Rubik's on the Torus

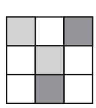
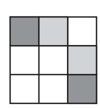
Jeremy Alm, Michael Gramelspacher, and Theodore Rice

Abstract. Rubik's Slide is an electronic puzzle in the same style as other puzzles in the Rubik's family. It will be of interest to puzzle enthusiasts as well as to teachers and students of mathematics. This paper analyzes the puzzle using tools from group theory and graph theory. We discuss solution heuristics and model game-play using group theory. We briefly give results concerning solutions requiring a minimal number of moves.

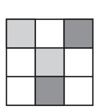
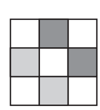
1. INTRODUCTION. Rubik's Slide[1] is a new electronic game that will be of interest to many mathematicians and teachers of mathematics. The flat display is a 3×3 grid of square lights that is reminiscent of both the original Rubik's Cube and the classic 15-slide-number puzzle. Rubik's Slide is much more accessible than the Rubik's Cube and is much richer than the 15-puzzle. Indeed, this puzzle lends itself to analysis using group-theoretic and graph-theoretic techniques which should be accessible to students who have taken a semester of abstract algebra; their instructors might find useful ideas here—even a vehicle, in the Rubik's Slide, for introducing group theory in the first place. (The interested reader may wish to consult David Joyner's *Adventures in Group Theory* [**4**] for exposition on the Rubik's Cube and the 15-puzzle.)

The goal of the game is for the player to transform a given initial state into a given final state using a set of allowed moves. At the beginning of play, a puzzle consisting of an initial state and final state is generated randomly. The player is shown the initial state on the display: some of the squares are illuminated in red or blue. The press of a button displays the final state. The number of squares of each color varies with each particular puzzle, but the total number of illuminated squares will not exceed six.

The allowed moves are a shift by one space, either up, down, left, or right, and a rotation of the border squares by one space, either clockwise or counterclockwise. The 3×3 grid is topologically equivalent to a torus, so when the squares shift to the right, the rightmost column wraps around to the left, and so on. Note that these moves are quite different from the moves on the Rubik's Cube; on the Rubik's Slide, there is but *one* face. To make a move, the player manipulates the display relative to the base of the device. For example, for a clockwise rotation, the player twists the display clockwise against the base. Let us consider an illustration. Applying a shift of one space to the right,

 becomes  ;

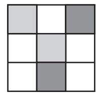
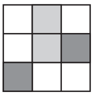
applying a shift of one space down,

 becomes  ;

http://dx.doi.org/10.4169/amer.math.monthly.120.02.150
MSC: Primary 00A08, Secondary 20B99; 05C90

[1] www.rubiksslide.com. There is a tutorial available on the web site.

and applying a rotation of one space clockwise,

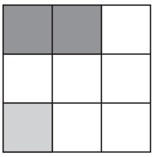
 becomes  .

Players can choose from three levels of difficulty: easy, medium, and hard. What distinguishes the easy mode from the others is that rotation is by *two* spaces, not one—one “twist” of the display rotates the grid of squares by 90°. This significantly reduces the difficulty of puzzles. As we will show in Section 3, puzzles in the easy mode really are much easier to solve than those in the other modes.

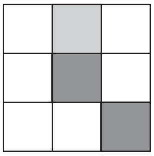
In the medium mode, only one color—sometimes red, sometimes blue—is used in any given puzzle. In the hard mode, both red and blue squares may appear in a puzzle, making that mode’s puzzles potentially more challenging. In Sections 5 and 6, we give strategies for solving all medium-mode and some hard-mode puzzles.

The reader may wish to try a sample (hard-mode) puzzle to experiment with the game. A solution is given at the end of the paper.

