Large Margin Classifiers: Convexity and Classification

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The Pattern Classification Problem

• i.i.d. \((X, Y), (X_1, Y_1), \ldots, (X_n, Y_n)\) from \(\mathcal{X} \times \{\pm 1\}\).

• Use data \((X_1, Y_1), \ldots, (X_n, Y_n)\) to choose \(f_n : \mathcal{X} \to \mathbb{R}\) with small risk,

\[ R(f_n) = \Pr(\text{sign}(f_n(X)) \neq Y) = \mathbb{E}\ell(Y, f(X)). \]

• Natural approach: minimize empirical risk,

\[ \hat{R}(f) = \hat{\mathbb{E}}\ell(Y, f(X)) = \frac{1}{n} \sum_{i=1}^{n} \ell(Y_i, f(X_i)). \]

• Often intractable...

• Replace 0-1 loss, \(\ell\), with a convex surrogate, \(\phi\).
Large Margin Algorithms

- Consider the margins, $Y f(X)$.
- Define a margin cost function $\phi : \mathbb{R} \rightarrow \mathbb{R}^+$.
- Define the $\phi$-risk of $f : \mathcal{X} \rightarrow \mathbb{R}$ as $R_\phi(f) = \mathbb{E}_\phi(Y f(X))$.
- Choose $f \in \mathcal{F}$ to minimize $\phi$-risk.
  (e.g., use data, $(X_1, Y_1), \ldots, (X_n, Y_n)$, to minimize empirical $\phi$-risk,
  
  $$\hat{R}_\phi(f) = \hat{\mathbb{E}}_\phi(Y f(X)) = \frac{1}{n} \sum_{i=1}^{n} \phi(Y_i f(X_i)),$$
  
or a regularized version.)
Large Margin Algorithms

- **Adaboost:**
  - $\mathcal{F} = \text{span}(\mathcal{G})$ for a VC-class $\mathcal{G}$,
  - $\phi(\alpha) = \exp(-\alpha)$,
  - Minimizes $\hat{R}_\phi(f)$ using greedy basis selection, line search.

- **Support vector machines** with 2-norm soft margin.
  - $\mathcal{F} = \text{ball in reproducing kernel Hilbert space, } \mathcal{H}$.
  - $\phi(\alpha) = (\max(0, 1 - \alpha))^2$.
  - Algorithm minimizes $\hat{R}_\phi(f) + \lambda\|f\|^2_{\mathcal{H}}$. 
Large Margin Algorithms

- Many other variants
  - Neural net classifiers
    \[ \phi(\alpha) = \max(0, (0.8 - \alpha)^2). \]
  - Support vector machines with 1-norm soft margin
    \[ \phi(\alpha) = \max(0, 1 - \alpha). \]
  - L2Boost, LS-SVMs
    \[ \phi(\alpha) = (1 - \alpha)^2. \]
  - Logistic regression
    \[ \phi(\alpha) = \log(1 + \exp(-2\alpha)). \]
Large Margin Algorithms

- 0–1
- exponential
- hinge
- logistic
- truncated quadratic

\[ \alpha \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \]

\[ -2 \quad -1 \quad 0 \quad 1 \quad 2 \]

\[ \alpha \]
Statistical Consequences of Using a Convex Cost

- Bayes risk consistency? For which $\phi$?
  - (Lugosi and Vayatis, 2004), (Mannor, Meir and Zhang, 2002): regularized boosting.
  - (Zhang, 2004), (Steinwart, 2003): SVM.
  - (Jiang, 2004): boosting with early stopping.
Statistical Consequences of Using a Convex Cost

- How is risk related to $\phi$-risk?
  - (Lugosi and Vayatis, 2004), (Steinwart, 2003): asymptotic.
  - (Zhang, 2004): comparison theorem.

- Convergence rates? With low noise?

- Estimating conditional probabilities?

- Multiclass?
Overview

• Relating excess risk to excess $\phi$-risk.
• The approximation/estimation decomposition and universal consistency.
• Convergence rates: low noise.
• Kernel classifiers: sparseness versus probability estimation.
• Structured multiclass classification.
**Definitions and Facts**

\[
R(f) = \Pr(\text{sign}(f(X)) \neq Y) \quad \text{Risk},
\]

\[
R^* = \inf_{f} R(f) \quad \text{Bayes risk},
\]

\[
\eta(x) = \Pr(Y = 1|X = x) \quad \text{conditional probability}.
\]

- \(\eta\) defines an optimal classifier:

\[
R^* = R(\text{sign}(\eta(x) - 1/2)).
\]

- **Excess risk of** \(f : \mathcal{X} \to \mathbb{R}\) **is**

\[
R(f) - R^* = \mathbb{E}(\mathbb{1}[\text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2)]|2\eta(X) - 1|).
\]
Definitions

Risk: \[ R(f) = \Pr(\text{sign}(f(X)) \neq Y). \]

\(\phi\)-Risk: \[ R_\phi(f) = \mathbb{E}\phi(Y f(X)). \]

\[ R_\phi(f) = \mathbb{E} (\mathbb{E} [\phi(Y f(X))|X]). \]

Conditional \(\phi\)-risk:

\[ \mathbb{E} [\phi(Y f(X))|X = x] = \eta(x) \phi(f(x)) + (1 - \eta(x)) \phi(-f(x)). \]
Conditional $\phi$-risk: Example

\[ \phi(\alpha) = (\max(0, 1 - \alpha))^2. \]

\[ C_{0.3}(\alpha) = 0.3\phi(\alpha) + 0.7\phi(-\alpha) \]

\[ C_{0.7}(\alpha) = 0.7\phi(\alpha) + 0.3\phi(-\alpha) \]
\[ R(f) = \Pr(\text{sign}(f(X)) \neq Y) \quad R^* = \inf_f R(f) \quad \text{(Bayes risk)} \]

\[ R_{\phi}(f) = \mathbb{E}\phi(Y f(X)) \quad R^*_{\phi} = \inf_f R_{\phi}(f) \quad \text{(optimal \(\phi\)-risk)} \]

**Conditional \(\phi\)-risk:**

\[ \mathbb{E}[\phi(Y f(X))|X = x] = \eta(x)\phi(f(x)) + (1 - \eta(x))\phi(-f(x)). \]

