# Probabilistic & Unsupervised Learning

## Variational EM
## and Variational Bayesian Learning

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# Integrals in Statistical Modelling

- **Parameter estimation**

$$\hat{\theta} = \underset{\theta}{\text{argmax}} \int d\mathcal{Y}\ P(\mathcal{Y}|\theta) P(\mathcal{X}|\mathcal{Y}, \theta)$$

(or using EM)

$$\theta^{\text{new}} = \underset{\theta}{\text{argmax}} \int d\mathcal{Y}\ P(\mathcal{Y}|\mathcal{X}, \theta^{\text{old}}) \log P(\mathcal{X}, \mathcal{Y}|\theta)$$

- **Prediction**

$$p(x|\mathcal{D}, m) = \int d\theta\ p(\theta|\mathcal{D}, m) p(x|\theta, \mathcal{D}, m)$$

- **Model selection or weighting** (by marginal likelihood)

$$p(\mathcal{D}|m) = \int d\theta\ p(\theta|m) p(\mathcal{D}|\theta, m)$$

These integrals are often intractable:

- **Analytic intractability**: integrals may not have closed form in non-linear, non-Gaussian models $\Rightarrow$ numerical integration.
- **Computational intractability**: Numerical integral (or sum if $\mathcal{Y}$ or $\theta$ are discrete) may be exponential in data or model size.

# Examples of Intractability

- Bayesian marginal likelihood/model evidence for Mixture of Gaussians: exact computations are exponential in number of data points

$$p(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \int d\theta \; p(\theta) \prod_{i=1}^{N} \sum_{s_i} p(\mathbf{x}_i | s_i, \theta) p(s_i | \theta)$$

$$= \sum_{s_1} \sum_{s_2} \ldots \sum_{s_N} \int d\theta \; p(\theta) \prod_{i=1}^{N} p(\mathbf{x}_i | s_i, \theta) p(s_i | \theta)$$

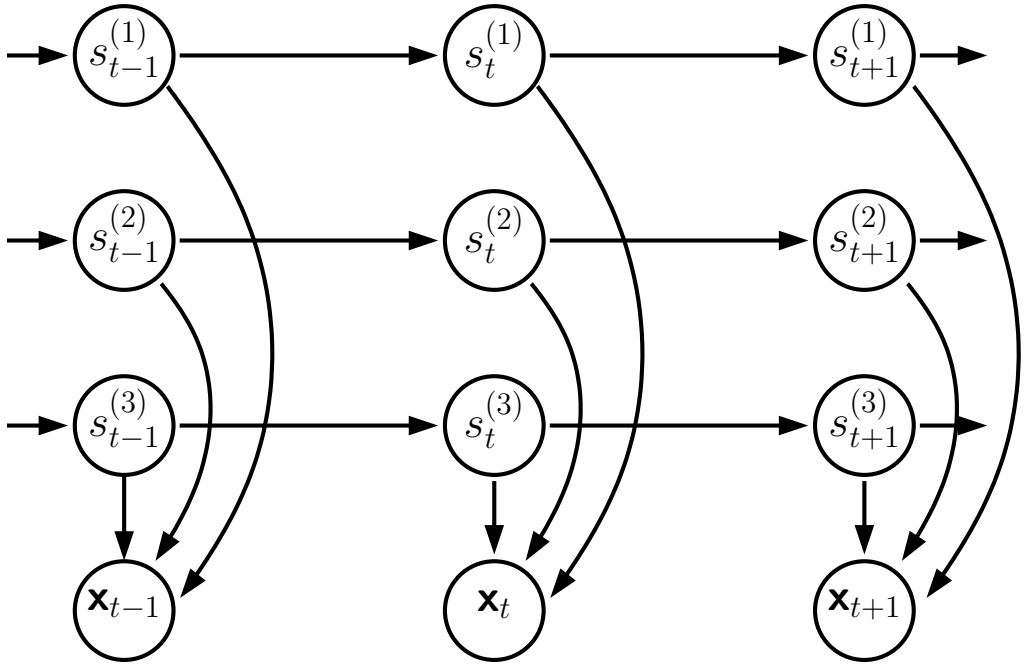
- Computing the conditional probability of a variable in a very large multiply connected directed graphical model:

$$p(x_i | X_j = a) = \sum_{\text{all settings of } \mathbf{y} \setminus \{i,j\}} p(x_i, \mathbf{y}, X_j = a) / p(X_j = a)$$

- Computing the hidden state distribution in a general nonlinear dynamical system

$$p(\mathbf{y}_t | \mathbf{x}_1, \ldots, \mathbf{x}_T) \propto \int p(\mathbf{y}_t | \mathbf{y}_{t-1}) p(\mathbf{x}_t | \mathbf{y}_t) p(\mathbf{y}_{t-1} | \mathbf{x}_1, \ldots, \mathbf{x}_{t-1}) p(\mathbf{x}_{t+1}, \ldots, \mathbf{x}_t | \mathbf{y}_t) d\mathbf{y}_{t-1}$$

# Distributed models




In the FHMM, moralisation puts simultaneous states $s_t^{(1)}, s_t^{(2)}, s_t^{(3)}$ into a single clique.