Initial State



Final State



One advantage of the Rubik’s Slide is that it is electronic—while there are countless Rubik’s Cubes stored away in boxes, hopelessly scrambled and never to be solved by their owners, Rubik’s Slide can simply be reset by a frustrated player. Also, since the display is two-dimensional, it can be represented easily on a chalkboard. The sizes of the various state spaces are manageable, and the moves can be represented as permutations in S_9, the symmetric group on nine letters. The third author has introduced some of these concepts in his classes, including classes for non-majors, and his students have responded with enthusiasm. We hope that the Rubik’s Slide will enliven abstract algebra classes everywhere.

2. SET-UP. We begin by defining permutations in S_9, using cycle notation, that represent the six allowed moves in the game. Let ε denote the identity permutation. We label the squares in the 3×3 grid as follows:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&6\\ \hline 7&8&9\\ \hline\end{array}\,.$$

We define h (for *horizontal*) to be the permutation $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$ in S_9. Clearly, h shifts each of the numbered squares one space to the right. The action of this move on the numbered squares is depicted below:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&6\\ \hline 7&8&9\\ \hline\end{array} \overset{h}{\longmapsto} \begin{array}{|c|c|c|}\hline 3&1&2\\ \hline 6&4&5\\ \hline 9&7&8\\ \hline\end{array}\,.$$

Similarly, we define v to be the permutation $(1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \xmapsto{h} \begin{array}{|c|c|c|}\hline 7 & 8 & 9 \\ \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}.$$

Finally, we define c to be $(1\ 2\ 3\ 6\ 9\ 8\ 7\ 4)$, a clockwise rotation of the outer squares.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \xmapsto{h} \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 7 & 5 & 3 \\ \hline 8 & 9 & 6 \\ \hline \end{array}.$$

Throughout the rest of this paper, we will use the term *move* to refer to any of h, h^{-1}, v, v^{-1}, c, c^{-1}. (Naturally, when discussing easy-mode play, the term *move* will refer to any of h, h^{-1}, v, v^{-1}, c^2, c^{-2}.) An element of S_9 that is a product of these, such as $h^{-1}vc^3$, will be called a *sequence of moves*.

For ease of reading, we adopt the convention of multiplying permutations from left to right, though this differs from the practice found in many standard texts, such as [**2**].

We focus for now on the medium and hard modes of the game. A natural question arises: Which subgroup of S_9 is generated by the moves $h,\ v$, and c? We begin with some lemmas.

Lemma 1. *The subgroup $\langle h, v\rangle$ generated by h and v is Abelian and of order 9.*

Proof. By inspection, $hv = (1\ 5\ 9)(2\ 6\ 7)(3\ 4\ 8) = vh$. Since $h^3 = v^3 = \varepsilon$, $\langle h, v\rangle = \{h^i v^j : i, j = 0, 1, 2\}$. A quick check of cycle structures reveals that all such $h^i v^j$ are distinct. ∎

A second lemma may be proven computationally.

Lemma 2. $c^3h = (1\ 4)(2\ 7\ 3\ 9\ 5\ 6\ 8)$ *and* $(c^3h)^7 = (1\ 4)$.

Theorem 3. *The group $\langle h, v, c\rangle$ contains every transposition in S_9; hence $\langle h, v, c\rangle = S_9$.*

Proof. To transpose two squares, x and y, repeatedly apply h or v to move x to Position 5; call this sequence of moves σ. Then use an appropriate power of c to rotate y around to Position 2; call this sequence of moves τ. Now apply h^{-1}; this places x in Position 4 and y in Position 1. Swap x and y by applying $(c^3h)^7$, then apply $h\tau^{-1}\sigma^{-1}$. The resulting product of moves, $\sigma\tau h^{-1}(c^3h)^7h\tau^{-1}\sigma^{-1}$, equals the transposition $(x\ y)$. ∎

Thus, every medium- or hard-mode puzzle is solvable, even in the hypothetical situation in which every square is a different color. In fact we have this theorem, which follows from the easily-verifiable fact that $v = c^2h^2c^{-2}$.