**Optimal conditional \(\phi\)-risk for \(\eta \in [0, 1]\):**

\[ H(\eta) = \inf_{\alpha \in \mathbb{R}} (\eta\phi(\alpha) + (1 - \eta)\phi(-\alpha)). \]

\[ R^*_{\phi} = \mathbb{E}H(\eta(X)). \]
Optimal Conditional $\phi$-risk: Example

![Graph showing various functions and their transformations](image)
Definitions

Optimal conditional $\phi$-risk for $\eta \in [0, 1]$:

$$H(\eta) = \inf_{\alpha \in \mathbb{R}} (\eta \phi(\alpha) + (1 - \eta) \phi(-\alpha)).$$

Optimal conditional $\phi$-risk with incorrect sign:

$$H^-(\eta) = \inf_{\alpha : \alpha (2\eta - 1) \leq 0} (\eta \phi(\alpha) + (1 - \eta) \phi(-\alpha)).$$

Note:

$$H^-(\eta) \geq H(\eta) \quad H^-(1/2) = H(1/2).$$
Example: $H^-(\eta) = \phi(0)$
Definitions

\[ H(\eta) = \inf_{\alpha \in \mathbb{R}} (\eta \phi(\alpha) + (1 - \eta) \phi(-\alpha)) \]

\[ H^{-}(\eta) = \inf_{\alpha : \alpha(2\eta - 1) \leq 0} (\eta \phi(\alpha) + (1 - \eta) \phi(-\alpha)) . \]

**Definition:** \( \phi \) is **classification-calibrated** if, for \( \eta \neq 1/2 \),

\[ H^{-}(\eta) > H(\eta). \]

i.e., pointwise optimization of conditional \( \phi \)-risk leads to the correct sign.
(c.f. Lin (2001))
Definition: Given $\phi$, define $\psi : [0, 1] \to [0, \infty)$ by $\psi = \tilde{\psi}^{**}$, where

$$\tilde{\psi}(\theta) = H^{-}\left(\frac{1 + \theta}{2}\right) - H\left(\frac{1 + \theta}{2}\right).$$

Here, $g^{**}$ is the Fenchel-Legendre biconjugate of $g$,

$$\text{epi}(g^{**}) = \text{co}(\text{epi}(g)),$$

$$\text{epi}(g) = \{(x, y) : x \in [0, 1], g(x) \leq y\}.$$
• \( \psi \) is the best convex lower bound on
\[
\tilde{\psi}(\theta) = H^-((1 + \theta)/2) - H((1 + \theta)/2),
\]
the excess conditional \( \phi \)-risk when the sign is incorrect.
• \( \psi = \tilde{\psi}^{**} \) is the biconjugate of \( \tilde{\psi} \),
\[
epi(\psi) = \overline{\text{co}}(\text{epi}(\tilde{\psi})),
\]
\[
epi(\psi) = \{ (\alpha, t) : \alpha \in [0, 1], \psi(\alpha) \leq t \}.
\]
• \( \psi \) is the functional convex hull of \( \tilde{\psi} \).

\[0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0\]
\[-2 \quad -1 \quad 0 \quad 1 \quad 2\]
The Relationship between Excess Risk and Excess $\phi$-risk

**Theorem:**

1. For any $P$ and $f$, $\psi(R(f) - R^*) \leq R_\phi(f) - R_\phi^*$.
2. This bound cannot be improved.
3. Near-minimal $\phi$-risk implies near-minimal risk precisely when $\phi$ is classification-calibrated.
The Relationship between Excess Risk and Excess $\phi$-risk

**Theorem:**

1. For any $P$ and $f$, \( \psi(R(f) - R^*) \leq R_\phi(f) - R_\phi^* \).
2. This bound cannot be improved:
   For $|\mathcal{X}| \geq 2$, $\epsilon > 0$ and $\theta \in [0, 1]$, there is a $P$ and an $f$ with
   \[
   R(f) - R^* = \theta \\
   \psi(\theta) \leq R_\phi(f) - R_\phi^* \leq \psi(\theta) + \epsilon.
   \]
3. Near-minimal $\phi$-risk implies near-minimal risk precisely when $\phi$ is classification-calibrated.
The Relationship between Excess Risk and Excess $\phi$-risk

**Theorem:**

1. For any $P$ and $f$, $\psi(R(f) - R^*) \leq R_\phi(f) - R^*_\phi$.
2. This bound cannot be improved.
3. The following conditions are equivalent:
   (a) $\phi$ is classification calibrated.
   (b) $\psi(\theta_i) \to 0$ iff $\theta_i \to 0$.
   (c) $R_\phi(f_i) \to R^*_\phi$ implies $R(f_i) \to R^*$. 
Excess Risk Bounds: Proof Idea

Facts:

- \( H(\eta), H^-(\eta) \) are symmetric about \( \eta = 1/2 \).
- \( H(1/2) = H^-(1/2) \), hence \( \psi(0) = 0 \).
- \( \psi(\theta) \) is convex.
- \( \psi(\theta) \leq \tilde{\psi}(\theta) = H^- \left( \frac{1+\theta}{2} \right) - H \left( \frac{1+\theta}{2} \right) \).
Excess Risk Bounds: Proof Idea

Recall:

\[ R(f) - R^* = \mathbb{E} (\mathbf{1} [\text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2)] |2\eta(X) - 1|) . \]

Thus,

\[ \psi(R(f) - R^*) \leq \mathbb{E} (\mathbf{1} [\text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2)] \psi (|2\eta(X) - 1|)) \]

\[ \leq \mathbb{E} \left( \mathbf{1} [\text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2)] \tilde{\psi} (|2\eta(X) - 1|) \right) \]

\[ = \mathbb{E} (\mathbf{1} [\text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2)] (H^- (\eta(X)) - H(\eta(X)))) \]

\[ \leq \mathbb{E} (\phi(Yf(X)) - H(\eta(X))) \]

\[ = R_\phi(f) - R^*_{\phi} . \]
Excess Risk Bounds: Proof Idea

Recall:

\[ R(f) - R^* = \mathbb{E} \left( 1 \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] |2\eta(X) - 1| \right). \]