- $M$ state variables, $K$ values $\Rightarrow$ sums over $K^{2M}$ terms.
- Factorial *prior* $\not\Rightarrow$ Factorial *posterior* (explaining away).

Variational methods approximate the posterior, often in a factored form. To see how they work, we need to review the free-energy interpretation of EM.

# The Free Energy for a Latent Variable Model

Observed data $\mathcal{X} = \{\mathbf{x}_i\}$; Latent variables $\mathcal{Y} = \{\mathbf{y}_i\}$; Parameters $\theta$.

**Goal:** Maximize the log likelihood (i.e. ML learning) wrt $\theta$:

$$\ell(\theta) = \log P(\mathcal{X}|\theta) = \log \int P(\mathcal{Y}, \mathcal{X}|\theta) d\mathcal{Y},$$

Any distribution, $q(\mathcal{Y})$, over the hidden variables can be used to obtain a lower bound on the log likelihood using Jensen's inequality:

$$\ell(\theta) = \log \int q(\mathcal{Y}) \frac{P(\mathcal{Y}, \mathcal{X}|\theta)}{q(\mathcal{Y})} \, d\mathcal{Y} \geq \int q(\mathcal{Y}) \log \frac{P(\mathcal{Y}, \mathcal{X}|\theta)}{q(\mathcal{Y})} \, d\mathcal{Y} \stackrel{\text{def}}{=} \mathcal{F}(q, \theta).$$

Now,

$$\begin{aligned} \int q(\mathcal{Y}) \log \frac{P(\mathcal{Y}, \mathcal{X}|\theta)}{q(\mathcal{Y})} \, d\mathcal{Y} &= \int q(\mathcal{Y}) \log P(\mathcal{Y}, \mathcal{X}|\theta) \, d\mathcal{Y} - \int q(\mathcal{Y}) \log q(\mathcal{Y}) \, d\mathcal{Y} \\ &= \int q(\mathcal{Y}) \log P(\mathcal{Y}, \mathcal{X}|\theta) \, d\mathcal{Y} + \mathbf{H}[q], \end{aligned}$$

where $\mathbf{H}[q]$ is the entropy of $q(\mathcal{Y})$.
So:

$$\mathcal{F}(q, \theta) = \langle \log P(\mathcal{Y}, \mathcal{X}|\theta) \rangle_{q(\mathcal{Y})} + \mathbf{H}[q]$$

# The E and M steps of EM

The log likelihood is bounded below (Jensen) by:

$$\mathcal{F}(q, \theta) = \langle \log P(\mathcal{Y}, \mathcal{X}|\theta) \rangle_{q(\mathcal{Y})} + \mathbf{H}[q],$$

EM alternates between:

