$1+2+3+4+5+ \ldots = ?$

BEN HUH?

TEA TALK?

01/31/2014?

DIVERGENT SUM?

DIVERGENT SUM?

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = \ ?$$

DIVERGENT SUM?

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots = \ ?$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = \ ?$$

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + \cdots = \ ?$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \ ?$$

DIVERGENT SUM?

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots = \ ?$$

$$1 - 1 + 1 - 1 + \cdots = \ ?$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = \ ?$$

$$1 - 2 + 3 - 4 + \cdots = \ ?$$

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + \cdots = \ ?$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \ ?$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ ?$$

DIVERGENT SUM?

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots = \ ?$$

$$1 - 1 + 1 - 1 + \cdots = \ ?$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = \ ?$$

$$1 - 2 + 3 - 4 + \cdots = \ ?$$

CAN THESE SUMS HAVE ANY SENSIBLE VALUES?

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + \cdots = \ ?$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \ ?$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ ?$$

DIVERGENT SUM?

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots = -\frac{1}{2}$$

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

YES!

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + \cdots = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2 \quad ?$$

DIVERGENT SUM?

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

Central to quantum field theory and string theory.

Physics[edit]

In bosonic string theory, the attempt is to compute the possible energy levels of a string, in particular the lowest energy level. Speaking informally, each harmonic of the string can be viewed as a collection of independent quantum harmonic oscillators, one for each transverse direction, where is the dimension of spacetime. If the fundamental oscillation frequency is then the energy in an oscillator contributing to the th harmonic is . So using the divergent series, the sum over all harmonics is . Ultimately it is this fact, combined with the Goddard–Thorn theorem, which leads to bosonic string theory failing to be consistent in dimensions other than 26.

A similar calculation, using the Epstein zeta-function in place of the Riemann zeta function, is involved in computing the Casimir force.[9]

PROOF 1

$S_1 = 1 - 1 + 1 - 1 + \cdots$

$S_2 = 1 - 2 + 3 - 4 + \cdots$

PROOF 1

$$S_1 = 1 - 1 + 1 - 1 + \cdots$$

$$\begin{aligned} 2 \cdot S_1 &= 1 - 1 + 1 - 1 + \cdots \\ &\quad + 1 - 1 + 1 - \cdots \\ &= 1 \end{aligned}$$

$$S_1 = \frac{1}{2}$$

$$S_2 = 1 - 2 + 3 - 4 + \cdots$$

$$\begin{aligned} 2 \cdot S_2 &= 1 - 2 + 3 - 4 + \cdots \\ &\quad + 1 - 2 + 3 - 4 + \cdots \\ &= 1 - 1 + 1 - 1 + \cdots \\ &= S_1 \end{aligned}$$