Thus,

\[
\psi(R(f) - R^*) \leq \tilde{\psi} \leq 2 \mathbb{E} \left( 1 \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] \psi (|2\eta(X) - 1|) \right) \\
\leq \mathbb{E} \left( 1 \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] \tilde{\psi} (|2\eta(X) - 1|) \right) \\
= \mathbb{E} \left( 1 \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] (H^- (\eta(X)) - H(\eta(X))) \right) \\
\leq \mathbb{E} (\phi(Y f(X)) - H(\eta(X))) \\
= R_\phi(f) - R^*_\phi.
\]
**Excess Risk Bounds: Proof Idea**

Recall:

\[ R(f) - R^* = \mathbb{E} \left( \mathbf{1} \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] |2\eta(X) - 1| \right). \]

Thus,

\[
\psi(R(f) - R^*) \leq \mathbb{E} \left( \mathbf{1} \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] \psi (|2\eta(X) - 1|) \right) \\
\leq \mathbb{E} \left( \mathbf{1} \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] \tilde{\psi} (|2\eta(X) - 1|) \right) \\
= \mathbb{E} \left( \mathbf{1} \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] (H^{-}(\eta(X)) - H(\eta(X))) \right) \\
\leq \mathbb{E} \left( \phi(Yf(X)) - H(\eta(X)) \right) \\
= R_\phi(f) - R_\phi^*.
\]
**Excess Risk Bounds: Proof Idea**

Recall:

\[ R(f) - R^* = \mathbb{E} \left( \mathbf{1} \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] |2\eta(X) - 1| \right). \]

Thus,

\[ \psi(R(f) - R^*) \quad (H^- \text{ minimizes conditional } \phi\text{-risk}) \]

\[ \leq \mathbb{E} \left( \mathbf{1} \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] \psi(|2\eta(X) - 1|) \right) \]

\[ \leq \mathbb{E} \left( \mathbf{1} \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] \tilde{\psi}(|2\eta(X) - 1|) \right) \]

\[ = \mathbb{E} \left( \mathbf{1} \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] (H^-(\eta(X)) - H(\eta(X))) \right) \]

\[ \leq \mathbb{E} (\phi(Yf(X)) - H(\eta(X))) \]

\[ = R_\phi(f) - R^*_\phi. \]
Excess Risk Bounds: Proof Idea

Recall:

\[ R(f) - R^* = \mathbb{E} \left( 1 \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] |2\eta(X) - 1| \right). \]

Thus,

\[
\begin{align*}
\psi(R(f) - R^*) & \leq \mathbb{E} \left( 1 \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] \psi(|2\eta(X) - 1|) \right) \\
& \leq \mathbb{E} \left( 1 \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] \tilde{\psi}(|2\eta(X) - 1|) \right) \\
& = \mathbb{E} \left( 1 \left[ \text{sign}(f(X)) \neq \text{sign}(\eta(X) - 1/2) \right] (H^-(\eta(X)) - H(\eta(X))) \right) \\
& \leq \mathbb{E} (\phi(\text{Yf}(X)) - H(\eta(X))) \\
& = R_\phi(f) - R^*_\phi.
\end{align*}
\]
Excess Risk Bounds: Proof Idea

Converse:

1. If $\tilde{\psi}$ is convex, $\psi = \tilde{\psi}$.
   Fix $P(x_1) = 1$ and choose $\eta(x_1) = (1 + \theta)/2$.
   Each inequality is clearly tight.

2. If $\tilde{\psi}$ is not convex:
   Choose $\theta_1$ and $\theta_2$ so that $\psi(\theta_i) = \tilde{\psi}(\theta_i)$ and $\theta \in \text{co}\{\theta_1, \theta_2\}$.
   Set $\eta(x_1) = (1 + \theta_1)/2$ and $\eta(x_2) = (1 + \theta_2)/2$.
   Again, each inequality is clearly tight.
Classification-calibrated $\phi$

**Theorem:** If $\phi$ is convex,

$\phi$ is classification calibrated $\iff$ \begin{align*}
\phi & \text{ is differentiable at } 0 \\
\phi'(0) & < 0.
\end{align*}

**Theorem:** If $\phi$ is classification calibrated, \begin{align*}
\exists \gamma > 0, \forall \alpha \in \mathbb{R}, \gamma \phi(\alpha) & \geq 1 \; [\alpha \leq 0].
\end{align*}
Overview

- Relating excess risk to excess $\psi$-risk.
- The approximation/estimation decomposition and universal consistency.
- Convergence rates: low noise.
- Kernel classifiers: sparseness versus probability estimation.
- Structured multiclass classification.
The Approximation/Estimation Decomposition

Algorithm chooses

\[ f_n = \arg \min_{f \in F_n} \hat{E}_n R_{\phi}(f) + \lambda_n \Omega(f). \]

We can decompose the excess risk estimate as

\[ \psi (R(f_n) - R^*) \leq R_{\phi}(f_n) - R_{\phi}^* \]

\[ = R_{\phi}(f_n) - \inf_{f \in F_n} R_{\phi}(f) + \inf_{f \in F_n} R_{\phi}(f) - R_{\phi}^* . \]

\[ \begin{align*}
\text{estimation error} & \quad \text{approximation error}
\end{align*} \]
The Approximation/Estimation Decomposition

\[
\psi (R(f_n) - R^*) \leq R_\phi(f_n) - R_\phi^* \\
= R_\phi(f_n) - \inf_{f \in \mathcal{F}_n} R_\phi(f) + \inf_{f \in \mathcal{F}_n} R_\phi(f) - R_\phi^*.
\]

- Approximation and estimation errors are in terms of \( R_\phi \), not \( R \).
- Like a regression problem.
- With a rich class and appropriate regularization, \( R_\phi(f_n) \to R_\phi^* \).
  (e.g., \( \mathcal{F}_n \) gets large slowly, or \( \lambda_n \to 0 \) slowly.)
- Universal consistency (\( R(f_n) \to R^* \)) iff \( \phi \) is classification calibrated.
Overview

• Relating excess risk to excess $\Phi$-risk.
• The approximation/estimation decomposition and universal consistency.
• Convergence rates: low noise.
• Kernel classifiers: sparseness versus probability estimation.
• Structured multiclass classification.
Low Noise

\[ \hat{f}(x) = 2\eta(x) - 1 \]

\[ R(\hat{f}) - R^* \]
**Definition:** [Tsybakov] The distribution $P$ on $\mathcal{X} \times \{\pm 1\}$ has *
noise exponent* $0 \leq \alpha < \infty$ if there is a $c > 0$ such that

$$
\Pr (0 < |2 \eta(X) - 1| < \epsilon) \leq c \epsilon^\alpha.
$$

- Equivalently, there is a $c$ such that for every $f : \mathcal{X} \rightarrow \{\pm 1\}$,

$$
\Pr (f(X) (\eta(X) - 1/2) < 0) \leq c (R(f) - R^*)^\beta,
$$

where $\beta = \frac{\alpha}{1 + \alpha}$.