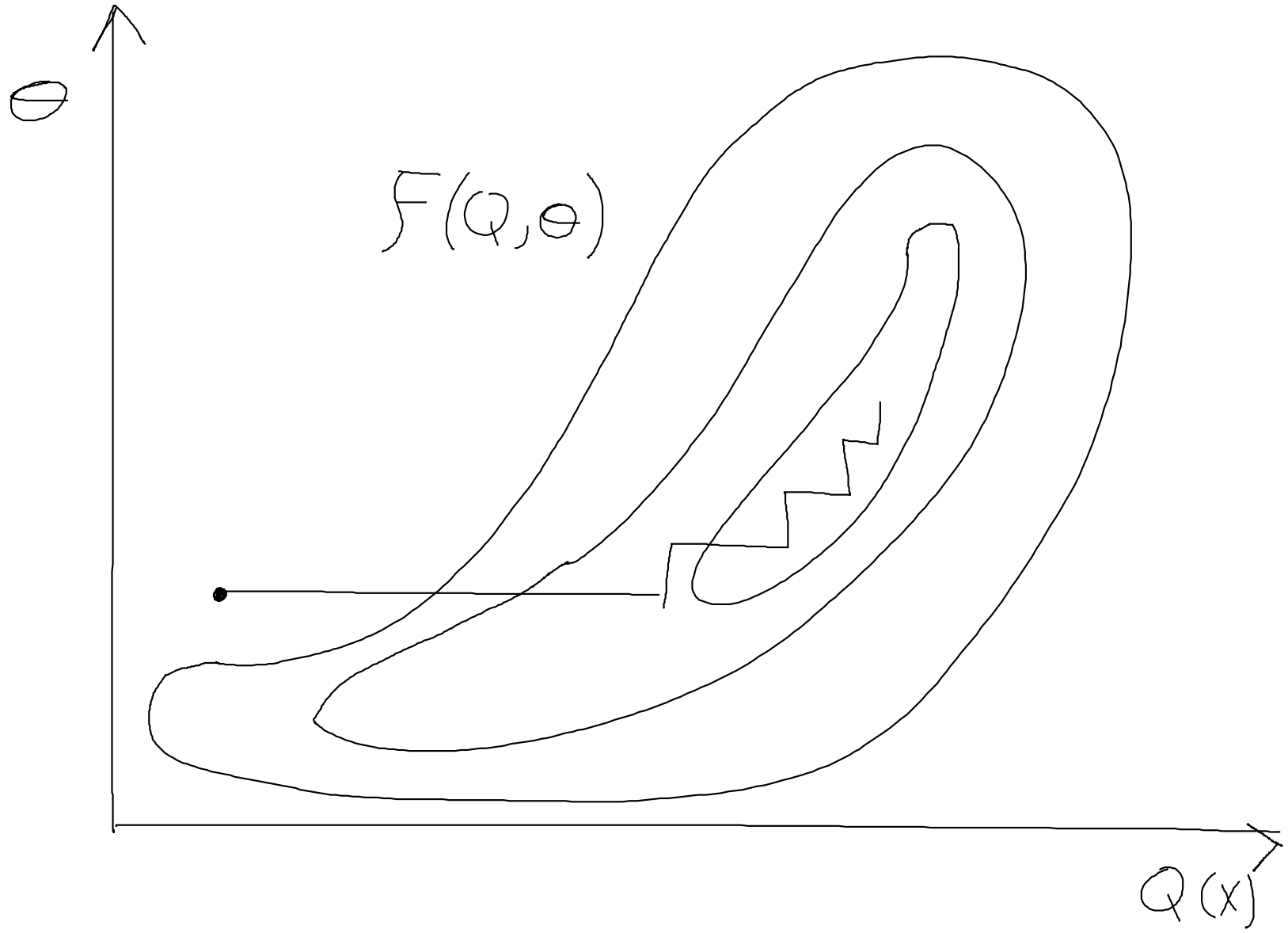
**E step:** optimise $\mathcal{F}(q, \theta)$ wrt distribution over hidden variables holding parameters fixed:

$$q^{(k)}(\mathcal{Y}) := \underset{q(\mathcal{Y})}{\text{argmax}} \; \mathcal{F}\big(q(\mathcal{Y}), \theta^{(k-1)}\big) = P(\mathcal{Y}|\mathcal{X}, \theta^{(k-1)})$$

**M step:** maximise $\mathcal{F}(q, \theta)$ wrt parametersholding hidden distribution fixed:

$$\theta^{(k)} := \underset{\theta}{\text{argmax}} \; \mathcal{F}\big(q^{(k)}(\mathcal{Y}), \theta\big) = \underset{\theta}{\text{argmax}} \; \langle \log P(\mathcal{Y}, \mathcal{X}|\theta) \rangle_{q^{(k)}(\mathcal{Y})}$$

# EM as Coordinate Ascent in $\mathcal{F}$



# EM Never Decreases the Likelihood

The E and M steps together never decrease the log likelihood:

$$\ell\big(\theta^{(k-1)}\big) \underset{\text{E step}}{=} \mathcal{F}\big(q^{(k)}, \theta^{(k-1)}\big) \underset{\text{M step}}{\leq} \mathcal{F}\big(q^{(k)}, \theta^{(k)}\big) \underset{\text{Jensen}}{\leq} \ell\big(\theta^{(k)}\big),$$

- The E step brings the free energy to the likelihood.
- The M-step maximises the free energy wrt $\theta$.
- $\mathcal{F} \leq \ell$ by Jensen – or, equivalently, from the non-negativity of KL

If the M-step is executed so that $\theta^{(k)} \neq \theta^{(k-1)}$ iff $\mathcal{F}$ increases, then the overall EM iteration will step to a new value of $\theta$ iff the likelihood increases.

# Variational Approximations to the EM algorithm

What if finding expected sufficient stats under $P(\mathcal{Y}|\mathcal{X},\theta)$ is computationally intractable?