Theorem 4. *Any puzzle may be solved using only c and h and their inverses.*

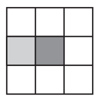
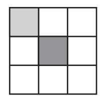
This reminds the authors of the fact that the standard $3 \times 3 \times 3$ Rubik's Cube may be solved without using the bottom face. See David Benson's nice proof of this fact on page 768 in [**1**].

In the following sections, we focus more explicitly on puzzles in the game's easy, medium, and hard modes.

3. THE EASY MODE. Recall that the easy-mode puzzle moves are c^2, h, v and their inverses.

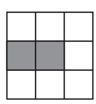
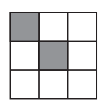
Definition 5. Two puzzle states are said to be *easy-equivalent* if they are reachable from one another by a sequence of moves from $\langle c^2, h, v\rangle$. The equivalence classes resulting from this equivalence relation are called *easy-mode families*.

In the easy mode, there are several varieties of puzzles.[2] The first puzzle that comes up when the easy mode is activated usually contains a single colored square in its initial (and hence in its final) state. This is trivial to solve, and can be done in two or fewer moves from $\{h^{\pm 1}, v^{\pm 1}\}$. A simple interesting case is when a puzzle's initial state contains exactly one red and one blue square. There are 72 such states, which comprise two easy-mode families, each of size 36. To prove that in this case there are at least two easy-mode families, we show that the states

 and 

are not easy-equivalent. To see this, recall that we can view the face as being on a torus so that, for instance, squares in Positions 2 and 8 share an edge. Any easy-mode move will preserve the property of two squares sharing an edge. In the state on the left, the red and blue squares share an edge; in the state on the right, they do not. It is tedious but straightforward to check that each of these two states lies in an easy-mode family of size 36. Hence there are exactly two easy-mode families in this case.

By the same reasoning, we can see that

 and 

are not easy-equivalent. These states, however, each lie in easy-mode families of size $\frac{36}{2} = 18$.

The following theorem, along with its proof, provides powerful tools for analysis of easy-mode puzzles.

Theorem 6. *No easy-mode family contains more than 36 states, and every easy-mode puzzle can be solved in four or fewer moves.*

Proof. Let $\sigma \in \langle c^2, h, v\rangle$. Then there are integers k, m, n with $-1 \le k \le 2$, $-1 \le m \le 1$, $-1 \le n \le 1$, so that $\sigma = (c^2)^k h^m v^n$. To see this, note that σ can be converted into the form $(c^2)^k h^m v^n$ using the following easily-verified relations, along with the fact that h and v commute:

$$\begin{aligned}
c^2 h &= v^2 c^2, \\
c^2 h^2 &= v c^2, \\
c^2 v &= h c^2, \\
c^2 v^2 &= h^2 c^2, \\
c^{-2} h &= v c^{-2},
\end{aligned}$$

[2] To the best of our knowledge, the Rubik's Slide never gives the user an unsolvable easy-mode puzzle.

$$
\begin{aligned}
c^{-2}h^2 &= v^2c^{-2}, \\
c^{-2}v &= h^2c^{-2}, \\
c^{-2}v^2 &= hc^{-2}.
\end{aligned}
$$

Hence, σ is equivalent to a sequence of moves that uses at most two rotations, at most one horizontal shift, and at most one vertical shift. Given any initial state, therefore, any state in its easy-mode family can be reached using one of the 36 sequences of moves of the form $(c^2)^k h^m v^n$. ■

Because the form $(c^2)^k h^m v^n$, $-1 \leq k \leq 2$, $-1 \leq m, n \leq 1$ is so useful, we will call it the *canonical form* of the permutation σ. The power of the canonical form will be illustrated in Section 4, where it will be used to describe the relationship between two states, without any need to specify how many red and blue squares are present.

