The Khintchine Constant and Friends

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Gatsby Unit, Tea Talk
September 18, 2015

A few days ago

$$\left\| k - \hat{k} \right\|_{L^{s}(\mathcal{D})} \leq ?$$

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$$\left\|k-\hat{k}\right\|_{L^s(\mathcal{D})} \le \; ? \xrightarrow{\text{after a bit of formula manipulation}}$$

$$\mathbb{E}_{\boldsymbol{\epsilon}} \left\| \sum_{i=1}^{n} \epsilon_i f_i \right\| \le B_s \left(\sum_{i=1}^{n} \|f_i\|^p \right)^{\frac{1}{p}},$$

where

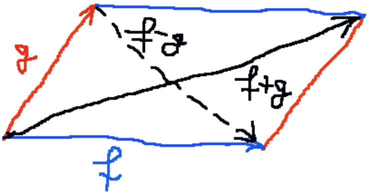
- $\boldsymbol{\epsilon}$: Rademacher sequence, $\mathbb{P}(\epsilon_i = \pm 1) = 0.5$, i.i.d.
- $\|\cdot\| = \|\cdot\|_{L^s(\mathcal{D})}$, $p = \min(s, 2)$, $f_i = \cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot \rangle)$.

Today

Khintchine constant

$$\mathbb{E}_\epsilon \left\| \sum_{i=1}^{n} \epsilon_i f_i \right\| \leq B \left(\sum_{i=1}^{n} \|f_i\|^p \right)^{\frac{1}{p}} .$$

Parallelogram rule



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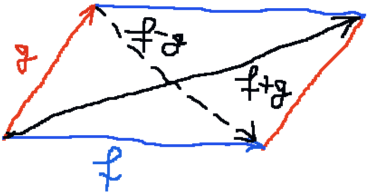

- Statement: $\|f+g\|_2^2 + \|f-g\|_2^2 = 2\left(\|f\|_2^2 + \|g\|_2^2\right)$.

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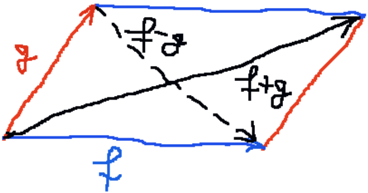

- Statement: $\|f+g\|_2^2 + \|f-g\|_2^2 = 2\left(\|f\|_2^2 + \|g\|_2^2\right)$. Indeed

$$\begin{aligned}\|f+g\|_2^2 + \|f-g\|_2^2 &= \langle f+g, f+g\rangle_2 + \langle f-g, f-g\rangle_2 \\ &= 2\left(\|f\|_2^2 + \|g\|_2^2\right) \pm 2\langle f, g\rangle_2 .\end{aligned}$$

- We only used: $\mathbb{R}^2$ is a normed space, $\|f\| = \sqrt{\langle f, f\rangle}$.

Example when the parallelogram rule fails

$X = C[0,1]$ with $\|h\|_\infty = \max_{y \in [0,1]} |h(y)|$:

$$f(y) := 1 - y, \; g(y) := y,$$

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$$2\left(\|f\|_\infty^2 + \|g\|_\infty^2\right) = 2\|\underbrace{1-y}_{\in[0,1]}\|_\infty^2 + 2\|\underbrace{y}_{\in[0,1]}\|_\infty^2 = 2+2 = 4.$$

Parallelogram rule $\Leftrightarrow$ inner product

Results: An X

- normed space is Euclidean $\Leftrightarrow$ parallelogram rule $(\forall f, g \in X)$.
- Banach space is Hilbert $\Leftrightarrow$ parallelogram rule $(\forall f, g \in X)$.

We are interested in Banach spaces; today in L^s.