In the **generalised EM** algorithm, we argued that intractable maximisations could be replaced by gradient M-steps. For the E-step we could:

- Parameterise $q = q_\rho(\mathcal{Y})$ and take a gradient step in $\rho$.
- Assume some simplified form for $q$, usually factored: $q = \prod_i q_i(\mathcal{Y}_i)$ where $\mathcal{Y}_i$ partition $\mathcal{Y}$, and maximise within this form.

In both cases, we assume $q \in \mathcal{Q}$, and optimise within this class:

**VE step**: maximise $\mathcal{F}(q,\theta)$ wrt restricted latent distribution given parameters:

$$q^{(k)}(\mathcal{Y}) := \underset{q(\mathcal{Y})\in\mathcal{Q}}{\text{argmax}}\ \mathcal{F}\big(q(\mathcal{Y}), \theta^{(k-1)}\big).$$

**M step**: unchanged

$$\theta^{(k)} := \underset{\theta}{\text{argmax}}\ \mathcal{F}\big(q^{(k)}(\mathcal{Y}), \theta\big) = \underset{\theta}{\text{argmax}}\ \int q^{(k)}(\mathcal{Y}) \log p(\mathcal{Y},\mathcal{X}|\theta) d\mathcal{Y},$$

This maximises a lower bound on the log likelihood.

# What do we lose?

What does restricting $q$ to $\mathcal{Q}$ cost us?

- Recall that the free-energy is bounded above by Jensen:

$$\mathcal{F}(q, \theta) \leq \ell(\theta^{\text{ML}})$$

  Thus, as long as every step increases $\mathcal{F}$, convergence is still guaranteed.

- But, since $P(\mathcal{Y}|\mathcal{X}, \theta^{(k)})$ may not lie in $\mathcal{Q}$, we no longer saturate the bound after the E-step. Thus, the likelihood may not increase on each full EM step.

$$\ell\big(\theta^{(k-1)}\big) \underset{\text{E step}}{\not=} \mathcal{F}\big(q^{(k)}, \theta^{(k-1)}\big) \underset{\text{M step}}{\leq} \mathcal{F}\big(q^{(k)}, \theta^{(k)}\big) \underset{\text{Jensen}}{\leq} \ell\big(\theta^{(k)}\big),$$

- Thus, we may not converge to a maximum of $\ell$.

The hope is that by *increasing a lower bound* on $\ell$ we will find a decent solution.
[Note that if $P(\mathcal{Y}|\mathcal{X}, \theta^{\text{ML}}) \in \mathcal{Q}$, then $\theta^{\text{ML}}$ is a fixed point of the variational algorithm.]

# KL divergence

Recall that

$$\begin{aligned}\mathcal{F}(q,\theta) &= \langle \log P(\mathcal{X},\mathcal{Y}|\theta)\rangle_{q(\mathcal{Y})} + \mathbf{H}[q] \\ &= \langle \log P(\mathcal{X}|\theta) + \log P(\mathcal{Y}|\mathcal{X},\theta)\rangle_{q(\mathcal{Y})} - \langle \log q(\mathcal{Y})\rangle_{q(\mathcal{Y})} \\ &= \langle \log P(\mathcal{X}|\theta)\rangle_{q(\mathcal{Y})} - \mathbf{KL}[q\|P(\mathcal{Y}|\mathcal{X},\theta)].\end{aligned}$$

Thus,

**E step** maximise $\mathcal{F}(q,\theta)$ wrt the distribution over latents, given parameters:

$$q^{(k)}(\mathcal{Y}) := \underset{q(\mathcal{Y})\in\mathcal{Q}}{\text{argmax}}\ \mathcal{F}\big(q(\mathcal{Y}), \theta^{(k-1)}\big).$$

is equivalent to:

**E step** minimise $\mathbf{KL}[q\|p(\mathcal{Y}|\mathcal{X},\theta)]$ wrt distribution over latents, given parameters:

$$q^{(k)}(\mathcal{Y}) := \underset{q(\mathcal{Y})\in\mathcal{Q}}{\text{argmin}} \int q(\mathcal{Y}) \log \frac{q(\mathcal{Y})}{p(\mathcal{Y}|\mathcal{X},\theta^{(k-1)})} d\mathcal{Y}$$

So, in each E step, the algorithm is trying to find the best approximation to $P(\mathcal{Y}|\mathcal{X})$ in $\mathcal{Q}$.

This is related to ideas in *information geometry*.