4. THE STATE-SPACE GRAPH. We associate with each easy-mode family a graph, whose vertices are labeled with the states in the family, and in which two vertices are connected by an edge if one is obtained by applying a single legal move (c^2, h, v or one of their inverses) to the other. (The edges may be viewed as undirected, since every legal move is invertible.) We call this graph the *state-space graph* of the family.

Theorem 7. *Let G be the state-space graph for any easy-mode family of size 36. Then G admits a Hamilton cycle, that is, a cycle that contains each vertex of G exactly once.*

Proof. Choose any vertex V_0. For any other vertex V, there exists a sequence of legal moves that transform V_0 into V. This sequence of legal moves has a unique canonical form. Label V with that canonical form. (Thus V_0 gets the label ε.) Let k_1 and k_2 be integers, with $0 \leq k_1 < k_2 \leq 35$. It is tedious but straightforward to check that for the following word,

$$
\begin{aligned}
w = hc^{-2}c^{-2}h^{-1}c^2c^2c^2v^{-1}v^{-1}h^{-1}vvc^{-2}c^{-2}c^{-2}hc^2c^2 \\
h^{-1}c^{-2}c^{-2}vc^2h^{-1}c^{-2}c^{-2}vvc^{-2}c^{-2}c^{-2}hc^2c^2h^{-1}v,
\end{aligned}
$$

the product of the first k_1 terms has a different canonical form than the product of the first k_2. Let $w_0 = \varepsilon$, $w_1 = h$, $w_2 = hc^{-2}$, and so on up to $w_{36} = w$. For each i, w_i corresponds to a vertex in G, so we may consider the sequence $\{w_i\}$ as a sequence of vertices. Since each w_{i+1} is one legal move from w_i, those two vertices are adjacent. Thus the sequence $\{w_i\}_{i=0}^{35}$ is a path that visits each vertex once and only once. Since $w_0 = w_{36}$, $\{w_i\}_{i=0}^{36}$ is a Hamilton cycle. ■

For an illustration of the Hamilton cycle, see Figure 1, in which straight lines represent either h or h^{-1}, dotted lines represent either c^2 or c^{-2}, and wavy lines represent v or v^{-1}. The heads on the arrows represent one direction in which the cycle can be traversed.

The Hamilton cycle allows us readily to solve any puzzle in the easy mode by simply following the cycle until the final state is reached. If we had memorized the sequence of moves above, we could even solve any easy-mode puzzle blindfolded! We note that in the case of puzzles situated in smaller easy-mode families, while the sequence above may not give a Hamilton cycle, it will still visit each vertex at least once and hence will still solve any puzzle.

