_{0 \le k < n} \mathbb{C}[x]/(x - \alpha_k)$ $s(x) \mapsto (s(\alpha_0), \ldots, s(\alpha_{n-1}))$ Chinese Remainder Theorem	Discrete Fourier Transform
Matrix representation	$\mathcal{F} = [\alpha_k^\ell]_{0 \le k,\ell < n}$ (Vandermonde)	$\mathcal{F} = [\omega_n^{k\ell}]_{0 \le k,\ell < n} = \mathbf{DFT}_n$
Algorithms	$O(n \log^2(n))$	$O(n \log(n))$

Algebraic theory allows us to reformulate original problem in terms of polynomials.

Find transforms that approach DTFT as $n \to \infty$ ⇒ **Find polynomial families whose sets of roots approach the unit circle**

Alternative DFTs

Goal Find polynomial algebras $\mathbb{C}[x]/p_n(x)$ such that the set of zeros of $p_n(x)$ converges to the unit circle as $n \to \infty$. This will produce transforms asymptotically approaching DTFT.

Definition Let $\{p_n(x) \mid n \geq 0\}$ be a family of complex polynomials, of increasing degree (e.g., $\deg(p_n) = n$) . We say that $z \in \mathbb{C}$ is a limit of zeros for this family if there is a sequence $\{z_n \mid n \geq 0\}$ such that $p_n(z_n) = 0$ and $\lim_{n \to \infty} z_n = z$.

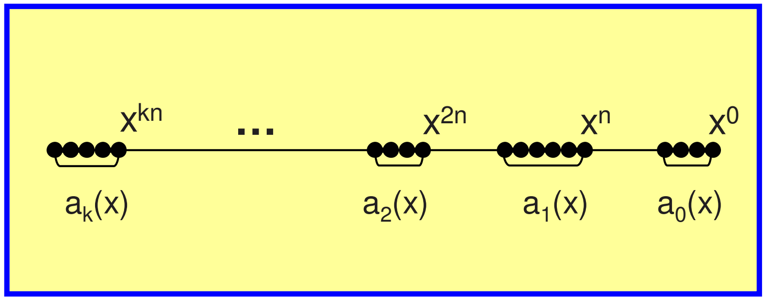
Main Theorem

- **Theorem** Let

$$q_n(x) = a_k(x)x^{kn} + a_{k-1}(x)x^{(k-1)n} + \ldots + a_1(x)x^n + a_0(x)$$

where $a_i(x) \in \mathbb{C}[x]$ and $a_0, a_k \neq 0$. Then $z \in \mathbb{C}$ is a limit of zeros if and only if one of the following holds:

(*i*) $|z| = 1$.
(*ii*) $|z| < 1$ and $a_0(z) = 0$.
(*iii*) $|z| > 1$ and $a_k(z) = 0$.



- **Corollary** Let $p_n(x) = \sum_{i=0}^{k} a_i(x)x^{\lfloor \frac{i(n-d)}{k} \rfloor}$ with $a_i(x) \in \mathbb{C}[x]$, $a_0, a_k \neq 0$, $d = \deg(a_k)$. Then $z \in \mathbb{C}$ is a limit of zeros for this sequence if and only if one of (*i*) - (*iii*) above holds.

- Proof uses the Beraha-Kahane-Weiss Theorem

Theorem (Beraha-Kahane-Weiss'78)

- $\{q_n | n \geq 0\}$ sequence of polynomials satisfying m^{th} degree recursion

$$q_{n+m}(x) = -\sum_{j=1}^{m} f_j(x) q_{n+m-j}(x)$$

where $f_j \in \mathbb{C}[x]$ are (constant) polynomials.

- *characteristic equation* of the recurrence

$$Q_x(\lambda) = \lambda^m + \sum_{j=1}^{m} f_j(x) \lambda^{m-j} = 0$$

with *roots* $\lambda_1(x), \ldots, \lambda_m(x)$.

- if for some x, $\lambda_1(x), \ldots, \lambda_m(x)$ are distinct*, then

$$q_n(x) = \sum_{j=1}^{m} \alpha_j(x) \lambda_j(x)^n$$

α_j's computed by letting n=0,1,…m-1

- assume nondegeneracy conditions hold

Theorem (Beraha-Kahane-Weiss'78) (cont.)