# Factored Variational E-step

The most common form of variational approximation partitions $\mathcal{Y}$ into disjoint sets $\mathcal{Y}_i$ with

$$\mathcal{Q} = \{q \mid q(\mathcal{Y}) = \prod_i q_i(\mathcal{Y}_i)\}.$$

In this case the E-step is itself iterative:

**(Factored VE step)**$_i$: maximise $\mathcal{F}(q, \theta)$ wrt $q_i(\mathcal{Y}_i)$ given other $q_j$ and parameters:

$$q_i^{(k)}(\mathcal{Y}_i) := \underset{q_i(\mathcal{Y}_i)}{\text{argmax}} \; \mathcal{F}\big(q_i(\mathcal{Y}_i)\prod_{j\neq i} q_j(\mathcal{Y}_j), \theta^{(k-1)}\big).$$

The $q_i$s can be updated iteratively until convergence before moving on to the M-step. Alternatively, we can make a single pass over all $q_i$ (starting from values at the last step) and then perform an M-step. Each VE step increases $\mathcal{F}$, so convergence is still guaranteed.

# Factored Variational E-step

The Factored Variational E-step has a general form.

The free energy is:

$$\mathcal{F}\Big(\prod_j q_j(\mathcal{Y}_j), \theta^{(k-1)}\Big) = \Big\langle \log P(\mathcal{X}, \mathcal{Y}|\theta^{(k-1)}) \Big\rangle_{\prod_j q_j(\mathcal{Y}_j)} + \mathbf{H}\Big[\prod_j q_j(\mathcal{Y}_j)\Big]$$

$$= \int d\mathcal{Y}_i \; q_i(\mathcal{Y}_i) \Big\langle \log P(\mathcal{X}, \mathcal{Y}|\theta^{(k-1)}) \Big\rangle_{\prod_{j\neq i} q_j(\mathcal{Y}_j)} + \mathbf{H}[q_i] + \sum_{j\neq i} \mathbf{H}[q_j]$$

Now, taking the variational derivative of the Lagrangian (enforcing normalisation of $q_i$):

$$\frac{\delta}{\delta q_i}\left(\mathcal{F} + \lambda\left(\int q_i - 1\right)\right) = \Big\langle \log P(\mathcal{X}, \mathcal{Y}|\theta^{(k-1)}) \Big\rangle_{\prod_{j\neq i} q_j(\mathcal{Y}_j)} - \log q_i(\mathcal{Y}_i) - 1 + \lambda$$

$$(= 0) \quad \Rightarrow \quad q_i(\mathcal{Y}_i) \propto \exp \Big\langle \log P(\mathcal{X}, \mathcal{Y}|\theta^{(k-1)}) \Big\rangle_{\prod_{j\neq i} q_j(\mathcal{Y}_j)}$$

In general, this depends only on the expected sufficient statistics under $q_j$. Thus, once again, we don't actually need the *entire* distributions, just the *relevant* expectations.

# Mean-field Approximations

If $\mathcal{Y}_i = y_i$ (*i.e.*, $q$ is factored over all variables) then the variational technique is often called a "mean field" approximation.

Suppose $P(\mathcal{X}, \mathcal{Y})$ is log-linear, *e.g.* the Boltzmann machine:

$$P(\mathcal{X}, \mathcal{Y}) = \frac{1}{Z} \exp\left( \sum_{ij} W_{ij} s_i s_j + \sum_i b_i s_i \right)$$

with some $s_i \in \mathcal{Y}$ and others observed.

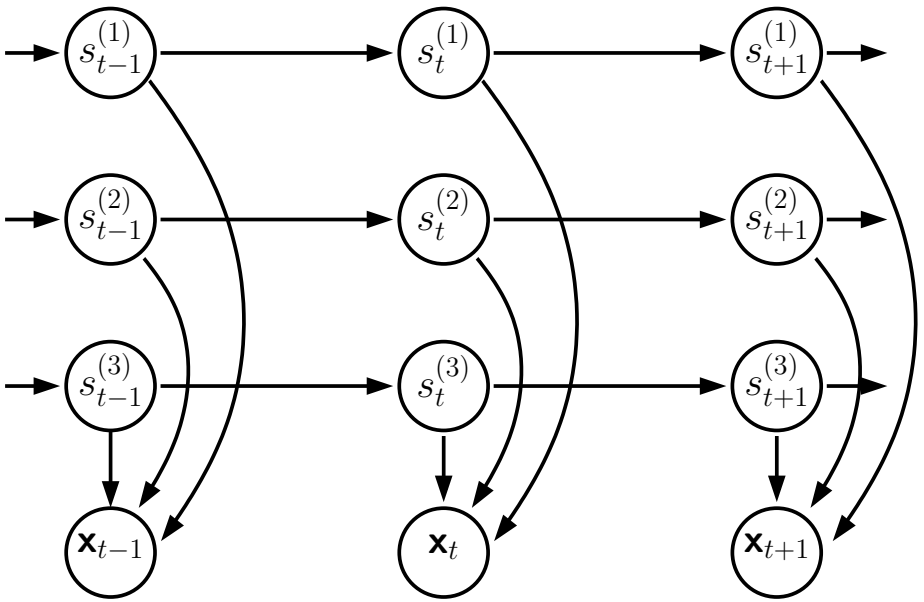
Expectations wrt a fully factored $q$ distribute over all $s_i \in \mathcal{Y}$

$$\langle \log P(\mathcal{X}, \mathcal{Y}) \rangle_{\prod q_i} = \sum_{ij} W_{ij} \langle s_i \rangle_{q_i} \langle s_j \rangle_{q_j} + \sum_i b_i \langle s_i \rangle_{q_i}$$