$$S_2 = \frac{1}{4}$$

PROOF 1

$$S_1 = 1 - 1 + 1 - 1 + \cdots$$

$$S_1 = \frac{1}{2}$$

$$S_2 = 1 - 2 + 3 - 4 + \cdots$$

$$S_2 = \frac{1}{4}$$

$$S = 1 + 2 + 3 + 4 + \cdots = \ ?$$

PROOF 1

$S_1 = 1 - 1 + 1 - 1 + \cdots$

$$S_1 = \frac{1}{2}$$

$S_2 = 1 - 2 + 3 - 4 + \cdots$

$$S_2 = \frac{1}{4}$$

$$S = 1 + 2 + 3 + 4 + \cdots = \ ?$$

$$-4 \cdot S = -4 - 8 - 12 - 16 - \cdots$$

$$= \ - 2 \cdot 2 \ - \ 2 \cdot 4 \ - \cdots$$

$$+S = 1 + 2 + 3 + 4 + \cdots$$

$$-3 \cdot S = 1 - 2 + 3 - 4 + \cdots$$

$$S = -\frac{1}{12}$$

PROOF 1

$$S_1 = 1 - 1 + 1 - 1 + \cdots$$

$$S_1 = \frac{1}{2}$$

$$S_2 = 1 - 2 + 3 - 4 + \cdots$$

$$S_2 = \frac{1}{4}$$

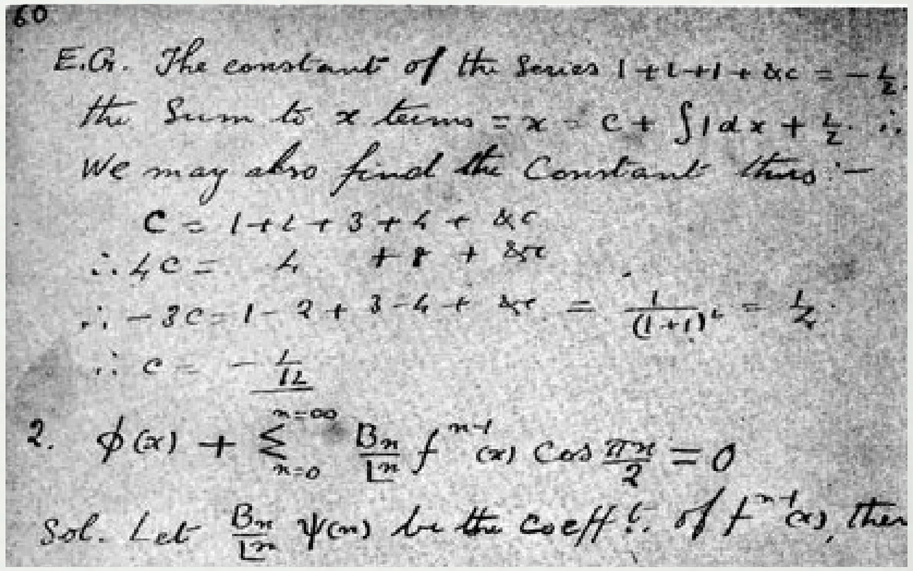
$$S = 1 + 2 + 3 + 4 + \cdots = \ ?$$

$$S = -\frac{1}{12}$$

EULER (1768)

RAMANUJAN'S LETTER TO HARDY

- "Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's *Infinite Series* and not fall into the pitfalls of divergent series. … I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \cdots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. I dilate on this simply to convince you that you will not be able to follow my methods of proof if I indicate the lines on which I proceed in a single letter. …"
- "You may ask how you can accept results based upon wrong premises. What I tell you is this... Go check it for yourself."
- "I am already a half starving man. To preserve my brains I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from the Government."


60
E.G. The constant of the Series $1+1+1+\&c = -\frac{1}{2}$ ∴
the Sum to x terms $= x + c + \int 1\,dx + \frac{1}{2}$ ∴
We may also find the Constant thus:—
$c = 1+2+3+4+\&c$
$\therefore 4c = 4 + 8 + \&c$
$\therefore -3c = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$
$\therefore c = -\frac{1}{12}$
2. $\phi(x) + \sum_{n=0}^{n=\infty} \frac{B_n}{\lfloor n} f^{n-1}(x) \cos\frac{\pi n}{2} = 0$
Sol. Let $\frac{B_n}{\lfloor n}\psi(n)$ be the coeff^t. of $f^{n-1}(x)$, then

Facebook.com/pages/Srinivasa-Ramanujan

INFINITE SUM

- Conventional infinite sum method only works for converging series

$$\sum_{k=1}^{\infty} a_k = \ ? \qquad S_n = \sum_{k=1}^{n} a_k \qquad \lim_{n\to\infty} S_n$$

- **GENERALIZED SUMMATION METHODS FOR DIVERGENT SERIES**
  EULER SUMMATION, BOREL SUMMATION, CESÀRO SUMMATION, ABEL SUMMATION, RAMANUJAN SUMMATION, ZETA FUNCTION REGULARIZATION

- **Ramanujan summation** is a technique for assigning a value to infinite divergent series. Although it is not a sum in the traditional sense, it has properties which make it mathematically useful in the study of divergent infinite series, for which conventional summation is undefined.

- "The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever." (N. ABEL, 1832)