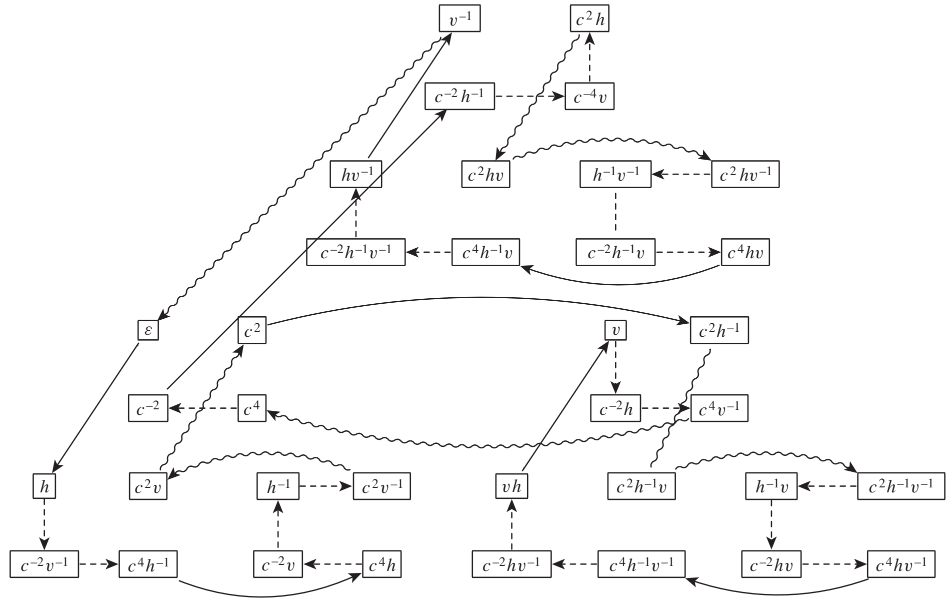


Figure 1. Hamilton cycle

5. THE MEDIUM MODE. Now we consider solutions to medium-mode puzzles. We remind the reader that in this mode only one color—either red or blue—is used in any given puzzle, but that we may use the move c (not just c^2, as in the easy mode).

If just one or two squares are colored, the resulting puzzles are not difficult to solve; the solutions to such puzzles are left to the reader! If three (respectively, four) squares are colored, consider all possible such states with the center square colored. Call any two such states equivalent if one can be reached from another by a power of c, i.e., by a rotation.

There are four such equivalence classes in the case of three colored squares, and seven equivalence classes in the case of four colored squares. We construct diagrams with representatives of these classes as vertices, and call them respectively the "Wheel of 4" and the "Wheel of 7." See Figure 2 and Figure 3.

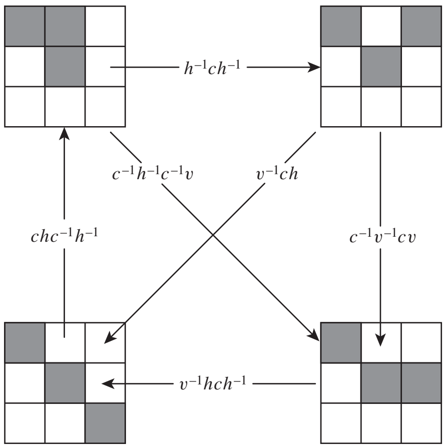


Figure 2. The Wheel of 4

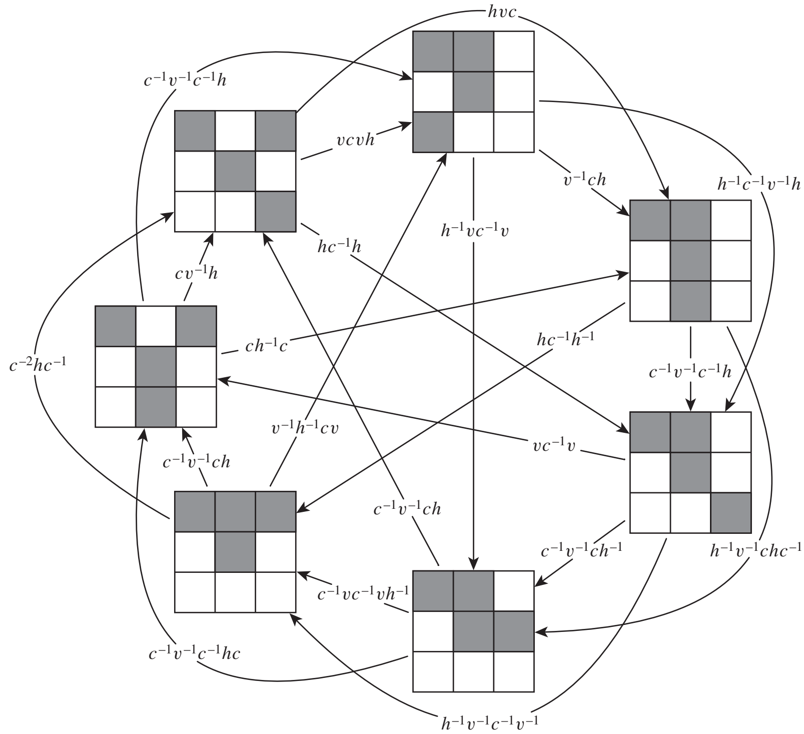


Figure 3. The Wheel of 7

The edges of the diagrams are labeled with sequences of moves that change one state to another. These diagrams will be crutches for us to lean on. Our strategy, then, is as follows:

Given an initial state S^0 and a final state S^ω, we want to find a path from S^0 to S^ω through the Wheel of 4 (respectively, Wheel of 7). We will label the intermediate states $S^1, S^2, \ldots, S^{\omega-1}$.