(where $q_i$ for $s_i \in \mathcal{X}$ is a delta function on observed value).

Thus, we can update each $q_i$ in turn given the means of the others. Each variable is seeing the *mean* field imposed by its neighbours. We update these fields until they all agree.

# Mean-field FHMM




The mean-field approach to the FHMM with

$$q(s^{1:M}_{1:T}) = \prod_{m,t} q^m_t(s^m_t)$$

yields a variant of the usual forward-backward algorithm. Coupling between the different chains only takes place through the joint output distribution. Each update depends only on the immediate neighbours.

$$q^{m'}_{t'}(s^{m'}_{t'}) \propto \exp \left\langle \log P(\mathbf{s}^{1:M}_{1:T}, \mathbf{x}_{1:T}) \right\rangle_{\prod_{\neg(m',t')} q^m_t(s^m_t)}$$

$$= \exp \left\langle \sum_m \sum_t \log P(s^m_t | s^m_{t-1}) + \sum_t \log P(\mathbf{x}_t | s^{1:M}_t) \right\rangle_{\prod_{\neg(m',t')} q^m_t}$$

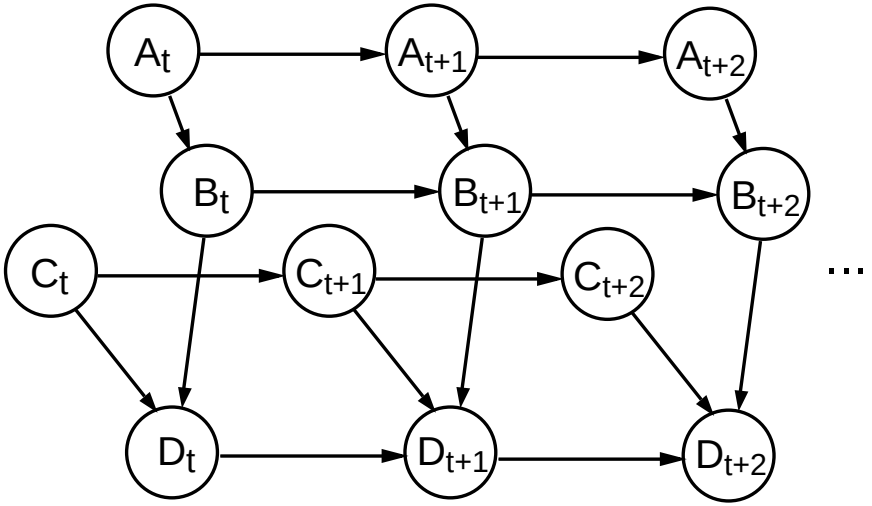
$$\propto \exp \left[ \left\langle \log P(s^{m'}_{t'} | s^{m'}_{t'-1}) \right\rangle_{q^{m'}_{t'-1}} + \left\langle \log P(s^{m'}_{t+1'} | s^{m'}_{t'}) \right\rangle_{q^{m'}_{t'+1}} + \left\langle \log P(\mathbf{x}_{t'} | s^{1:M}_{t'}) \right\rangle_{\prod_{\neg m} q^m_{t'}} \right]$$