Deviation from the parallelogram rule

Randomized signs in the parallelogram rule ($p = 2$):

$$\underset{\epsilon_i = \pm 1}{\text{average}} \left\| \sum_{i=1}^{n} \epsilon_i x_i \right\|^p$$

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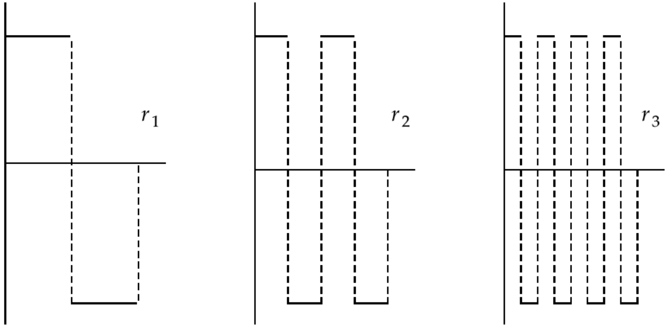
where $r_i(u) = sgn\left(\sin\left(2^i \pi u\right)\right) \in L^2[0,1]$, ONS,



Hilbert space: randomized parallelogram rule

In a Hilbert space:

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Result: X Banach space is Hilbert $\Leftrightarrow$ this rule holds $(\forall n, \{x_i\}_{i=1}^n \subset X)$.

Type-, cotype definition: Hilbert space $\Leftrightarrow p = q = 2$

X Banach space is of

1. type p if $\forall n, \forall \{x_i\}_{i=1}^n \subset X$:

$$\sqrt{\mathbb{E}_\epsilon \left\| \sum_{i=1}^n \epsilon_i x_i \right\|^2} = \sqrt{\int_0^1 \left\| \sum_{i=1}^n r_i(u) x_i \right\|^2 \mathrm{d}u} \leq B \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}.$$

2. cotype q if $\forall n, \forall \{x_i\}_{i=1}^n \subset X$:

$$\sqrt{\mathbb{E}_\epsilon \left\| \sum_{i=1}^n \epsilon_i x_i \right\|^2} = \sqrt{\int_0^1 \left\| \sum_{i=1}^n r_i(u) x_i \right\|^2 \mathrm{d}u} \geq A \left(\sum_{i=1}^n \|x_i\|^q \right)^{\frac{1}{q}}.$$

Relevant intervals: $p \in [1, 2]$, $q \in [2, \infty]$.

Classical Khintchine inequality: $X = \mathbb{R}$

For $\forall s \in (0, \infty)$, $\exists A_s > 0, B_s > 0$ s.t. $\forall \{x_i\}_{i=1}^n \subset \mathbb{R}$

$$A_s \left\|\mathbf{x}\right\|_2 \leq \left(\mathbb{E}_\epsilon \left| \sum_{i=1}^n \epsilon_i x_i \right|^s \right)^{\frac{1}{s}} \leq B_s \left\|\mathbf{x}\right\|_2 .$$

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Interpretation:

- $\mathbb{R}$ is of type 2, cotype 2 (simplest Hilbert space). $s \neq 2$ (too).
- $\left(\mathbb{E}_\epsilon \left|\sum_{i=1}^n \epsilon_i x_i\right|^s\right)^{\frac{1}{s}} = \left(\int_0^1 \left|\sum_{i=1}^n r_i(u) x_i\right|^s \mathrm{d}u\right)^{\frac{1}{s}} = \left\|\sum_{i=1}^n x_i r_i\right\|_{L^s[0,1]}$, i.e.
- $(r_i) \subset L^s[0,1] \Leftrightarrow (e_i) \subset \ell^2$ basis.
- A_s, B_s: Khintchine constants.

The exponent is irrelevant in the type/cotype definition

$\xrightarrow{s\text{-conjecture holds generally}}$ Kahane theorem: For $\forall s \in (1, \infty)$ $\exists K_s$ s.t. for *every* Banach space X, $\forall n, \{x_i\}_{i=1}^n \subset X$:

$$\mathbb{E}_\epsilon \left\| \sum_{i=1}^n \epsilon_i x_i \right\| \leq \left(\mathbb{E}_\epsilon \left\| \sum_{i=1}^n \epsilon_i x_i \right\|^s \right)^{\frac{1}{s}} \leq K_s \mathbb{E}_\epsilon \left\| \sum_{i=1}^n \epsilon_i x_i \right\|.$$

Note (proof $\Rightarrow$): $K_s = \left(\frac{2s-1}{s-1} \right)^{s-1}$ is good.

