Hex, and Brouwer's fixed point theorem

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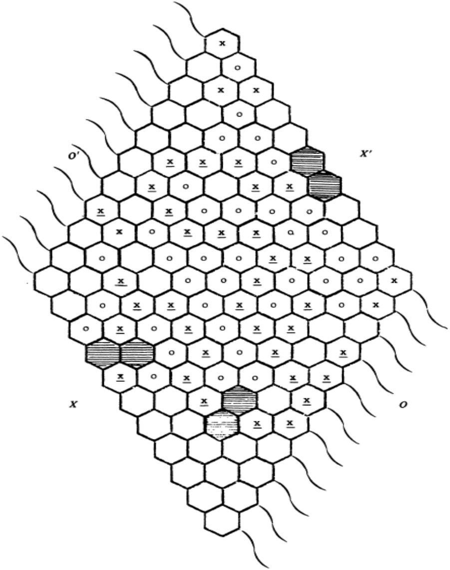
The game of Hex

What is Hex?

- Hex is a **strategy board game** played on a hexagonal grid, theoretically of any size and several possible shapes, but *traditionally* laid out as an 11x11 rhombus.
- It was invented in 1942 by Piet Hein, but was rediscovered by John Nash around 1947.

The game of Hex



Brouwer's fixed point theorem

Why is it important?

- Brouwer's fixed-point theorem is probably the most widely used fixed-point theorem in topology, named after Luitzen Brouwer.
- It is the precursor of the Kakutani fixed point theorem, and was initially used by Nash for his first description of Nash equilibria.

Statement

- Any continuous function mapping a compact, convex set to itself has a fixed point. i.e. $f : [0, 1] \to [0, 1]$ has a fixed point $f(x) = x$.
- This is easy to see in one dimension. Not that obvious in more dimensions (and it indeed holds in n dimensions).

So what's the link?

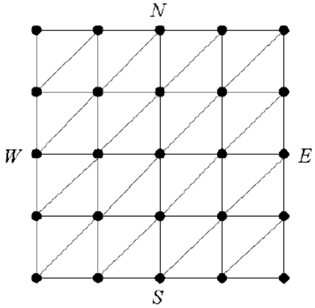
What's nice about these?

- The Hex theorem states that Hex can never end in a draw...
- Curiously, the Hex theorem **implies** Brouwer's fixed point theorem (and vice-versa) !
- This result had been discussed informally for many years but was formalised by David Gale in 1970 who also generalised to n dimensions (n dimensional Hex, n players).
- In fact, the Hex theorem and Brouwer's theorem are actually equivalent.

Let's have a go at proving this. Hex → FPT

The idea

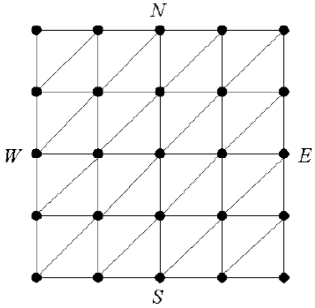
- **The key idea is to find a graph representation of the board (below)**. The two dimensional Hex board of size k, call it B_k, is a graph whose vertices is the set of all $z \in Z^2$ with $(1,1) \leq z \leq (k,k)$. Two vertices z and z_0 are adjacent (i.e., an edge in B_k connects z and z_0) if $|z - z_0| = 1$ and z and z_0 are comparable.



Proof (1)

The Hex theorem now states..

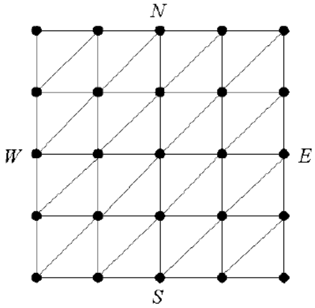
Let B_k be covered by two sets H and V . Then either **H** contains a connected set meeting E and W or **V** contains a connected set meeting N and S. **We are going to prove that** this is equivalent to saying that a function $f : I^2 \to I^2$ has a fixed point. Where I is the unit square.



Proof (2)

Basic observations

The set I^2 is compact, so it suffices to show that for any ϵ, there exists $x \in I^2$ for which $|f(x) - x| < \epsilon$. The compactness of I^2 also implies that f is uniformly continuous, so we know that for $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$. Also we can pick $\delta < \epsilon$. **Crucially**, we choose a Hex board large enough so that $\frac{1}{k} < \delta$.



Proof (3)

Let's define...

- $H^+ = \{z | f_1(z/k) - z_1/k > \epsilon\}$
- $H^- = \{z | z_1/k - f_1(z/k) > \epsilon\}$
- $V^+ = \{z | f_2(z/k) - z_2/k > \epsilon\}$
- $V^- = \{z | z_2/k - f_2(z/k) > \epsilon\}$

Why?

- The **goal** is to show that these do not make it to cover the whole of B_k!
- Intuitively, H^+ is the set of all points which are shuffled by f, towards E, by more than ϵ. Same applies to other sets.

Proof (4)

H^+ and H^- (V^+ and V^-) are disjoint and not contiguous

- Take $z \in H^+$ and $z' \in H^-$ to be adjacent. Then by def.

$$f_1(z/k) - z_1/k > \epsilon$$

$$z'_1/k - f_1(z'/k) > \epsilon$$

.

- Add them to obtain: $f_1(z/k) - f_1(z'/k) - z_1/k + z'_1/k > 2\epsilon$.
- Because z and z' are adjacent we know that

$$z'_1/k - z_1/k \leq |z'_1/k - z_1/k| = 1/k < \delta < \epsilon$$

- This implies, if we add it with point 2 in this slide, that $f_1(z/k) - f_1(z'/k) > \epsilon$ which is a contradiction since z and z' were adjacent.

Proof (5)

Finally...

- We have to prove that the sets $H = H^+ \cup H^-$ and $V = V^+ \cup V^-$ do not cover B_k completely. Here, we use the Hex theorem.
- Let Q be a connected set on H. Now, this must lie entirely on either H^+ or H^-. But H^+ does not touch E, and H^- does not touch W (f is an automorphism). Similarly, a connected set on V cannot touch both W and E. The theorem is proved!

Discussion

- A nice example of how concepts that are seemingly quite different can turn out to be equivalent!
- There are some cases in which you can see mathematical facts through games. I.e. You can also prove that

$$\frac{1}{4}+\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\frac{5}{64}+\ldots=1$$

via a solitaire game.