Some Counterexamples in Probability

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Gatsby Tea Talk
3 Feb 2017

1. Correlation & independence

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- Let $X \sim \mathcal{N}(0, 1)$.
- Let $Y := X^2$.
- Then, $\mathrm{cov}(X, Y) = 0$:

$$\begin{aligned} \mathrm{cov}(X, Y) &= \mathbb{E}[XY] - \overbrace{\mathbb{E}[X]}^{0}\,\mathbb{E}[Y] \\ &= \mathbb{E}[XX^2] \\ &= 0 \text{ (a Gaussian has 0 skewness).} \end{aligned}$$

- X, Y are clearly dependent.

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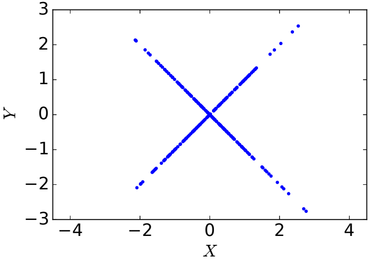
Will construct a counterexample such that

- X and Y are marginally Gaussian (not jointly).
- $\mathrm{cov}(X, Y) = 0$.
- But, $X \not\perp Y$.

Counterexample

- Let $X \sim \mathcal{N}(0, 1)$.
- Let $W \in \{-1, 1\}$ s.t. $P(W = 1) = P(W = -1) = 0.5$.
- Let $Y := WX$. Clearly, $X \not\perp Y$.



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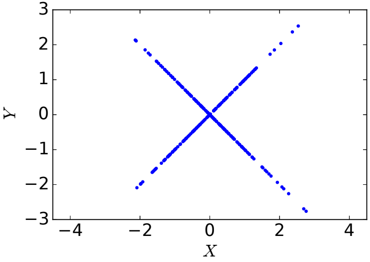

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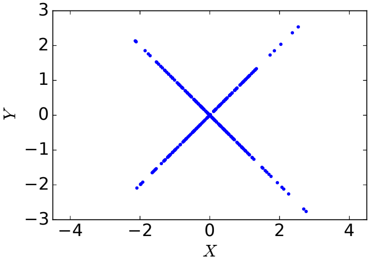

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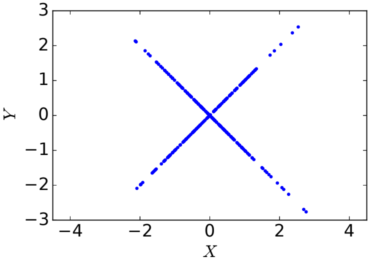

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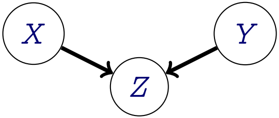
Summary: If X, Y are only marginally Gaussian, and $\mathrm{cov}(X,Y) = 0$, then X and Y are not necessarily independent.

3. A three-variable graph

Consider  .

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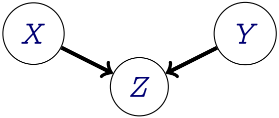
Consider 

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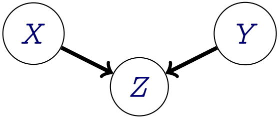
Consider  .

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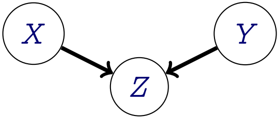
Consider  .

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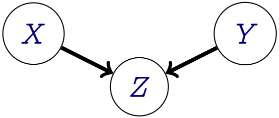
Consider  .

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1. $X \perp Y$, and $Z \not\perp (X, Y)$ and $X \not\perp Y \mid Z$, by the semantics of the graph.
2. $Z \perp X$ and $Z \perp Y$? **Yes**

- $X, Y \in \{-1, 1\}$ (i.i.d.) with probability 0.5 i.e., Rademacher variables.
- $Z := XY$.
-

	$Y = -1$	$Y = 1$
$X = -1$	$Z = 1$	$Z = -1$
$X = 1$	$Z = -1$	$Z = 1$

- Knowing that $X = 1$ does not say anything about Z.

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Reason:

- If $f, g : \mathbb{R} \to \mathbb{R}$ are one-to-one, then $f(X) \perp g(Y) \implies X \perp Y$.
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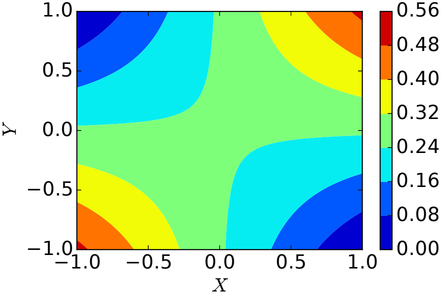
Counterexample:

- Let $X, Y \in (-1, 1)$.
- Consider the joint density

$$p(x, y) = (1 + xy)/4.$$



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$$\begin{aligned} p(x) &= \frac{1}{4}\int_{-1}^{1}(1 + xy)\,\mathrm{d}y \\ &= \frac{1}{4}[y]_{-1}^{1} + \frac{x}{4}\left[\frac{y^2}{2}\right]_{-1}^{1} \\ &= \frac{1}{2} + \frac{x}{4}\left(\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2} = p(y). \end{aligned}$$

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- Clearly, $p(x,y) \neq p(x)p(y)$. So, $X \not\perp Y$.
- To see that $X^2 \perp Y^2$, consider the joint CDF:

$$P(X^2 < a, Y^2 < b) = P\left(-\sqrt{a} < X < \sqrt{a}, -\sqrt{b} < Y < \sqrt{b}\right)$$
$$= \frac{1}{4}\int_{-\sqrt{a}}^{\sqrt{a}}\int_{-\sqrt{b}}^{\sqrt{b}}(1+xy)\,\mathrm{d}x\mathrm{d}y$$
$$= \sqrt{a}\sqrt{b}$$
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- $C : 9, 9, 3, 3, 3, 3$

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- $D : 8, 8, 8, 2, 2, 2$

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- $D : 8, 8, 8, 2, 2, 2$
- **Extra**: All dice have an expected value of 5.
- So, summarizing a random quantity with its mean is not always good.

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- So, $\mathbb{E}[Y \,|\, X] = \mathbb{E}[Y]$ and $X \not\perp Y$.

References

- **Counterexamples in Probability**: Third Edition (Dover Books on Mathematics) by Jordan Stoyanov
- https://en.wikipedia.org/wiki/Normally_distributed_and_uncorrela
- https://en.wikipedia.org/wiki/Nontransitive_dice

Questions?

Thank you