(1) If necessary, shift S^0 by h or v so that the middle square is occupied. (We call the resulting state S^1.)

(2) Use a power of c to match S^1 to a state on the Wheel. (We call the resulting state S^2.)

(3) On paper, recording moves, shift S^ω so that the middle square is occupied, if necessary. (We call the resulting state $S^{\omega-1}$.)

(4) On paper, recording moves, use a power of c to match $S^{\omega-1}$ to a state on the Wheel. (We call the resulting state $S^{\omega-2}$.)

(5) Use the Wheel to move S^2 to $S^{\omega-2}$.

(6) Move $S^{\omega-2}$ to S^ω by reversing all the steps recorded in Steps (3) and (4).

We illustrate our strategy in Figure 4. The reader should note that this solution is *not* unique. A different square could be placed at the center in Step (1) or Step (3), resulting in different Wheel states being attained in Step (4).

6. HARD MODE. Now we consider solutions to hard-mode puzzles. Again, we leave puzzles with one or two colored squares to the reader to solve. We also leave to the reader puzzles with four or more colored squares, since our “cheat-sheet” meth-

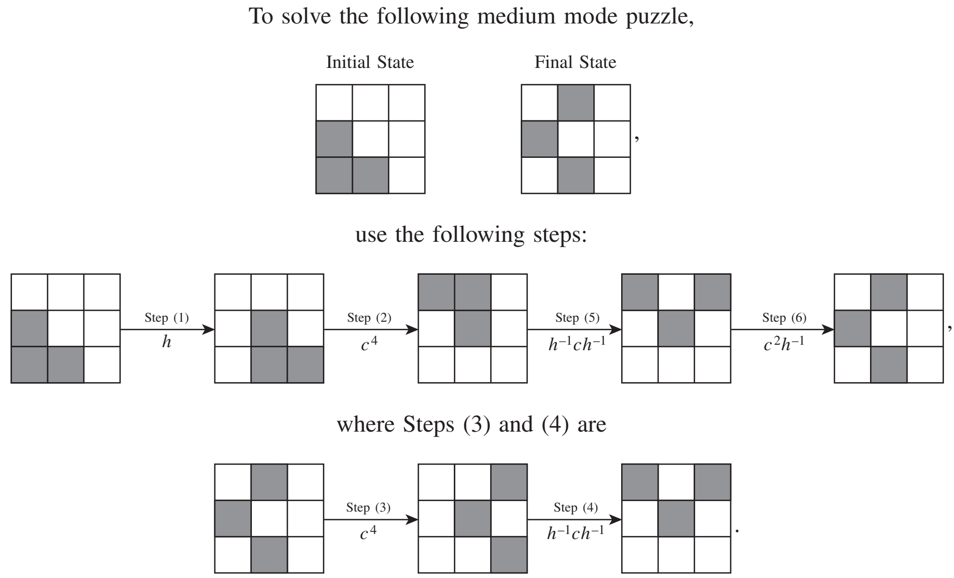


Figure 4. The strategy exemplified

ods would be too cumbersome in those cases. Theorem 3 does provide some help, however. (See the comments at the end of this section.)

Suppose that three squares are colored, two blue and one red. Our strategy is this: Pretend for the moment that we are color-blind, and "solve" the puzzle using the medium-mode strategy, ignoring the difference in colors. Then, shedding our color-blindness, we are in one of two situations. Either we got lucky, and the puzzle is solved, or the red square needs to be switched with one of the blue squares. In the latter case, we switch the red and blue squares as follows.