ANALYTIC CONTINUATION

$$S_n = 1 + x + x^2 + \cdots + x^n$$

$$S_\infty = 1 + x + x^2 + \cdots$$

ANALYTIC CONTINUATION

$$S_n = 1 + x + x^2 + \cdots + x^n$$

$$S_\infty = 1 + x + x^2 + \cdots$$

$$1 + xS_n = 1 + x + x^2 + \cdots + x^{n+1}$$
$$= S_n + x^{n+1}$$

$$1 + xS_\infty = S_\infty$$

$$S_n = \frac{1 - x^{n+1}}{1 - x}$$

$$S_\infty = \frac{1}{1 - x}$$

CONVERGES FOR $|x| < 1$

ANALYTIC CONTINUATION

$$S_\infty = 1 + x + x^2 + \cdots$$

$$S_\infty = \frac{1}{1-x}$$

APPLY IT OUTSIDE THE CONVERGENCE DOMAIN

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

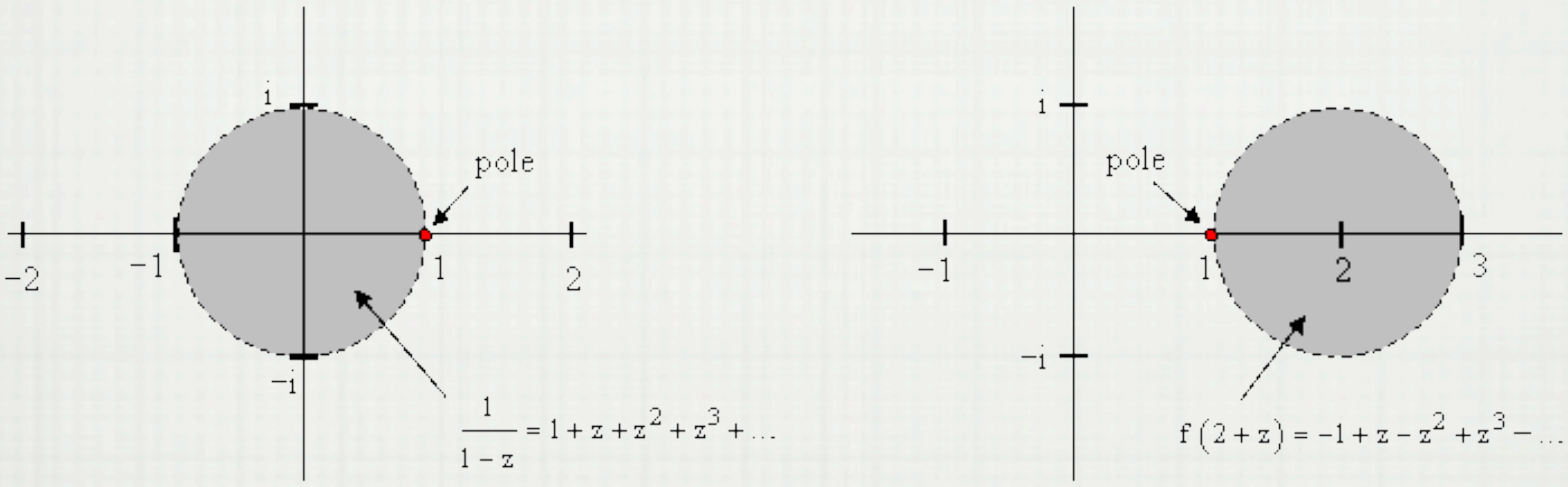
$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$$

$$(x = -1)$$

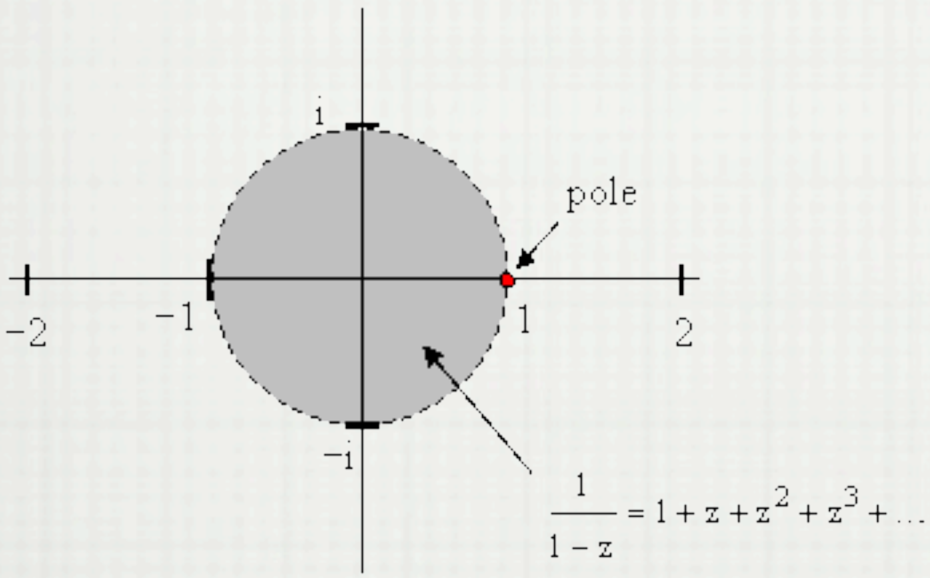
$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