With above notations, $z \in \mathbb{C}$ is a limit of zeros for $(q_n(x))$ if and only if there is an ordering of the characteristic roots $\lambda_1(z), \ldots, \lambda_m(z)$ such that one of the following conditions holds:

(i) $|\lambda_1(z)| > |\lambda_j(z)|, 2 \leq j \leq m,$ and $\alpha_1(z) = 0$

(ii) $|\lambda_1(z)| = |\lambda_2(z)| = \ldots = |\lambda_l(z)| > |\lambda_j(z)|,\ l+1 \leq j \leq m$ for some $l \geq 2$.

How to apply BKW?

- Polynomial family:

$$q_n(x) = a_k(x)x^{kn} + a_{k-1}(x)x^{(k-1)n} + \ldots + a_1(x)x^n + a_0(x)$$

- Minimal recurrence degree:

$$m = |I| \qquad I = \{i \mid 0 \le i \le k, a_i(x) \ne 0\} = \{i_1, \ldots, i_m\}$$

- Linear recurrence:

$$q_n(x) = -\sum_{j=1}^{m} f_j(x) q_{n-j}(x) \qquad \text{with} \quad f_j(x) = (-1)^j \sum_{J \subset I, |J|=j} \prod_{\ell \in J} x^\ell.$$

- Characteristic equation:

$$Q_x(\lambda) = \lambda^m + \sum_{j=1}^{m} f_j(x)\lambda^{m-j} = \prod_{i \in I} (\lambda - x^i)$$

Recurrence depends only on *I*, not on $(a_i(x))_i$ (!!!)

- "Roots":

$$\lambda_j(x) = x^{i_j}, \; 1 \le j \le m$$

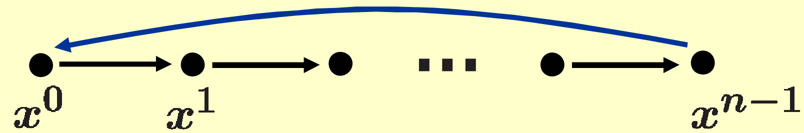
$|\lambda_s(z)| > |\lambda_t(z)| \Leftrightarrow |z^u| > |z^v| \quad (u = i_s, v = i_t)$

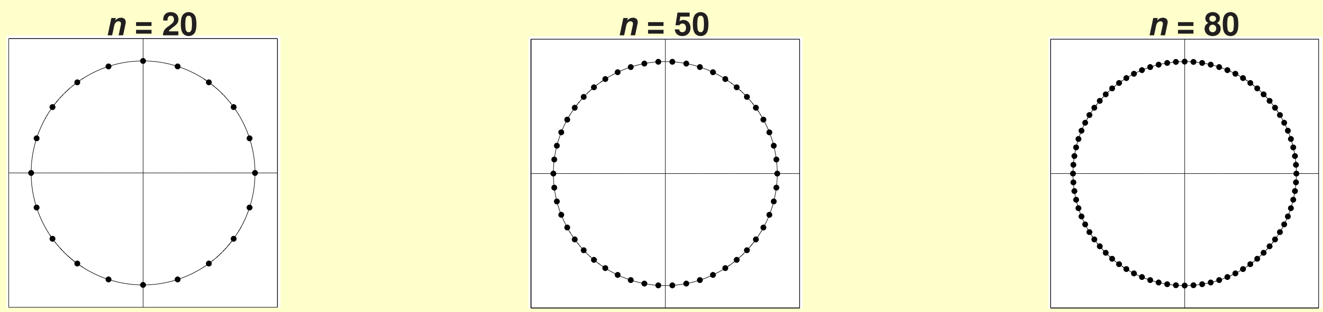
- "Coefficients":