1. Using h and v, shift the blue square that does *not* need to be switched with red into the center, recording moves on paper.
2. Recording moves, rotate using c until the current state matches one in what we've called the *hard-mode chart* (see Figure 5).
3. Execute the sequence of moves indicated in the hard-mode chart.
4. Reverse all the moves done in Steps (1) and (2).

Some final comments are in order. It may be that some players find that these strategies "take the fun out" of playing. We do not disagree. Such players are encouraged to play without using the strategies; when one gets stuck, one can always return to the strategies for help. We wish to note that all three authors have frequently encountered the situation that, when playing without resorting to the strategies, the puzzle was almost solved, save that a red square needed to be swapped with a blue square. The proof of Theorem 3 is in essence an algorithm for swapping two squares while leaving all others fixed, and is quite handy in such situations. The use of this algorithm also gives us an alternative to using the hard-mode chart, though it would require using a much larger number of moves.

7. MINIMAL-MOVE SOLUTIONS. Given a puzzle P, there must be some solution of P requiring a minimal number of moves; denote this minimal number by $\mu(P)$. We call $\mu(P)$ the *length* of P. It is easy to see that any medium-mode puzzle having

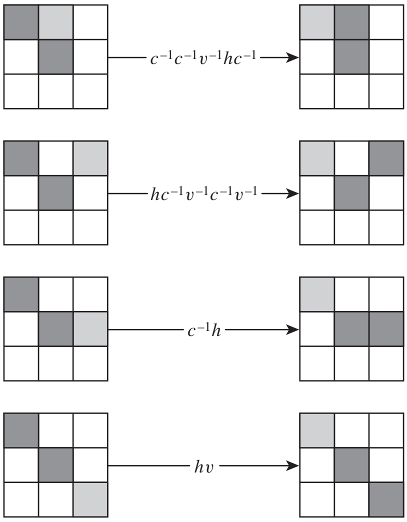


Figure 5. Hard-mode chart

just one colored square can be solved in two or fewer moves. Having two colored squares in the medium mode, it turns out that all such puzzles can solved in five or fewer moves; furthermore, there are puzzles that require five moves. In Table 1, we give all such color combinations for medium- and hard-mode puzzles and the corresponding maximum value for $\mu(P)$ as P ranges over all such puzzles.

Table 1. Maxima for $\mu(P)$

Color combination	max $\mu(P)$
(1 Red, 0 Blue)	2
(1 Red, 1 Blue)	5
(2 Red, 0 Blue)	4
(2 Red, 1 Blue)	6
(2 Red, 2 Blue)	8
(3 Red, 0 Blue)	5
(3 Red, 1 Blue)	7
(3 Red, 2 Blue)	9
(3 Red, 3 Blue)	9
(4 Red, 0 Blue)	7
(4 Red, 1 Blue)	8

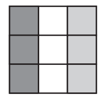
The data in Table 1 were generated using the GRAPE package [**6**] for GAP [**3**] and creating a medium- or hard-mode state-space graph, where the edges are labeled c, h, or v.

These data remind the authors of the fact that the Rubik's Cube can be solved in twenty moves, regardless of its initial state.[3] In [**5**], Tomas Rokicki discusses an algorithm that solves the Rubik's Cube in twenty-two or fewer moves, regardless of its initial state. The reader is invited to look for a natural algorithm for solving Rubik's Slide puzzles in a minimal number of moves.

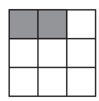
[3] This result is published online at http://www.cube20.org, but it has yet to appear in a peer-reviewed journal.

8. PROBLEMS AND QUESTIONS. We conclude with a list of problems that could be given to undergraduates. The first three might be appropriate as exercises in an abstract algebra class; the final three are more exploratory and might be more appropriate for an undergraduate research project. We offer these merely as a starting point, and encourage the reader to invent some questions of her own.