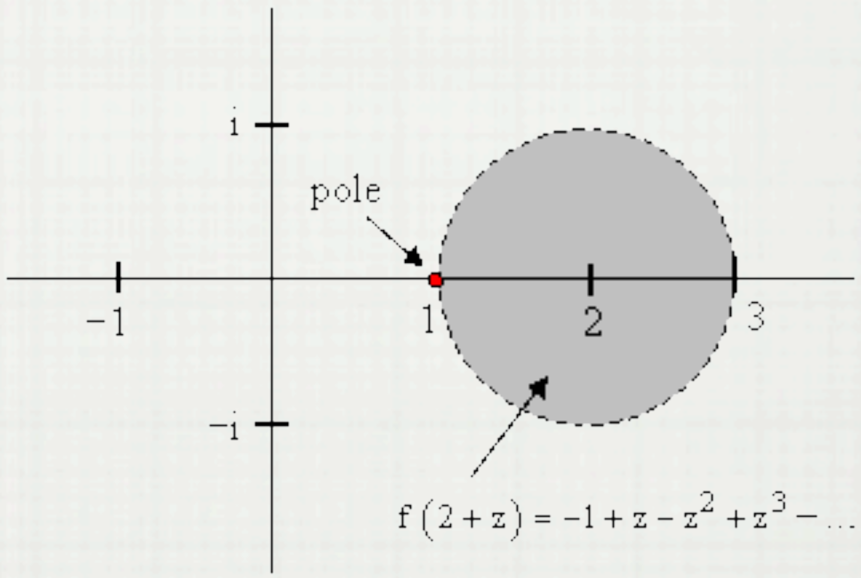
$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

