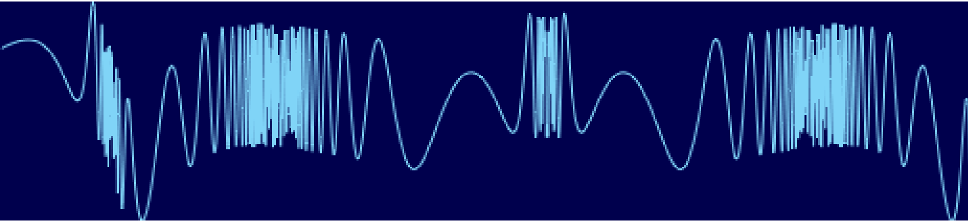

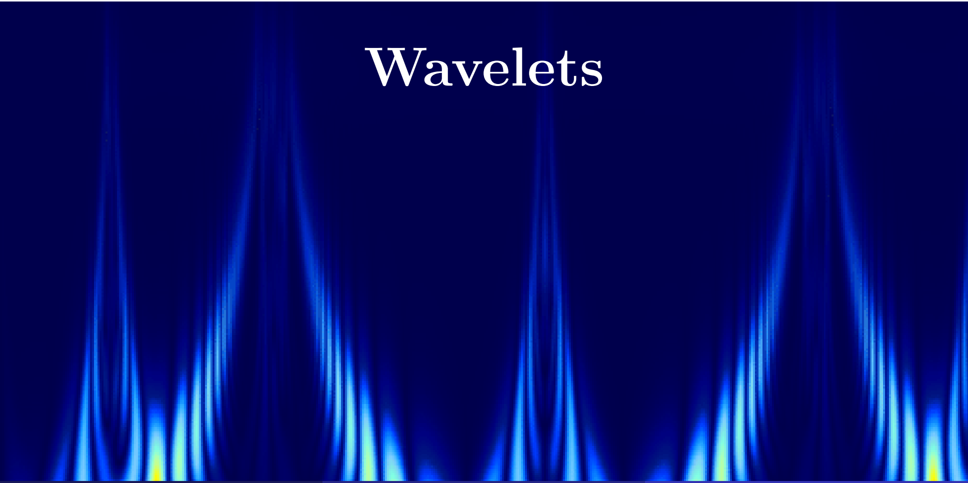

Wavelets

What are Wavelets?

Localised "signals" in space which, under translation and scaling, span $\mathbb{L}_2(\mathbb{R})$.

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A *Mother Wavelet* is a $\psi \in \mathbb{L}_2(\mathbb{R})$ with the property that the set:

$$\left\{\psi_{jk}(x) := 2^{j/2}\psi(2^j x - k) \middle|\; j,k \in \mathbb{Z}\right\}$$

constitutes an **orthonormal basis** of $\mathbb{L}_2(\mathbb{R})$.

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Quick description of Wavelet Transforms - via Multi-Resolution Analysis

Think of nested subspaces:

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Let V_1 be the subspace spanned by integer translations of $\varphi(2x)$:

$$\overline{span}\{\varphi_{1k}(x) := \sqrt{2}\varphi(2x-k) | k \in \mathbb{Z}\} = V_1.$$

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Multi-Resolution Analysis Equation

For a given set $h(n)$,

- existence of φ?
- uniqueness?

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ψ is our mother wavelet! It's a 'characteristic' function in the orthogonal complement.

Two equations, two filters:

$$\text{Multi. Res. equation: } \varphi(x) = \sum_n h(n)\sqrt{2}\varphi(2x - n)$$

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$$g(n) = (-1)^n h(N-1-n)$$

So given our hierarchy of embedded functions spaces:

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Wavelet Transforms

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Can construct $f(x)$:

$$f(x) = \sum_{k}^{N} c_{j_0}(k) 2^{j_0/2} \phi(2^{j_0} x - k) + \sum_{k}^{N} \sum_{j=j_0}^{\infty} d_j(k) 2^{j/2} \psi(2^j x - k)$$