# Structured Variational Approximations

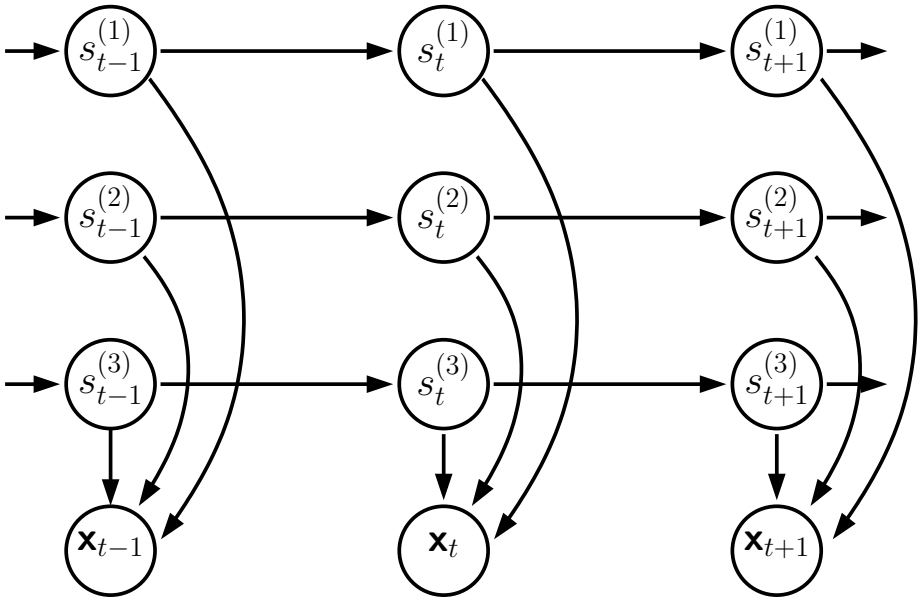
$q(\mathcal{Y})$ need not be completely factorized.

For example, suppose you can partition $\mathcal{Y}$ into sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ such that computing the expected sufficient statistics under $q(\mathcal{Y}_1)$ and $q(\mathcal{Y}_2)$ is tractable.
Then $q(\mathcal{Y}) = q(\mathcal{Y}_1)q(\mathcal{Y}_2)$ is tractable.

If you have a graphical model, you may want to factorize $q(\mathcal{Y})$ into a product of trees, which are tractable distributions.



# Stuctured FHMM




The most natural structured approximation in the FHMM is to factor each chain from the others

$$q(s_{1:T}^{1:M}) = \prod_m q^m(s_{1:T}^m)$$

Updates within each chain are then found by a forward-backward algorithm, with a modified "likelihood" term.

$$\begin{aligned}
q^{m'}(s_{1:T}^{m'}) &\propto \exp\left\langle \log P(\mathbf{s}_{1:T}^{1:M}, \mathbf{x}_{1:T}) \right\rangle_{\prod_{\neg m'} q^m(s_{1:T}^m)} \\
&= \exp\left\langle \sum_m \sum_t \log P(s_t^m | s_{t-1}^m) + \sum_t \log P(\mathbf{x}_t | s_t^{1:M}) \right\rangle_{\prod_{\neg m'} q^m} \\
&\propto \exp\left[ \sum_t \log P(s_t^{m'} | s_{t-1}^{m'}) + \sum_t \left\langle \log P(\mathbf{x}_{t'} | s_{t'}^{1:M}) \right\rangle_{\prod_{\neg m} q^m s_{t'}^m} \right] \\
&= \prod_t P(s_t^{m'} | s_{t-1}^{m'}) \prod_t e^{\left\langle \log P(\mathbf{x}_{t'} | s_{t'}^{1:M}) \right\rangle_{\prod_{\neg m} q^m s_{t'}^m}}
\end{aligned}$$

# Variational Approximations and Graphical Models I

Let $q(\mathcal{Y}) = \prod_i q_i(\mathcal{Y}_i)$.

Variational approximation maximises $\mathcal{F}$:

$$\mathcal{F}(q) = \int q(\mathcal{Y}) \log p(\mathcal{Y}, \mathcal{X}) d\mathcal{Y} - \int q(\mathcal{Y}) \log q(\mathcal{Y}) d\mathcal{Y}$$

Focusing on one term, $q_j$, we can write this as:

$$\mathcal{F}(q_j) = \int q_j(\mathcal{Y}_j) \left\langle \log p(\mathcal{Y}, \mathcal{X}) \right\rangle_{\neg q_j(\mathcal{Y}_j)} d\mathcal{Y}_j + \int q_j(\mathcal{Y}_j) \log q_j(\mathcal{Y}_j) d\mathcal{Y}_j + \text{const}$$

Where $\langle \cdot \rangle_{\neg q_j(\mathcal{Y}_j)}$ denotes averaging w.r.t. $q_i(\mathcal{Y}_i)$ for all $i \neq j$

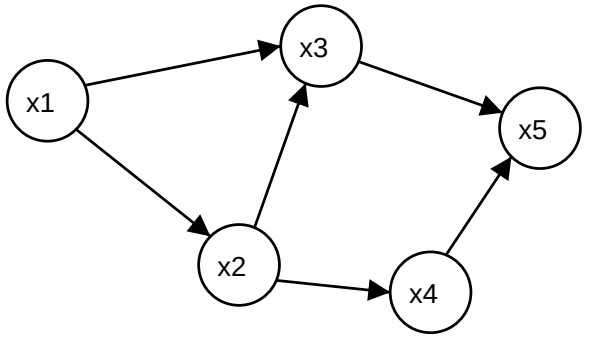
Optimum occurs when:

$$q_j^*(\mathcal{Y}_j) = \frac{1}{Z} \exp \left\langle \log p(\mathcal{Y}, \mathcal{X}) \right\rangle_{\neg q_j(\mathcal{Y}_j)}$$