- $\alpha = \infty$: for some $c > 0$, $\Pr (0 < |2 \eta(X) - 1| < c) = 0$. 
Low Noise

- Tsybakov considered empirical risk minimization. (But ERM is typically hard)

- With:
  - the noise assumption,
  - the Bayes classifier in the function class

the empirical risk minimizer has (true) risk converging surprisingly quickly to the minimum.  

(Tsybakov, 2001)
**Theorem:** If \( P \) has noise exponent \( \alpha \),
then there is a \( c > 0 \) such that for any \( f : \mathcal{X} \rightarrow \mathbb{R} \),

\[
c (R(f) - R^*)^\beta \psi \left( \frac{(R(f) - R^*)^{1-\beta}}{2c} \right) \leq R_\phi(f) - R_\phi^*,
\]

where \( \beta = \frac{\alpha}{1 + \alpha} \in [0, 1] \).

Notice that we only improve the rate, since the convexity of \( \psi \) implies

\[
c (R(f) - R^*)^\beta \psi \left( \frac{(R(f) - R^*)^{1-\beta}}{2c} \right) \geq c \psi \left( \frac{R(f) - R^*}{2c} \right).
\]
**Risk Bounds with Low Noise**

**Note:** Minimizing $R_\phi$ adapts to noise exponent: lower noise implies closer relationship between risk and $\phi$-risk.

**Proof idea**

Split $\mathcal{X}$:

1. Low noise region ($|\eta(X) - 1/2| > \epsilon$): bound risk using noise assumption.

2. High noise ($\leq \epsilon$): bound risk as before.
Fast Convergence Rates for Large Margin Classifiers

\[ \Psi(R(f_n) - R^*) \leq R_\phi(f_n) - R^* \]

\[ = R_\phi(f_n) - \inf_{f \in \mathcal{F}_n} R_\phi(f) + \inf_{f \in \mathcal{F}_n} R_\phi(f) - R^* . \]

- \( R(f_n) - R^* \) decreases with \( R_\phi(f_n) - \inf_f R_\phi(f) \).
  (Faster decrease with low noise.)

- How rapidly does \( R_\phi(f_n) \) converge?
Assume that $\phi$ satisfies

1. A Lipschitz condition:
   
   \[
   \text{for all } a, b \in \mathbb{R}, \quad |\phi(a) - \phi(b)| \leq L|a - b|.
   \]

2. A strict convexity condition: the modulus of convexity of $\phi$ satisfies

   \[
   \delta_\phi(\epsilon) \geq \epsilon^r, \quad \text{where}
   \]

   \[
   \delta_\phi(\epsilon) = \inf \left\{ \frac{\phi(\alpha_1) + \phi(\alpha_2)}{2} - \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) : |\alpha_1 - \alpha_2| \geq \epsilon \right\}.
   \]
Modulus of Convexity

Let $a$, $a + \frac{\epsilon}{2}$, and $a + \epsilon$ be points on the graph.
Fast Convergence Rates for Strictly Convex $\phi$, Convex $\mathcal{F}$

**Theorem:** Suppose that:
- $\phi$ is Lipschitz with constant $L$.
- $\phi$ has modulus of convexity $\delta_{\phi}(\epsilon) \geq \epsilon^r$. (Set $\alpha = \max(1, 2 - 2/r)$.)
- $\mathcal{F}$ is a convex set of uniformly bounded functions.
- $\mathcal{F}$ is finite dimensional ($\sup_{P} \log \mathcal{N}(\epsilon, \mathcal{F}, L_2(P)) \leq d \log(1/\epsilon)$).

Then with probability at least $1 - \delta$, the minimizer $\hat{f} \in \mathcal{F}$ of $\hat{R}_\phi$ satisfies

$$R_\phi(\hat{f}) - \inf_{f \in \mathcal{F}} R_\phi(f) \leq c \left( \frac{d \log n + \log(1/\delta)}{n} \right)^{1/\alpha}.$$
Fast Convergence Rates for Strictly Convex $\phi$, Convex $\mathcal{F}$

The key idea:
Strict convexity ensures that the variance of the excess $\phi$-loss is controlled.

Define $f^* = \arg\min_{f \in \mathcal{F}} R_{\phi}(f)$.

For $f \in \mathcal{F}$, define the excess $\phi$-loss as

$$g_f(x, y) = \phi(yf(x)) - \phi(yf^*(x)).$$

**Theorem:** If $\phi$ is Lipschitz with constant $L$ and uniformly convex with modulus of convexity $\delta_{\phi}(\epsilon) \geq \epsilon^r$, then for any $f$ in a convex set $\mathcal{F}$,

$$\mathbb{E}g_f^2 \leq L^2 \mathbb{E} (f - f^*)^2 \leq L^2 \left( \frac{\mathbb{E}g_f}{2} \right)^{\min(1,2/r)}.$$
Fast Convergence Rates for Strictly Convex $\phi$

\[ R_\phi(f) - R_\phi(f^*) \]

\[ \hat{R}_\phi(f) - \hat{R}_\phi(f^*) \]
An Aside: Tsybakov’s Condition Revisited

**Definition:** [Tsybakov] The distribution $P$ on $\mathcal{X} \times \{\pm 1\}$ has *noise exponent* $\alpha$ if there is a $c > 0$ such that every $f : \mathcal{X} \rightarrow \{\pm 1\}$ has

$$\Pr (f(X)(\eta(X) - 1/2) < 0) \leq c (R(f) - R^*)^\beta,$$

where $\beta = \frac{\alpha}{1 + \alpha} \in [0, 1]$.

