Teaching singular distributions to undergrads

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Density Police

$$P(a \leq X \leq b) = \int_{[a,b]} \mathrm{d}P(x)$$

(P: a probability distribution)

Q. Why not just write with a density?

$$\int_a^b p(x)\mathrm{d}x$$

Density Police

$$\mathbb{E}_{X\sim P}\left[f(X)\right] = \int f(x)\mathrm{d}P(x)$$

(P: a probability distribution)

Q. Why not just write with a density?

$$\int f(x)p(x)\mathrm{d}x$$

Singular Distribution (SD)

Sometimes you can't do this

= when you have SDs (no density available)

Any example?

- Cantor distribution with CDF:

$$c(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2^n}, & x = \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in \mathcal{C} \text{ for } a_n \in \{0,1\}; \\ \sup_{y \le x,\, y \in \mathcal{C}} c(y), & x \in [0,1] \setminus \mathcal{C}. \end{cases}$$

Singular Distribution (SD)

Goal of this talk:

- Give a more relatable example of SDs

At the end, you should

- Know there is more than disc./cont. dists
- Be able to teach SDs to undergrads

More relatable example

- Reward R from bandit  $\sim U[0,1]$

More relatable example

- Reward R from arm  $\sim U[0,1]$
- Reward R from arm  $\sim \delta_0 \; (=0)$

More relatable example

Repeat the following:

1. Uniformly pick from {  ,  }
2. Play and receive reward

Let R_1, R_2: resp. reward of 1st and 2nd trials.

Q. What is the joint distribution of R_1, R_2 ?

More relatable example

- CDF of ith reward (i.i.d.):

$$F_i(x) = \frac{1}{2} + \frac{1}{2}x, \quad \text{for } x \in [0,1]$$

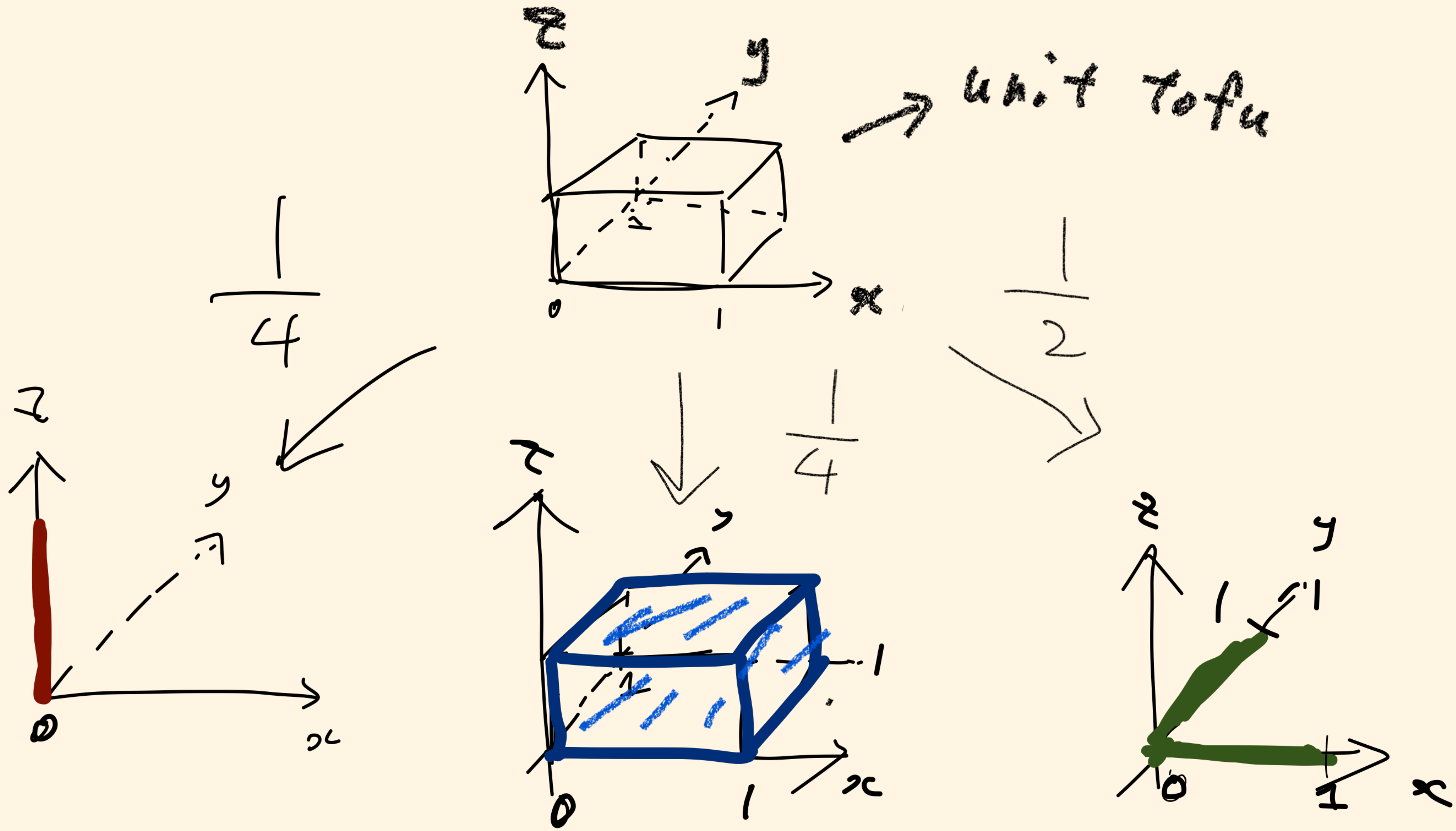
- CDF of joint (by independence):

$$F(x, y) = F_1(x)F_2(y)$$

$$= \frac{1}{4} + \frac{1}{4}xy + \frac{1}{4}(x + y), \quad (x, y) \in [0,1] \times [0,1]$$

① Discrete ② Continuous ③ Singular

Illustration



δ_0

$U[0,1]^2$

Conclusions

1. SDs can arise in our daily life:

   ➡ Sets with Lebesgue measure zero

   - Line in 2d space, e.g., GPS data (objects move along roads)

2. Good to know other integrals (e.g. Stieltjes/ Lebesgue integrals)

Reference

Teaching Singular Distributions to Undergraduates

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Singular distributions are seldom covered in undergraduate probability courses, although they are of interest in statistics and, as is shown by example, can easily arise through extending mixed discrete and continuous distributions to two or more dimensions. A representation is given that makes the construction of a class of singular distributions in two dimensions simple to carry out. This representation is also used to characterize the types of marginal distributions that members of this class can have.

KEY WORDS: Singular distributions; Mixtures of distributions.

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1. INTRODUCTION

While preparing an exam for an undergraduate course in probability some years ago, I came upon the following simple cumulative distribution function (cdf)

$$F(x,y) = (x+y)/2$$

$$0 \le x \le 1,\ 0 \le y \le 1. \qquad (1)$$

My students were equipped with the usual knowledge about analyzing cdf's: The probability function of a discrete distribution, $p(x,y)$, can be computed by a standard method at the jump points of $F(x,y)$, while the probability density function $f(x,y)$ is computed by taking the mixed second-order partial derivative of $F(x,y)$. I had even discussed mixed distributions for which both components can be present (i.e., nonzero) at the same time.