Coefficients from orthogonality:

$$\begin{aligned}\langle f(x), \varphi(x-m)\rangle =& : c_0(m)\\ =&\langle f(x), \sum_{z\in\mathbb{Z}} h(z)\sqrt{2}\varphi(2x-2m-z)\rangle\end{aligned}$$

let $n = 2m + z$:

$$\begin{aligned}=&\sum_{n\in\mathbb{Z}} h(n-2m)\langle f(x), \varphi_{1,n}\rangle\\ c_0(m) =& \sum_{n} h(n-2m)c_1(n)\end{aligned}$$

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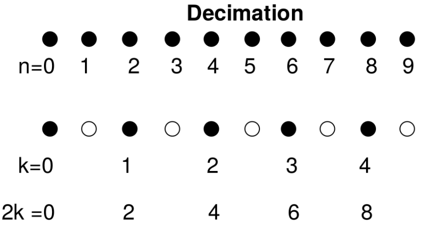
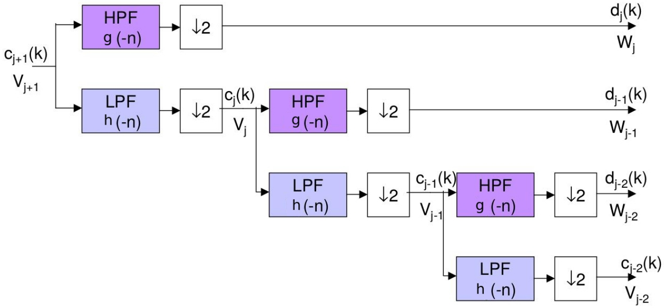
$$c_0(m) = \sum_{n} h(n-2m)c_1(n)$$

So we can iteratively obtain lower resolution coefficients. . .