This is the *variance condition*:

- Bayes classifier is in $\mathcal{F}$; set $f^* = \text{sign}(\eta - 1/2)$.
- $\mathbb{E}g_f^2 = \Pr (f(X)(\eta(X) - 1/2) < 0)$.
- $\mathbb{E}g_f = R(f) - R^*$.
- $\implies$ Assumption is equivalent to $\mathbb{E}g_f^2 \leq c (\mathbb{E}g_f)^\beta$. Fast rates follow.
**Risk Bounds with Low Noise: Examples**

- Adaboost: $\phi(\alpha) = e^{-\alpha}$.
- SVM with 2-norm soft-margin penalty: $\phi(\alpha) = (\max(0, 1 - \alpha))^2$.
- Quadratic loss: $\phi(\alpha) = (1 - \alpha)^2$.

All of these satisfy:
- convex.
- classification calibrated.
- quadratic modulus of convexity, $\delta_\phi$.
- quadratic $\psi$. 
**Risk Bounds with Low Noise**

**Theorem:** If $\phi$ has
- modulus of convexity $\delta_\phi(\alpha) \geq \alpha^2$,
- noise exponent $= \infty$ (that is, $|\Pr(Y = 1|X) - 1/2| \geq c_1$), and
- $\mathcal{F}$ is $d$-dimensional,

then with probability at least $1 - \delta$, the minimizer $\hat{f}$ of $\hat{L}_\phi$ satisfies

$$R(\hat{f}) - R^* \leq c \left( \frac{d \log(n/\delta)}{n} + \inf_{f \in \mathcal{F}} R_\phi(f) - R^*_\phi \right).$$

(And there are similar fast rates for larger classes.)
Summary: Large Margin Classifiers

- Relating excess risk to excess $\phi$-risk:
  - $\psi$ relates excess risk to excess $\phi$-risk.
  - Best possible.

- The approximation/estimation decomposition and universal consistency.

- Convergence rates: low noise.
  - Tighter bound on excess risk.
  - Fast convergence of $\phi$-risk for strictly convex $\phi$. 
Overview

- Relating excess risk to excess $\phi$-risk.
- The approximation/estimation decomposition and universal consistency.
- Convergence rates: low noise.
- **Kernel classifiers**: sparseness versus probability estimation.
- Structured multiclass classification.
Kernel Methods for Classification

\[ f_n = \arg \min_{f \in \mathcal{H}} \left( \hat{E} \phi(Yf(X)) + \lambda_n \|f\|^2 \right), \]

where \( \mathcal{H} \) is a reproducing kernel Hilbert space (RKHS), with norm \( \| \cdot \| \), and \( \lambda_n > 0 \) is a regularization parameter.

Example:

L1-SVM: \( \phi(\alpha) = (1 - \alpha)_+ \)

L2-SVM: \( \phi(\alpha) = ((1 - \alpha)_+)^2 \).
Kernel Methods for Classification

support of $P$ in $\{x : k(x, x) \leq B\}$.  

$\lambda_n \to 0$, suitably slowly. 

$\phi$ locally Lipschitz.  

RKHS suitably rich  

$\Rightarrow \inf_{f \in \mathcal{H}} R_{\phi}(f) = R^\ast$.  

$\phi$ classification calibrated  

$\Rightarrow R(f_n) \to R^\ast$.  

i.e., a universal kernel, suitable $\phi$, appropriate regularization schedule  

$\Rightarrow$ universal consistency.

e.g., (Steinwart, 2001)
Overview

- Relating excess risk to excess $\phi$-risk.
- The approximation/estimation decomposition and universal consistency.
- Convergence rates: low noise.
- Kernel classifiers
  - probability estimation
  - sparseness
- Structured multiclass classification.
Can we use a large margin classifier,

\[ f_n = \arg \min_{f \in \mathcal{H}} \left( \hat{E}_\phi(Yf(X)) + \lambda_n \|f\|^2 \right), \]

to estimate the conditional probability \( \eta(x) = \Pr(Y = 1|X = x) \)?

Does \( f_n(x) \) give information about \( \eta(x) \), say, asymptotically?

- Confidence-rated predictions are of interest for many decision problems.
- Probabilities are useful for combining decisions.
Estimating Conditional Probabilities

If $\phi$ is convex, we can write

$$H(\eta) = \inf_{\alpha \in \mathbb{R}} (\eta \phi(\alpha) + (1 - \eta) \phi(-\alpha))$$

$$= \eta \phi(\alpha^*(\eta)) + (1 - \eta) \phi(-\alpha^*(\eta)),$$

where $\alpha^*(\eta) = \arg \min_{\alpha} (\eta \phi(\alpha) + (1 - \eta) \phi(-\alpha)) \subset \mathbb{R} \cup \{\pm \infty\}.$

Recall:

$$R^*_\phi = \mathbb{E}H(\eta(X)) = \mathbb{E}\phi(Y \alpha^*(\eta(X)))$$

$$\eta(x) = \Pr(Y = 1|X = x).$$
Estimating Conditional Probabilities

\[ \alpha^*(\eta) = \arg\min_{\alpha} (\eta \phi(\alpha) + (1 - \eta) \phi(-\alpha)) \subset \mathbb{R} \cup \{\pm \infty\}. \]

Examples of \( \alpha^*(\eta) \) versus \( \eta \in [0, 1] \):

- L2-SVM: \( \phi(\alpha) = ((1 - \alpha)_+)^2 \)
- L1-SVM: \( \phi(\alpha) = (1 - \alpha)_+ \).
If $\alpha^*(\eta)$ is not invertible, that is, there are $\eta_1 \neq \eta_2$ with

$$\alpha^*(\eta_1) \cap \alpha^*(\eta_2) \neq \emptyset,$$

then there are distributions $P$ and functions $f_n$ with $R_\phi(f_n) \rightarrow R_\phi^*$ but $f_n(x)$ cannot be used to estimate $\eta(x)$.

E.g., $f_n(x) \rightarrow \alpha^*(\eta_1) \cap \alpha^*(\eta_2)$. Is $\eta(x) = \eta_1$ or $\eta(x) = \eta_2$?
Overview

- Relating excess risk to excess $\phi$-risk.
- The approximation/estimation decomposition and universal consistency.
- Convergence rates: low noise.
- Kernel classifiers: sparseness versus probability estimation
- Structured multiclass classification.
Sparseness

- Representer theorem: solution of optimization problem can be represented as:
  \[ f_n(x) = \sum_{i=1}^{n} \alpha_i k(x, x_i). \]

- Inputs \( x_i \) with \( \alpha_i \neq 0 \) are called support vectors (SV’s).