$X = L^s(Z, \mathcal{A}, \mu)$, $s \in [1, \infty)$: $p = \min(s, 2)$, $q = \max(2, s)$

$$\mathbb{E}_\varepsilon \left\| \sum_i \epsilon_i x_i \right\|_{L^s}^s$$

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$$(*)^{\frac{2}{s}} \overset{(e)}{\leq} \sum_i \left(\int_Z |x_i(z)|^s \mathrm{d}\mu(z)\right)^{\frac{2}{s}} = \sum_i \|x_i\|_{L^s}^2.$$

(a), (d): $\|\cdot\|_{L^s}$ def., $\int \leftrightarrow \sum$, (b) [$\mathbb{R}$-Khintchine]^s, (c): if $s\leq 2$ $\|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_s$, (e): if $s\geq 2$, triangle ineq. to $z\mapsto \sum_i |x_i(z)|^2$.

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$$B_s = \begin{cases} 1 & s \in (1,2], \\ \sqrt{2}\left[\frac{\Gamma\left(\frac{s+1}{2}\right)}{\sqrt{\pi}}\right]^{\frac{1}{s}} & s \in (2,\infty). \end{cases}$$

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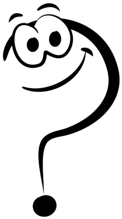
- B_s order $(s \to \infty)$: $\Gamma \sim !$ $\xrightarrow{\text{Stirling formula}}$ $B_s \le \mathcal{O}\left(\sqrt{s}\right)$.

Summary

- L^s guarantees: empirical processes, concentration, type.
- Type:
  - analytical formula for L^s.
- Classical Khintchine constant ($X = \mathbb{R}$):
  - It bounds the L^s-constant.
  - Its order & optimal value are known.

Thank you for the attention!



Contents

- Relevant (co)type intervals.
- L^s: type-cotype, $X - X^*$: type-cotype.
- Kahane theorem: l.h.s.
- Some additional (co)type properties.
- Optimal A_s.

Relevant (co)type intervals

Let $x_i = x$ $(\forall i)$, where $\|x\| = 1$. Then $(s = 1)$

$$\int_0^1 \left\| \sum_i r_i(u) x_i \right\| \mathrm{d}u \overset{\|x\|=1}{=} \int_0^1 \left| \sum_i r_i(u) \right| \mathrm{d}u =: (*),$$

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L^s: type-cotype relation

- $L^s(Z, \mathcal{A}, \mu)$: type $p = \min(s, 2)$. $L^{s^*}(Z, \mathcal{A}, \mu)$ $(\frac{1}{s} + \frac{1}{s^*} = 1)$: cotype $q = \max(2, s^*)$. Observation:

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1}$$

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- More generally, if X is of type $p \Rightarrow X^*$ is of cotype q satisfying (1).
- Note (converse is not true): ℓ^1 of cotype 2, $\left(\ell^1\right)^*=\ell^\infty$ is not of type $p\geq 1$.

X: type $p \Rightarrow X^*$: cotype q such that $1/p + 1/q = 1$

For $\forall \epsilon > 0$ and $\forall \{x_i^*\}_{i=1}^n \subset X^*$ $\exists \{x_i\}_{i=1}^n \subset X$, $\|x_i\| = 1$: $\|x_i^*\| < (1+\epsilon) x_i^*(x_i)$.

$$\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{\frac{1}{q}} \leq (1+\epsilon)\left[\sum_i x_i^*(x_i)^q\right]^{\frac{1}{q}} = (1+\epsilon)\left\|[x_i^*(x_i)]\right\|_q,$$

$$\left\|[x_i^*(x_i)]\right\|_q \overset{(a)}{=} \sup_{\|\mathbf{a}\|_p \leq 1} \Big\{ \underbrace{\sum_i a_i x_i^*(x_i)}_{\overset{(b)}{=} \int_0^1 \left[\sum_i r_i(u) x_i^*\right]\left[\sum_j r_j(u) a_j x_j\right] \mathrm{d}u = (*)} \Big\},$$

$$(*) \overset{(c)}{\leq} \left(\int_0^1 \left\|\sum_i r_i(u) x_i^*\right\|^q \mathrm{d}u\right)^{\frac{1}{q}} \left(\int_0^1 \left\|\sum_j r_j(u) a_j x_j\right\|^p \mathrm{d}u\right)^{\frac{1}{p}}$$

(a): dual of $\|\cdot\|_p$, (b): (r_i): ONS, (c): Hölder inequality.