$$\alpha_j(x) = a_{i_j}(x), \; 1 \le j \le m$$

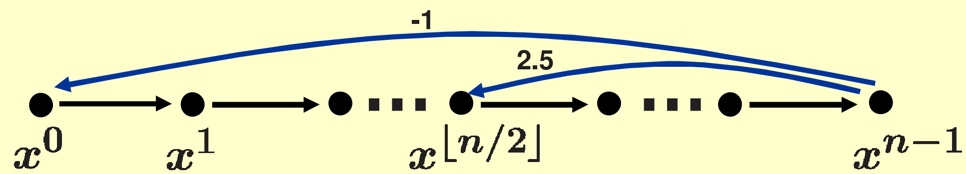
Examples and experiments

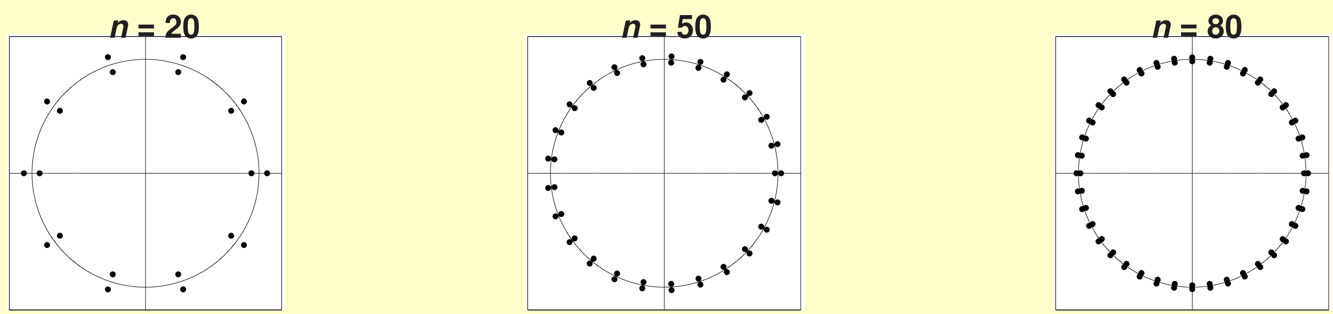
$$p_n(x) = x^n - 1$$





$$p_n(x) = x^n - \frac{5}{2}x^{\lfloor n/2 \rfloor} + 1$$

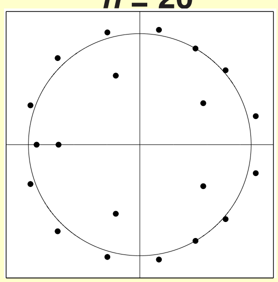




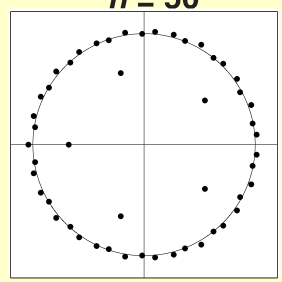
Examples and experiments

$$p_n(x) = (4x^3+1)x^{n-3} + (5x^2+1)x^{\lfloor \frac{n-3}{2} \rfloor} + \boxed{7x^5+1}$$