- Sparseness (number of support vectors \( \ll n \)) means faster evaluation of the classifier.
For L1 and L2-SVM, Steinwart proved that the asymptotic fraction of SV’s is $E G(\eta(X))$ (under some technical assumptions).

The function $G(\eta)$ depends on the loss function used:

L2-SVM doesn’t produce sparse solutions (asymptotically) while L1-SVM does.

Recall: L2-SVM can estimate $\eta$ while L1-SVM cannot.
Sparseness versus Estimating Conditional Probabilities

The ability to estimate conditional probabilities always causes loss of sparseness:

- Lower bound of the asymptotic fraction of data that become SV’s can be written as $\mathbb{E}G(\eta(X))$.
- $G(\eta)$ is 1 throughout the region where probabilities can be estimated.
- The region where $G(\eta) = 1$ is an interval centered at $1/2$. 
• Steinwart’s lower bound on the asymptotic fraction of SV’s: 
\[ \Pr[ 0 \notin \partial \phi(Y\alpha^*(\eta(X))) ] \]

• Write the lower bound as \( \mathbb{E}G(\eta(X)) \) where 
\[
G(\eta) = \eta \mathbf{1}[0 \notin \partial \phi(\alpha^*(\eta))] + (1 - \eta) \mathbf{1}[0 \notin \partial \phi(-\alpha^*(\eta))]
\]

\[
\frac{1}{3}((1 - t)_+)^2 + \frac{2}{3}(1 - t)_+
\]

\( \alpha^*(\eta) \) vs. \( \eta \)

\( G(\eta) \) vs. \( \eta \)
Sparseness vs. Estimating Probabilities

- In general, $G(\eta)$ is 1 on an interval around 1/2; outside that interval, $G(\eta) = \min\{\eta, 1 - \eta\}$.

- We know this gives a loose lower bound for L1-SVM:

- Sharp bound can be derived for loss functions of the form:
  $$\phi(t) = h((t_0 - t)_+)$$
  where $h$ is convex, differentiable and $h'(0) > 0$. 
Asymptotically Sharp Result

• Recall that our classifier can be expressed as \( \sum_i \alpha_i k(\cdot, x_i) \) and let \( \#SV = |\{i : \alpha_i \neq 0\}| \).

• If the kernel \( k \) is analytic and universal (and the underlying \( P_X \) is continuous and non-trivial), then for a regularization sequence \( \lambda_n \to 0 \) sufficiently slowly:

\[
\frac{\#SV}{n} \xrightarrow{P} \mathbb{E}G(\eta(X))
\]

where

\[
G(\eta) = \begin{cases} 
\eta/\gamma & 0 \leq \eta \leq \gamma \\
1 & \gamma < \eta < 1 - \gamma \\
(1 - \eta)/\gamma & 1 - \gamma \leq \eta \leq 1 
\end{cases}
\]
Example again

- $\gamma$ is given by $\frac{-\varphi'(t_0)}{-\varphi'(t_0) - \varphi'(-t_0)}$ and $\alpha^*(\eta)$ is invertible in the interval $(\gamma, 1 - \gamma)$.
- Below $h(t) = \frac{1}{3}t^2 + \frac{2}{3}t$, $-\varphi'(1) = \frac{2}{3}$, $-\varphi'(-1) = 2$ and hence $\gamma = \frac{1}{4}$.

\[
\frac{1}{3}((1 - t)_+)^2 + \frac{2}{3}(1 - t)_+
\]

$\alpha^*(\eta)$ vs. $\eta$

$G(\eta)$ vs. $\eta$
Overview

- Relating excess risk to excess $\phi$-risk.
- The approximation/estimation decomposition and universal consistency.
- Convergence rates: low noise.
- Kernel classifiers
  - No sparseness where $\alpha^*(\eta)$ is invertible.
  - Can design $\phi$ to trade off sparseness and probability estimation.
- Structured multiclass classification.

slides at http://www.stat.berkeley.edu/~bartlett/talks
Structured Classification: Optical Character Recognition

\[ X = \text{grey-scale image of a sequence of characters} \]
\[ Y = \text{sequence of characters} \]

This is an example of

This is an example of
Structured Classification: Parsing

\[ X = \text{sentence} \]
\[ Y = \text{parse tree} \]

The pedestrian crossed the road.

```
S
  /   \
NP   VP
  /   /   \\
DT N V NP
```

- The pedestrian
- crossed
- DT
- N
- the road
Structured Classification

- Data: i.i.d. \((X, Y), (X_1, Y_1), \ldots, (X_n, Y_n)\) from \(\mathcal{X} \times \mathcal{Y}\).
- Loss function: \(\ell : \mathcal{Y}^2 \to \mathbb{R}^+, \ell(\hat{y}, y) = \text{cost of mistake}\).
- Use data \((X_1, Y_1), \ldots, (X_n, Y_n)\) to choose \(f : \mathcal{X} \to \mathcal{Y}\) with small risk,

\[ R(f) = \mathbb{E}\ell(f(X), Y). \]

Often choose \(f\) from a fixed class \(\mathcal{F}\).
Structured Classification Problems

Key issue: $|\mathcal{Y}|$ is very large.