X: type $p \Rightarrow X^*$: cotype q

The remaining term:

$$\left(\int_0^1 \left\| \sum_j r_j(u) a_j x_j \right\|^p \mathrm{d}u\right)^{\frac{1}{p}} \overset{(a)}{\leq} K_p \underbrace{\int_0^1 \left\| \sum_j r_j(u) a_j x_j \right\| \mathrm{d}u}$$

$$\overset{(b)}{\leq} A_p \left(\sum_j \left\|a_j x_j\right\|^p\right)^{\frac{1}{p}} \overset{(c)}{=} A_p \underbrace{\left\|\mathbf{a}\right\|_p}_{\leq 1}.$$

(a): exponent is irrelevant (Kahane-T.), (b): X is of type p, (c): $\|a_j x_j\| = |a_j| \, \|x_j\| = |a_j|$ ($\|x_j\| = 1$). At the end: $\epsilon \to 0$.

Kahane theorem: l.h.s.

$$\mathbb{E}_\epsilon \left\|\sum_i \epsilon_i x_i\right\| = \int_0^1 \left\|\sum_i r_i(u) x_i\right\| \mathrm{d}u = \left\|\sum_i r_i(u) x_i\right\|_{L^1([0,1];B)},$$

$$\left(\mathbb{E}_\epsilon \left\|\sum_i \epsilon_i x_i\right\|^p\right)^{\frac{1}{p}} = \left(\int_0^1 \left\|\sum_i r_i(u) x_i\right\|^p \mathrm{d}u\right)^{\frac{1}{p}} = \left\|\sum_i r_i(u) x_i\right\|_{L^p([0,1];B)},$$

$$1 \leq a \leq b \leq \infty \Rightarrow \|f\|_{L^a(Z,\mu;B)} \leq \|f\|_{L^b(Z,\mu;B)}, \text{ if } \mu(Z) = 1.$$

Proof: $a := 1 \leq b := p$, $\lambda([0,1]) = 1$ gives the result.

Further (co)type properties - 1

- By triangle inequality & $|r_i(u)| = 1$: always
  - Type $p = 1$: $\left\| \sum_i r_i(u) x_i \right\| \le \sum_i \|r_i(u) x_i\| = \sum_i \|x_i\|$.
  - Cotype $q = \infty$: $\left\| \sum_i r_i(u) x_i \right\| \ge \|r_j(u) x_j\| = \|x_j\| \ (\forall j)$.
- ℓ^1 is of no type $p > 1$.
- ℓ^∞, c_0: is of no cotype $q < \infty$.

Further (co)type properties - 2

- X is of type p (cotype q) $\Rightarrow$
  - X is of type $p' \leq p$ (cotype $q' \geq q$).
  - all its subspaces are so.
  - quotients are of type p (with the same constant).
- Y: Banach of type p_Y, cotype $q_Y \Rightarrow L^s(Z, \mathcal{A}, \mu; Y)$ is of type $\min(s, p_Y)$, $\max(s, q_Y)$.
- L^∞ is of type 1 and cotype ∞ ($r \to \infty$: valid for cotype).

Stirling formula

Order estimation for $n!$:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}.$$

Optimal A_s

$$A_s = \begin{cases} 2^{\frac{1}{2}-\frac{1}{s}} & s \in (0, s_0], \\ \sqrt{2}\left[\frac{\Gamma\left(\frac{s+1}{2}\right)}{\sqrt{\pi}}\right]^{\frac{1}{s}} & s \in (s_0, 2), \\ 1 & s \in [2, \infty), \end{cases}$$

where s_0 is the solution of $\Gamma\left(\frac{s+1}{2}\right) = \frac{\sqrt{\pi}}{2}$ on $s \in (1, 2)$,
$s_0 \approx 1.84742$.