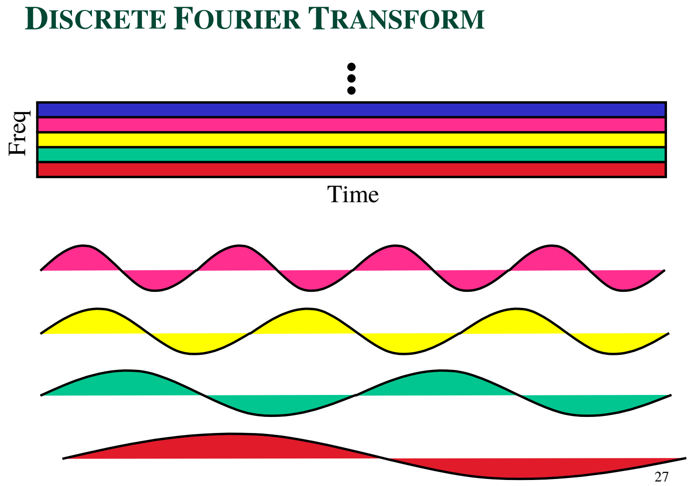
Convolve & Cascade

$$y_0(n) = c_{j_{max}}(n) * h(-n)$$
$$= \sum_m h(m-n) c_{j_{max}}(m)$$

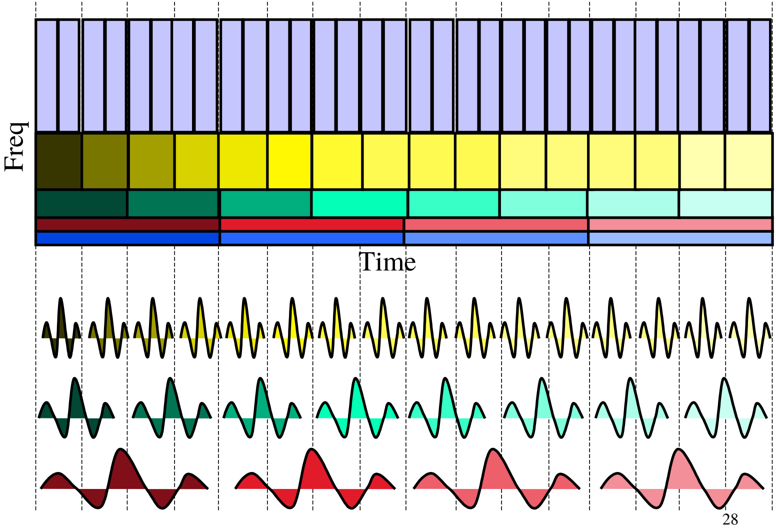




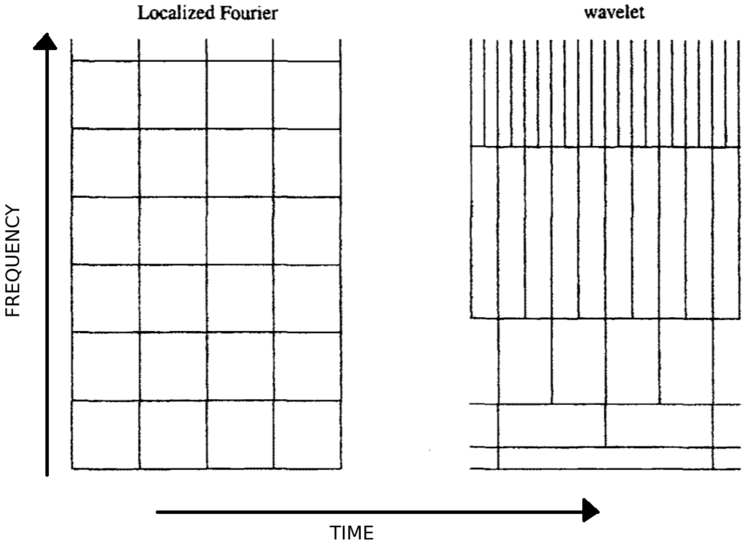
Time-Frequency trade-off



Wavelet Transform



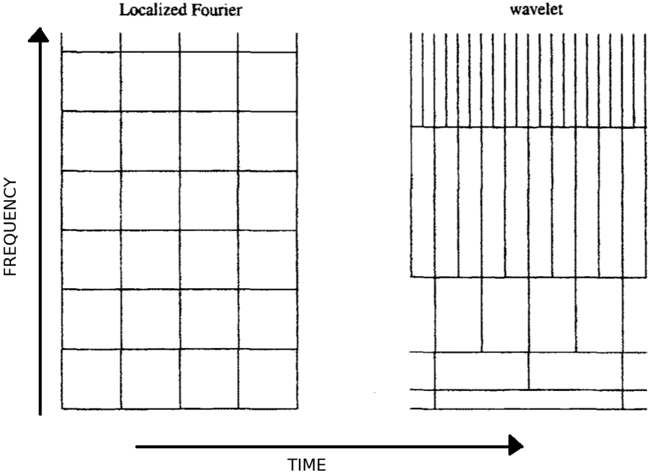
src: http://www.ws.binghamton.edu/fowler/fowler%20personal%20page/EE523_files/Wavelet_Charts.pdf



src: Statistical modeling by wavelets, Brani Vidakovic, Wiley & Sons, 1999

Uncertainty Principle

"Area of time-frequency tile is bounded from below…"



Wavelets naturally trade this off: different levels different $\frac{\text{Time resi}}{\text{freq res}}$…

Wavelet theory well developed, includes…

- Parameterised ways to “generate” wavelet bases with desired smoothness / orthogonality / fractal structure (see Daubechies wavelets…)

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Wavelet Networks

Qinghua Zhang and Albert Benveniste, *Fellow, IEEE*

***Abstract*— Based on the wavelet transform theory, the new notion of wavelet network is proposed as an alternative to feedforward neural networks for approximating arbitrary nonlinear functions. An algorithm of backpropagation type is proposed for wavelet network training and experimental results are reported.**

I. Introduction

THE approximation of general continuous functions by nonlinear networks such as discussed in [1], [2] is very useful for system modeling and identification. Such approximation methods can be used, for example, in black-box

A. Neural Networks

Fig. 1 depicts a so-called $(1 + \frac{1}{2})$—layer neural network. Recently, the ability of such neural networks to approximate continuous functions has been widely studied [3], [5]–[7]. In particular, the following result has been proved in [3]:

If $\sigma(\cdot)$ is a continuous discriminatory function[1], then finite sums of the form

$$g(\boldsymbol{x}) = \sum_{i=1}^{N} w_i \sigma(\boldsymbol{a}_i^T \boldsymbol{x} + b_i) \tag{1}$$