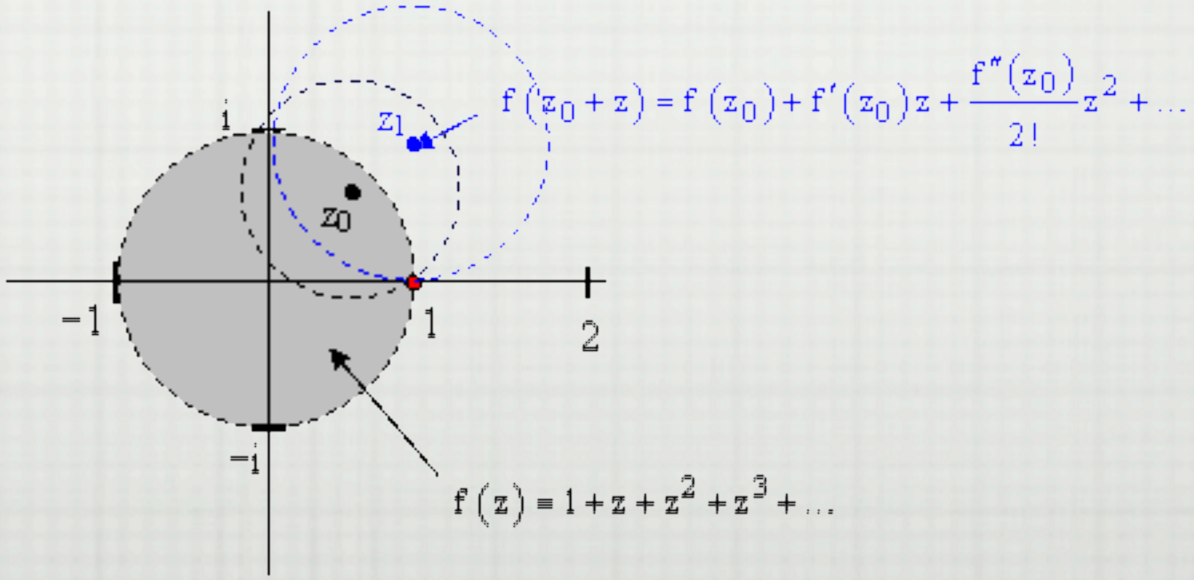
ANALYTIC CONTINUATION



ANALYTIC CONTINUATION

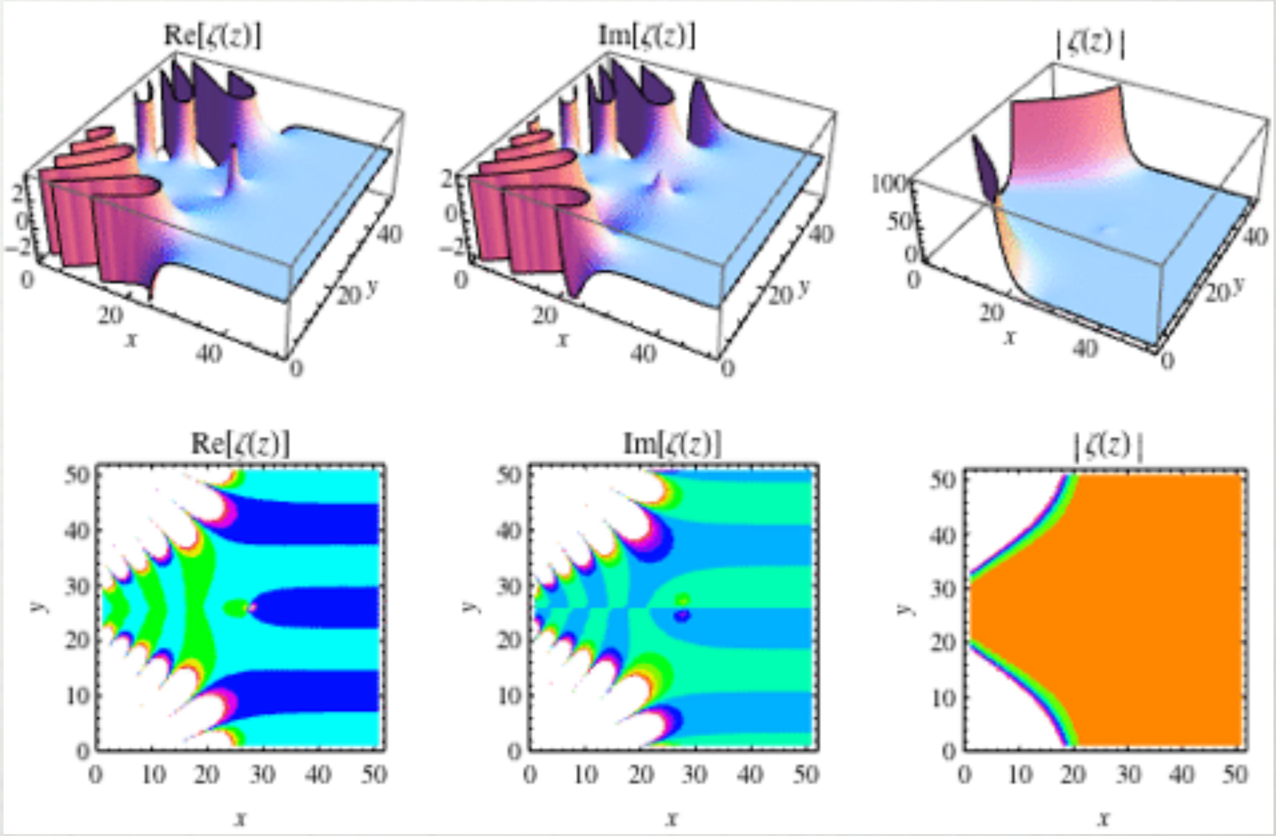






RIEMANN ZETA

$$\zeta(s) = \sum n^{-s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$



RIEMANN ZETA

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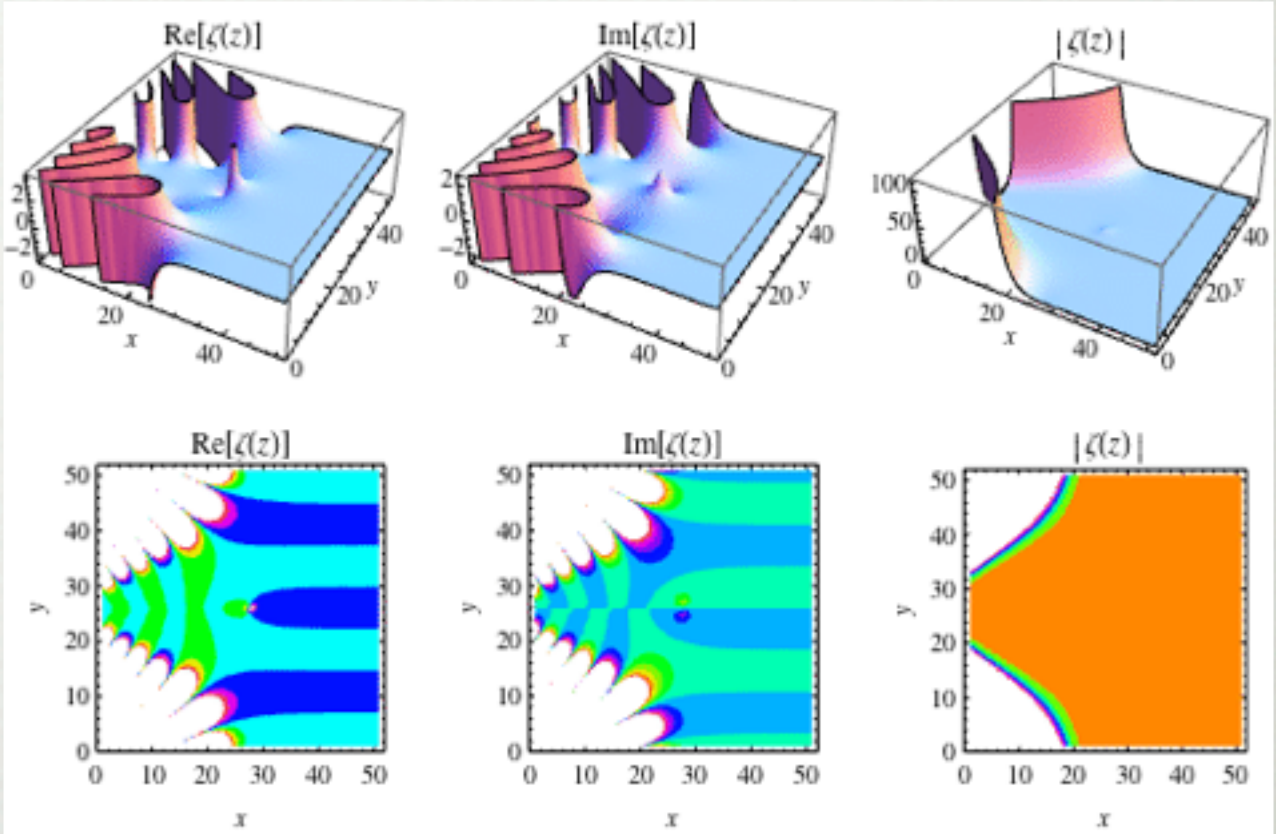
$$\zeta(2) = \sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$$