1. Given a state S, such as

   ,

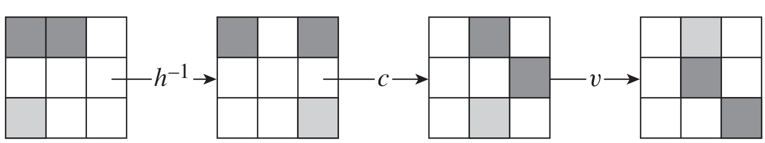
   that is invariant under v, show that its easy-mode family has size at most 12.
2. Prove that the size of any easy-mode family must be a divisor of 36.
3. Design an easy-mode puzzle that is solvable in four moves but not in three, and prove this is the case.
4. Do any of the smaller easy-mode families admit Hamilton cycles?
5. The state

   lies in a medium-mode family of size 36. Does the state-space graph of this family admit a Hamilton cycle?
6. The vertices of any state-space graph can be partitioned into easy-mode families, with the h-edges and v-edges lying within families and the c-edges going between families. (A move of c takes you from one easy-mode family to another.) Can the existence of a Hamilton cycle within easy-mode families be used to construct a Hamilton cycle for a hard-mode state-space graph? (This is admittedly a much more difficult question, one to which the authors do not know the answer.)

ACKNOWLEDGMENTS. We thank one of the referees for his/her extensive and helpful comments, which resulted in a significant improvement of the paper.

Solution to sample puzzle. The puzzle may be solved using the move sequence $h^{-1}cv$, as illustrated below:



We note that this is a solution of minimal length, and was not found using the methods given in Section 6.

REFERENCES

1. E. R. Berlekamp, J. H. Conway, R. K. Guy, *Winning Ways For Your Mathematical Plays. Vol. 1*, second edition, A K Peters, Natick, MA, 2001.
2. J. A. Gallian, *Contemporary Abstract Algebra*, Brooks Cole, Belmont, CA, 2010.
3. The GAP Group, *GAP—Groups, Algorithms, and Programming, Version 4.4.12*, 2008.

4. D. Joyner, *Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys*, second edition, Johns Hopkins University Press, Baltimore, MD, 2008.
5. T. Rokicki, Twenty-two moves suffice for Rubik's Cube, *The Mathematical Intelligencer* **32** (2010) 33–40, available at http://dx.doi.org/10.1007/s00283-009-9105-3.
6. L. H. Soicher, *GRAPE—GRaph Algorithms using PErmutation groups Version 4.3*, 2006.

JEREMY ALM is assistant professor of mathematics at Illinois College. He received a B.S. in mathematics education from the University of Wisconsin-Eau Claire in 2001 and a Ph.D. in mathematics from Iowa State University in 2006. He is addicted to combinatorial problems arising from considerations in algebraic logic and elsewhere. His two young daughters assist him in not thinking about mathematics from time to time.
Department of Mathematics, Illinois College, Jacksonville, IL 62650
alm.academic@gmail.com

MICHAEL GRAMELSPACHER is co-owner/founder of SG Labs, LLC, a product design, development, and consulting firm. He received a B.S. in electrical engineering from Rose-Hulman Institute of Technology with a focus in embedded systems. He has worked in the automotive, medical electronics, and consumer products fields and now is a full-time toy and game inventor. His most recent inventions are Air Picks by Ohio Art, Rubik's Slide by Techno Source, and Electra Doodle by Cra-Z-Art.
msgramel@sg-labs.com

THEODORE RICE received his B.A. from Gustavus Adolphus College in 1998 and his M.S. and Ph.D. from Iowa State in 2004 and 2007. He teaches at Illinois Wesleyan. He loves hiking and riding his bike. He is newly married and is excited about this new chapter in his life.
Department of Mathematics, Illinois Wesleyan, Normal, IL 61701
rice.academic@gmail.com