- OCR: exponential in number of characters
- Parsing: exponential in sentence length

Generative Modelling:

- Split $Y$ into parts/assume sparse dependencies.
  (e.g., graphical model; probabilistic context-free grammar.)
- Plug-in estimate:
  1. Simple model $\hat{p}(x, y; \theta)$ of $\Pr(Y = y | X = x)$.
  2. Use data to estimate parameters $\theta$. (e.g., ML)
  3. Compute $\arg \max_{y \in \mathcal{Y}} \hat{p}(x, y; \theta)$. (e.g., dynamic programming)
Generative Model

If each factor is a log-linear model, we compute a linear discriminant:

\[
\hat{y} = \arg \max_{y \in \mathcal{Y}} \log(p(x, y; \theta))
\]

\[
= \arg \max_{y \in \mathcal{Y}} \sum_{i} g_i(x, y) \theta_i.
\]
Structured Classification Problems: Sparse Representations

Suppose \( y \) naturally decomposes into parts:

\[
R(x, y) \text{ denotes the set of “parts” belonging to } (x, y) \in \mathcal{X} \times \mathcal{Y}
\]

\[
G(x, y) = \sum_{r \in R(x, y)} g(x, r)
\]

\[
\hat{y} = \arg \max_{y \in \mathcal{Y}} G(x, y) \theta = \arg \max_{y \in \mathcal{Y}} \sum_{r \in R(x, y)} g(x, r) \theta,
\]

- e.g. Markov random fields. Parts are configurations for cliques.
- e.g. PCFGs. Parts are rule-location pairs (rules of grammar applied at specific locations in the sentence).
Large Margin Methods for Structured Classification

• Choose $f$ as maximum of linear functions,

$$f(x) = \arg \max_{y \in \mathcal{Y}} G(x, y)' \theta,$$

to minimize empirical $\phi$-risk.

• e.g., Support Vector Machines:
\[ \mathcal{Y} = \{ \pm 1 \}, \ell(\hat{y}, y) = 1[\hat{y} \neq y], G(x, y) = yx: \]
Choose $\theta$ to minimize

$$\lambda \| \theta \|^2 + \frac{1}{n} \sum_{i=1}^{n} (1 - Y_i X_i' \theta)_+, \quad$$

where $(x)_+ = \max\{x, 0\}$.

This is a quadratic program (QP).
• For $Y = \{\pm 1\}$, $\ell(\hat{y}, y) = 1[\hat{y} \neq y]$, and $G(x, y) = yx$,

$$
(1 - 2Y_i X_i' \theta)_+ = \max_{\hat{y}} (\ell(\hat{y}, Y_i) - (Y_i - \hat{y})X_i' \theta)_+
$$

$$
= \max_{\hat{y}} (\ell(\hat{y}, Y_i) - (G(X_i, Y_i)' \theta - G(X_i, \hat{y})' \theta))_+ .
$$

• Think of $G'(x, y)' \theta - G(x, \hat{y})' \theta$ as an upper bound on the loss $l(\hat{y}, y)$ that we’ll incur when we choose the $\hat{y}$ that maximizes $G(x, \hat{y})' \theta$. 

\textbf{Large Margin Classifiers}
Choose $\theta$ to minimize

$$\lambda \| \theta \|^2 + \frac{1}{n} \sum_{i=1}^{n} \max_{\hat{y}} (\ell(\hat{y}, Y_i) - (G(X_i, Y_i)' \theta - G(X_i, \hat{y})' \theta))_+$$

$$= \lambda \| \theta \|^2 + \frac{1}{n} \sum_{i=1}^{n} \max_{\hat{y}} (\ell(\hat{y}, Y_i) - G_i'(\hat{y}) \theta)_+ ,$$

where $(x)_+ = \max\{x, 0\}$ and $G_i(\hat{y}) = G(X_i, Y_i) - G(X_i, \hat{y})$.

- Quadratic program.
Large Margin Multiclass Classification

Primal problem:

\[
\min_{\theta, \epsilon} \left( \frac{1}{2} \lambda \|\theta\|^2 + \frac{1}{n} \sum_i \epsilon_i \right)
\]

Subject to the constraints:

\[
\forall i, y \in \mathcal{Y}(X_i), \\
\theta' G_{i,y} \geq \ell(y, Y_i) - \epsilon_i \\
\forall i, \epsilon_i \geq 0
\]

Dual problem:

\[
\max_{\alpha} \left( C \sum_{i,y} \alpha_{i,y} \ell(y, Y_i) - \frac{C^2}{2} \sum_{i,y,j,z} \alpha_{i,y} \alpha_{j,z} G'_{i,y} G_{j,z} \right)
\]

Subject to the constraints:

\[
\forall i, \sum_y \alpha_{i,y} = 1 \\
\forall i, y, \alpha_{i,y} \geq 0
\]
Large Margin Multiclass Classification

Some observations:

- Quadratic program over $\alpha = (\alpha_{i,y})$, restricted to ($n$ copies of) the probability simplex:
  \[
  \max_{\alpha} \quad Q(\alpha) \\
  \text{s.t.} \quad \alpha_i \in \Delta.
  \]

- Number of variables is sum over data of number of possible labels. Very large: $n|\mathcal{Y}|$. 
Exponentiated Gradient Algorithm

Exponentiated gradients:

\[ \alpha^{(t+1)} = \arg \min_\alpha \left( D \left( \alpha, \alpha^{(t)} \right) + \eta \alpha' \nabla Q \left( \alpha^{(t)} \right) \right). \]

- \( D \) is Kullback-Liebler divergence.
- \( \nabla Q \) term moves \( \alpha \) in direction of decreasing \( Q \).
- KL term constrains it to be close to \( \alpha^{(t)} \).

Solution is

\[ \alpha^{(t)}_{i,y} = \frac{\exp(\theta^{(t)}_{i,y})}{\sum_{z} \exp(\theta^{(t)}_{i,z})}, \]

with \( \theta^{(t+1)} = \theta^{(t)} - \eta \nabla Q(\alpha^{(t)}) \).
Theorem: For all $u \in \Delta$,
\[
\frac{1}{T} \sum_{t=1}^{T} Q(\alpha^{(t)}) \leq Q(u) + \frac{D(u, \alpha^{(1)})}{\eta T} + c_{\eta,Q} \frac{Q(\alpha^{(1)})}{T}.
\]
Exponentiated Gradient Algorithm with Parts

Suppose \( y \) naturally decomposes into parts:

\[
R(x, y) \text{ denotes the set of "parts" belonging to } (x, y) \in \mathcal{X} \times \mathcal{Y}
\]

\[
G(x, y) = \sum_{r \in R(x, y)} g(x, r)
\]

\[
\ell(\hat{y}, y) = \sum_{r \in R(x, \hat{y})} L(r, y).
\]

- e.g. Markov random fields. Parts are configurations for cliques.
- e.g. PCFGs. Parts are rule-location pairs (rules of grammar applied at specific locations in the sentence).
Exponentiated Gradient Algorithm with Parts

\[ G(x, y) = \sum_{r \in R(x, y)} g(x, r) \]

\[ \ell(\hat{y}, y) = \sum_{r \in R(x, \hat{y})} L(r, y). \]

- Like a factorization of \( \Pr(Y | X) \), where log probabilities decompose as sums over parts.