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$$\zeta(-s) = \sum n^s = 1 + 2^s + 3^s + 4^s + \cdots$$

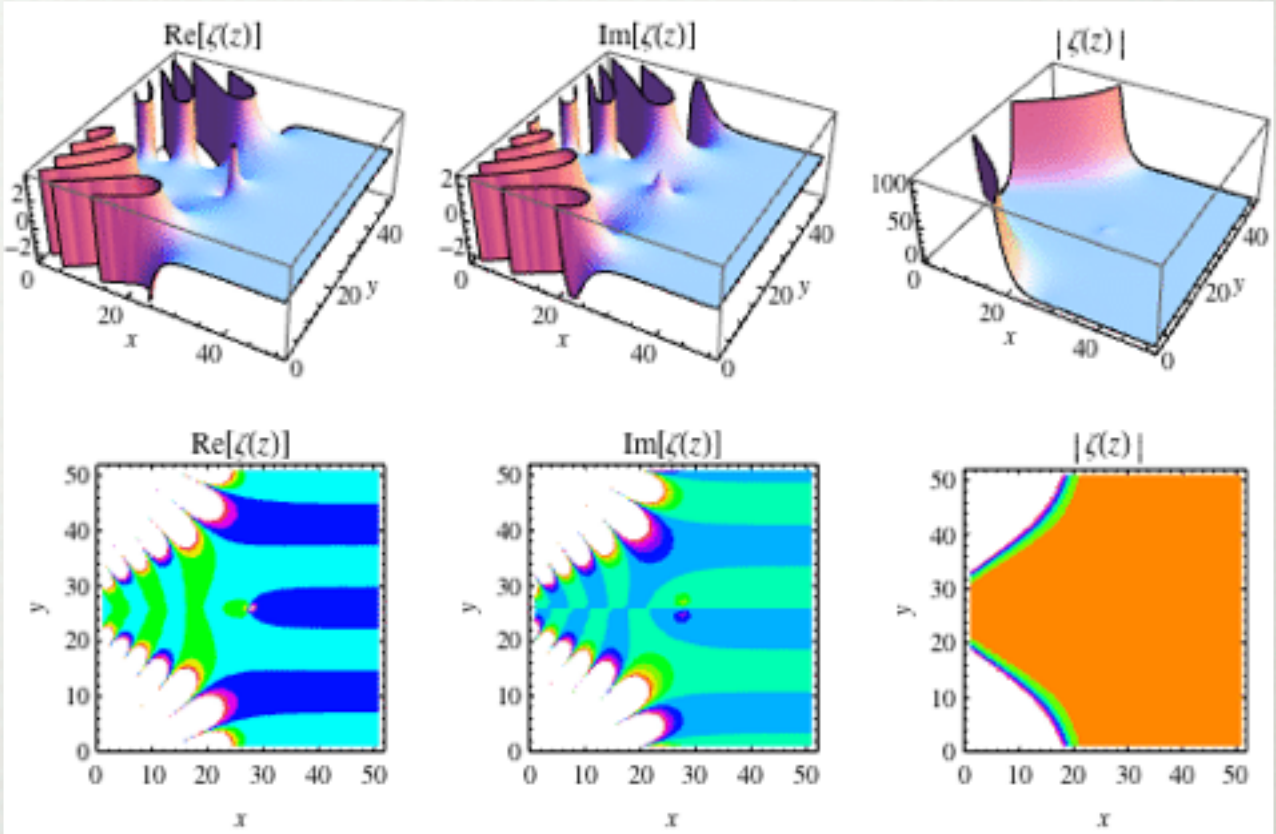
$$2 \cdot 2^s \zeta(-s) = \quad 2 \cdot 2^s \quad + \quad 2 \cdot 4^s + \cdots$$