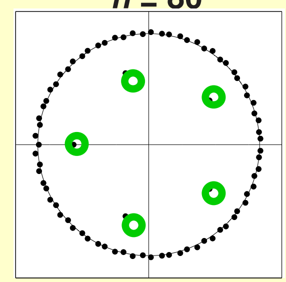
***n* = 20**



***n* = 50**

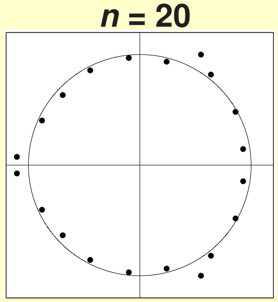


***n* = 80**

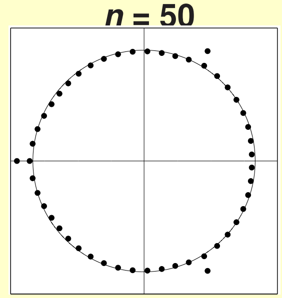


$$p_n(x) = \boxed{(2x^3+3)}x^{n-3} - (x^5-2)$$

***n* = 20**



***n* = 50**



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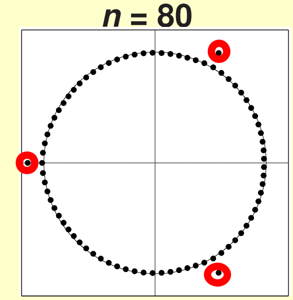
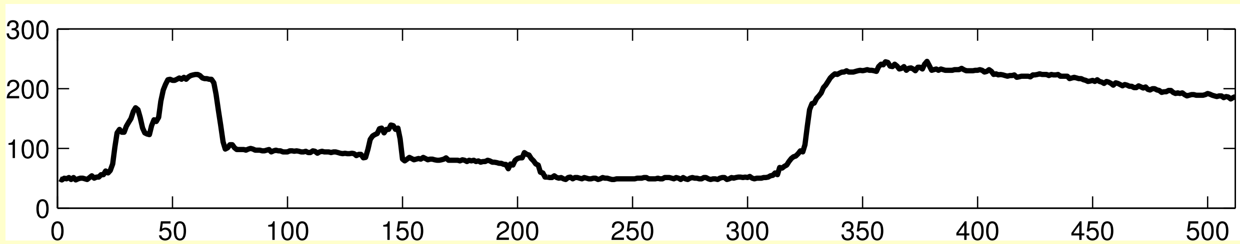
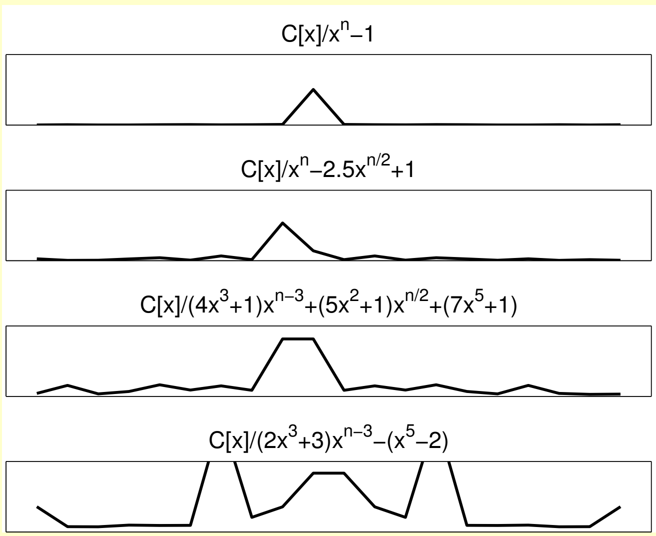


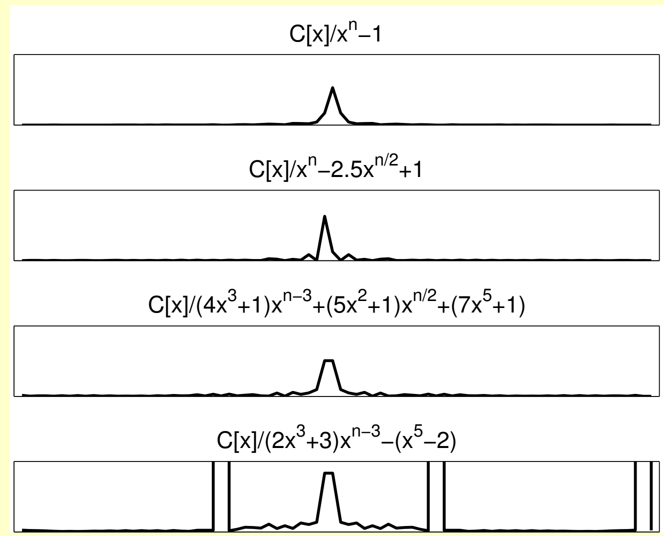
Magnitude of Fourier transform



Sample signal *s*



***n* = 20**



***n* = 80**

Summary

- Algebraic Signal Processing Theory
  - Derive a large class of transforms approaching DTFT asymptotically
  - Do not require periodic signal extension
  - Fast algorithms (due to Vandermonde structure)

- Applications
  - when periodic extension is not realistic (e.g., image blocks)
  - suggestions?

References

- **Papers**

D. Balcan, A. Sandryhaila, J. Gross, M. Püschel, “Alternatives to the Discrete Fourier Transform”, *IEEE Intl. Conf. Acoustics, Speech and Signal Processing* (ICASSP) 2008, pp. 3537-3540.

[1] S. Beraha, J. Kahane, and N. J. Weiss, “Limits of zeros of recursively defined families of polynomials,” in *Studies in Foundations and Combinatorics: Advances in Math., Suppl. Studies*, vol. I, pp. 213–232. Acad. Press, 1978.

[2] M. Püschel and J. M. F. Moura, “Algebraic signal processing theory,” available at http://arxiv.org/abs/cs.IT/0612077, parts of this manuscript submitted as [3].

[3] M. Püschel and J. M. F. Moura, “Algebraic signal processing theory: Foundation and 1-D time,” *IEEE Trans. Signal Proc.*, 56:3572-3588, 2008.

[4] V. Olshevsky and A. Shokrollahi, “Fast matrix-vector multiplication algorithms for confluent Cauchy-like matrices with applications,” in *Proc. ACM Symposium on Theory of Computing (STOC)*, 2000, pp. 573-581.

- More information on the **Algebraic Signal Processing Theory** is available at **http://www.ece.cmu.edu/~smart**