- We require that loss decomposes in the same way.
  - e.g., Markov random field: \( \ell(\hat{y}, y) = \sum_c L(\hat{y}_c, y_c). \)
  - e.g., PCFG: \( \ell(\hat{y}, y) = \sum_r 1[r \text{ in } \hat{y}, \text{not in } y] \).
Exponentiated Gradient Algorithm with Parts

In this case, $Q$ can be expressed as a function of the “marginal” variables, $Q(\alpha) = \tilde{Q}(\mu)$, with

$$
\mu_{i,r} = \sum_y \alpha_{i,y} 1[r \in R(x_i, y)].
$$

Exponentiated gradient algorithm:

$$
\mu_{i,r}^{(t)} = \sum_y \alpha_{i,y}^{(t)} 1[r \in R(x_i, y)]
$$

$$
\alpha_{i,y}^{(t)} = \frac{\exp(\sum_{r \in R(x_i, y)} \theta_{i,r}^{(t)})}{\sum_y \exp(\sum_{r \in R(x_i, y)} \theta_{i,r}^{(t)})}
$$

$$
\theta^{(t+1)} = \theta^{(t)} - \eta \nabla \mu \tilde{Q}(\mu^{(t)}).
$$
Exponentiated Gradient Algorithm: Sparse Representations

Efficiently computing $\mu$ from $\theta$:

- Markov random field: Computing clique marginals from exponential family parameters.
- PCFG: Computing rule probabilities from exponential family parameters.
Exponentiated Gradient Algorithm: Convergence

**Theorem:** For all $u \in \Delta$,

$$
\frac{1}{T} \sum_{t=1}^{T} Q(\alpha^{(t)}) \leq Q(u) + \frac{D(u, \alpha^{(1)})}{\eta T} + c_{\eta, Q} \frac{Q(\alpha^{(1)})}{T}.
$$

**Step 1:**

For any $u \in \Delta$,

$$
\eta Q(\alpha^{(t)}) - \eta Q(u) \leq D(u, \alpha^{(t)}) - D(u, \alpha^{(t+1)}) + D(\alpha^{(t)}, \alpha^{(t+1)}).
$$

Follows from convexity of $Q$, definition of updates. (Standard in analysis of online prediction algorithms.)
Exponentiated Gradient Algorithm: Convergence

Step 2:

\[
D(\alpha^{(t)}, \alpha^{(t+1)}) = \sum_{i=1}^{n} \log \mathbb{E} \left[ e^{\eta(X^{(t)}_i - \mathbb{E}X^{(t)}_i)} \right] \\
\leq \left( \frac{e^{\eta B} - 1 - \eta B}{B^2} \right) \sum_{i=1}^{n} \text{var}(X^{(t)}_i),
\]

where \( \text{Pr} \left( X^{(t)}_i = - (\nabla Q(\alpha^{(t)}))_{i,y} \right) = \alpha_{i,y}^{(t)} \).

Follows from definition of updates, Bernstein’s inequality.
Exponentiated Gradient Algorithm: Convergence

**Step 3a:** For some $\theta \in [\theta(t), \theta(t+1)]$,

$$
\eta \sum_{i=1}^{n} \text{var}(X_{i}^{(t)}) - \eta^2 (B + \lambda) \sum_{i=1}^{n} \text{var}(X_{i,\theta}^{(t)}) \leq Q(\alpha(t)) - Q(\alpha(t+1)),
$$

where $\text{Pr} \left( X_{i,\theta}^{(t)} = - (\nabla Q(\alpha(t)))_{i,y} \right) = \alpha(\theta)_{i,y}$.

- Variance of $X_{i}^{(t)}$ is first order term in Taylor series expansion (in $\theta$) for $Q$.
- Variance of $X_{i,\theta}^{(t)}$ is second order term.
- $B$ is infinity norm of centered version of $\nabla Q$
- $\lambda$ is largest eigenvalue of $\nabla^2 Q$. 

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Exponentiated Gradient Algorithm: Convergence

**Step 3b:** For all \( \theta \in [\theta^{(t)}, \theta^{(t+1)}] \),

\[
\text{var}(X_{i,\theta}^{(t)}) \leq e^{\eta B} \text{var}(X_i^{(t)}).
\]

Hence,

\[
\sum_{i=1}^{n} \text{var}(X_i^{(t)}) \leq \frac{1}{\eta (1 - \eta(B + \lambda)e^{2\eta B})} \left( Q(\alpha^{(t)}) - Q(\alpha^{(t+1)}) \right).
\]
**Exponentiated Gradient Algorithm: Convergence**

\[
\eta Q(\alpha(t)) - \eta Q(u) \\
\leq D(u, \alpha(t)) - D(u, \alpha^{(t+1)}) + D(\alpha(t), \alpha^{(t+1)}) \\
\leq D(u, \alpha(t)) - D(u, \alpha^{(t+1)}) + \left( \frac{e^{\eta B} - 1 - \eta B}{B^2} \right) \sum_{i=1}^{n} \text{var}(X_i^{(t)}) \\
\leq D(u, \alpha(t)) - D(u, \alpha^{(t+1)}) + c'_{\eta, Q} \left( Q(\alpha(t)) - Q(\alpha^{(t+1)}) \right).
\]

**Theorem:** For all \( u \in \Delta \),

\[
\frac{1}{T} \sum_{t=1}^{T} Q(\alpha(t)) \leq Q(u) + \frac{D(u, \alpha^{(1)})}{\eta T} + c_{\eta, Q} \frac{Q(\alpha^{(1)})}{T}.
\]
Large Margin Methods for Structured Classification

- Generative models
  - Markov random fields
  - Probabilistic context-free grammars
- Quadratic program for large margin classifiers
- Exponentiated gradient algorithm
- Convergence analysis
Overview

- Relating excess risk to excess $\phi$-risk.
- The approximation/estimation decomposition and universal consistency.
- Convergence rates: low noise.
- Kernel classifiers: sparseness versus probability estimation.
- Structured multiclass classification.

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