$$\left(1 - 2^{s+1}\right) \zeta(-s) = 1 - 2^s + 3^s - 4^s + \cdots$$

$$\zeta(-s) = \frac{1 - 2^s + 3^s - 4^s + \cdots}{1 - 2^{s+1}}$$

$$= -\frac{B_{s+1}}{s+1}$$

Bernoulli numbers



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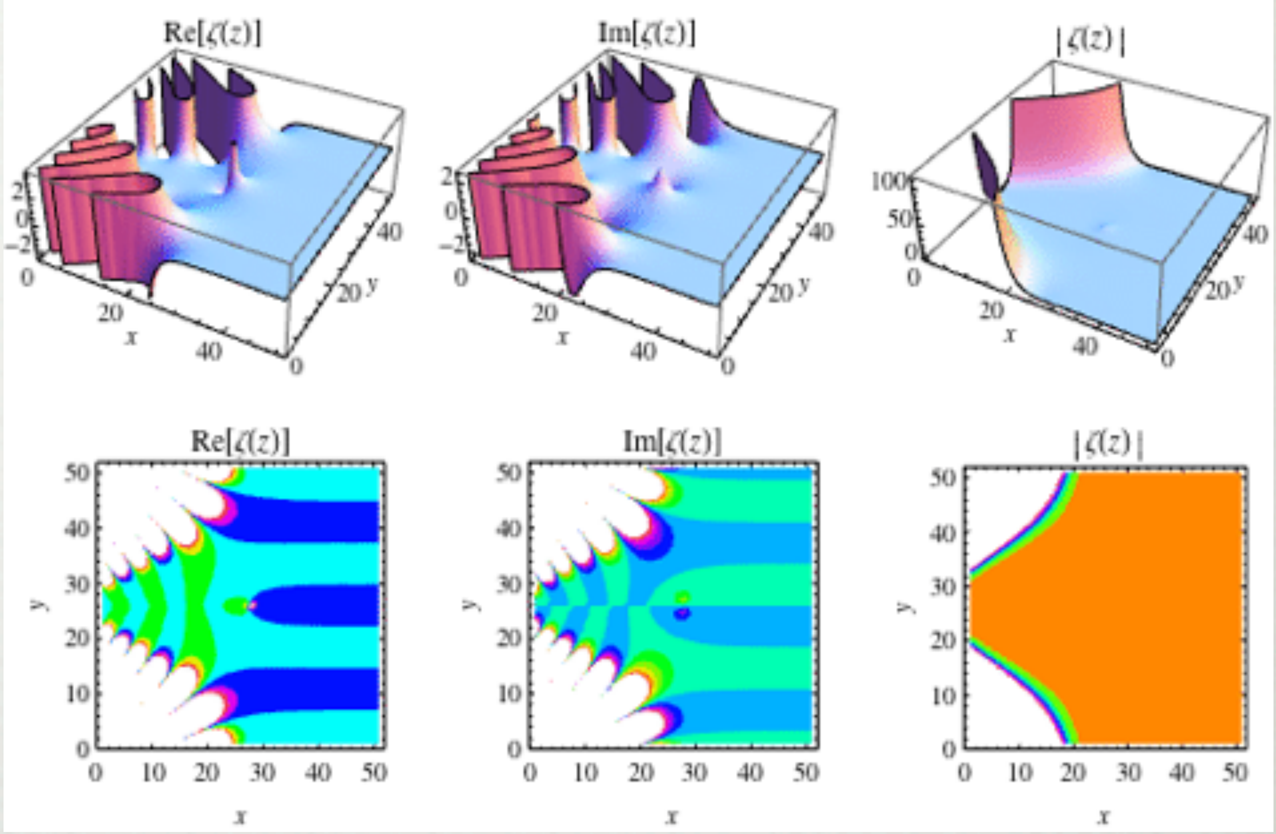
$$= -\frac{B_{s+1}}{s+1}$$

Bernoulli numbers

n	B_n
0	1
1	$\pm\frac{1}{2}$
2	$\frac{1}{6}$
3	0
4	$-\frac{1}{30}$
5	0
6	$\frac{1}{42}$
7	0
8	$-\frac{1}{30}$
9	0



$$\zeta(0) = 1 + 1 + 1 + \cdots = -\frac{1}{2}$$

$$\zeta(-1) = 1 + 2 + 3 + \cdots = -\frac{1}{12}$$

$$\zeta(-2) = 1 + 4 + 9 + \cdots = 0$$

CASIMIR EFFECT

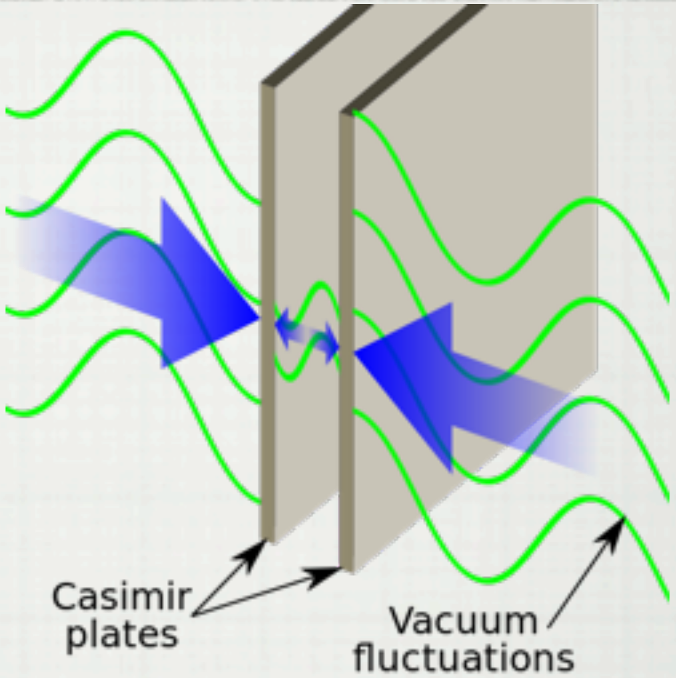
$$\langle E \rangle = \frac{\hbar}{2} \cdot 2 \int \frac{A dk_x dk_y}{(2\pi)^2} \sum_{n=1}^{\infty} \omega_n$$

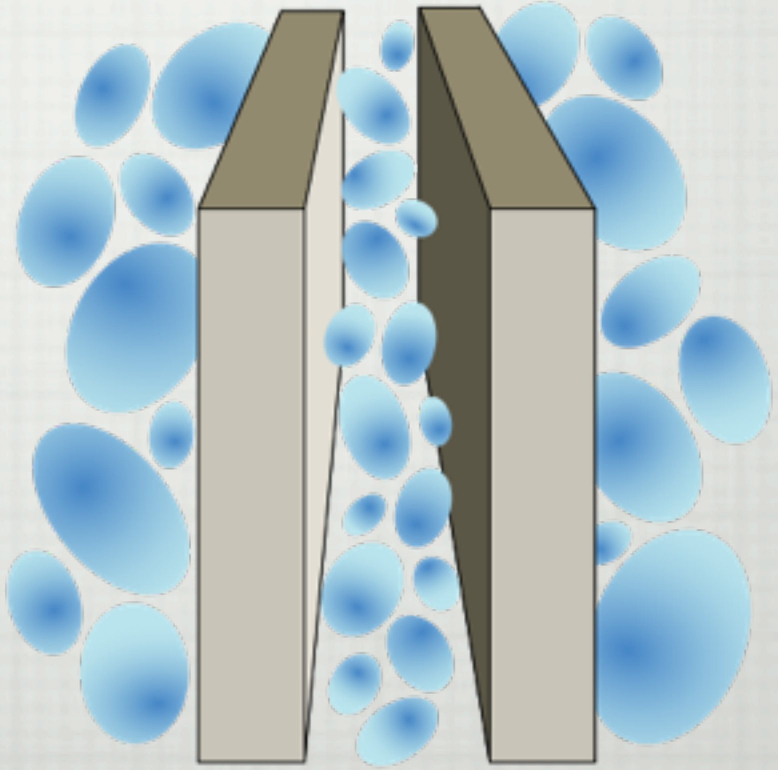
$$\frac{\langle E(s) \rangle}{A} = -\frac{\hbar c^{1-s} \pi^{2-s}}{2a^{3-s}} \frac{1}{3-s} \sum_n |n|^{3-s}.$$

$$\frac{\langle E \rangle}{A} = \lim_{s \to 0} \frac{\langle E(s) \rangle}{A} = -\frac{\hbar c \pi^2}{6a^3} \zeta(-3).$$

$$\frac{\langle E \rangle}{A} = \frac{-\hbar c \pi^2}{3 \cdot 240 a^3}.$$

$$\frac{F_c}{A} = -\frac{d}{da} \frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{240 a^4}$$





- Terrence Tao: *The Euler-Maclaurin formula, Bernoulli numbers, the zeta function, and real-variable analytic continuation*