# Variational Approximations and Graphical Models II

Optimum occurs when:

$$q_j^*(\mathcal{Y}_j) = \frac{1}{Z} \exp \langle \log p(\mathcal{Y}, \mathcal{X}) \rangle_{\neg q_j(\mathcal{Y}_j)}$$




Assume graphical model: $p(\mathcal{Y}, \mathcal{X}) = \prod_i p(X_i|\text{pa}_i)$

$$\begin{aligned}
\log q_j^*(\mathcal{Y}_j) &= \left\langle \sum_i \log p(X_i|\text{pa}_i) \right\rangle_{\neg q_j(\mathcal{Y}_j)} + \text{const} \\
&= \left\langle \log p(\mathcal{Y}_j|\text{pa}_j) \right\rangle_{\neg q_j(\mathcal{Y}_j)} + \sum_{k \in \text{ch}_j} \left\langle \log p(X_k|\text{pa}_k) \right\rangle_{\neg q_j(\mathcal{Y}_j)} + \text{const}
\end{aligned}$$

This defines messages that get passed between nodes in the graph. Each node receives messages from its Markov boundary: parents, children and parents of children.

Variational Message Passing (Winn and Bishop, 2004)

# Variational Approximations to Bayesian Learning

$$\begin{aligned}\log p(\mathcal{X}) &= \log \int\int p(\mathcal{X}, \mathcal{Y}|\boldsymbol{\theta}) p(\boldsymbol{\theta})\, d\mathcal{Y}\, d\boldsymbol{\theta} \\ &\geq \int\int q(\mathcal{Y}, \boldsymbol{\theta}) \log \frac{p(\mathcal{X}, \mathcal{Y}, \boldsymbol{\theta})}{q(\mathcal{Y}, \boldsymbol{\theta})}\, d\mathcal{Y}\, d\boldsymbol{\theta}\end{aligned}$$

Constrain $q \in \mathcal{Q}$ s.t. $q(\mathcal{Y}, \boldsymbol{\theta}) = q(\mathcal{Y})q(\boldsymbol{\theta})$.

This results in the **variational Bayesian EM algorithm**.

# Variational Bayesian Learning

### Lower Bounding the Marginal Likelihood

Let the hidden latent variables be $\mathcal{Y}$, data $\mathcal{X}$ and the parameters $\boldsymbol{\theta}$.

Lower bound the marginal likelihood (Bayesian model evidence) using Jensen's inequality:

$$\begin{aligned}
\log P(\mathcal{X}) &= \log \int d\mathcal{Y}\, d\boldsymbol{\theta}\; P(\mathcal{X}, \mathcal{Y}, \boldsymbol{\theta}) && ||m \\
&= \log \int d\mathcal{Y}\, d\boldsymbol{\theta}\; Q(\mathcal{Y}, \boldsymbol{\theta}) \frac{P(\mathcal{X}, \mathcal{Y}, \boldsymbol{\theta})}{Q(\mathcal{Y}, \boldsymbol{\theta})} \\
&\geq \int d\mathcal{Y}\, d\boldsymbol{\theta}\; Q(\mathcal{Y}, \boldsymbol{\theta}) \log \frac{P(\mathcal{X}, \mathcal{Y}, \boldsymbol{\theta})}{Q(\mathcal{Y}, \boldsymbol{\theta})}.
\end{aligned}$$

The saturating $Q(\mathcal{Y}, \boldsymbol{\theta}) = P(\mathcal{Y}, \boldsymbol{\theta}|\mathcal{X})$ is almost always intractable.
Use a simpler, factorised approximation $Q(\mathcal{Y}, \boldsymbol{\theta}) = Q_{\mathcal{Y}}(\mathcal{Y}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})$:

$$\begin{aligned}
\log P(\mathcal{X}) &\geq \int d\mathcal{Y}\, d\boldsymbol{\theta}\; Q_{\mathcal{Y}}(\mathcal{Y}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \log \frac{P(\mathcal{X}, \mathcal{Y}, \boldsymbol{\theta})}{Q_{\mathcal{Y}}(\mathcal{Y}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})} \\
&= \mathcal{F}(Q_{\mathcal{Y}}(\mathcal{Y}), Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}), \mathcal{X}).
\end{aligned}$$

Maximize this lower bound. The resulting value is the approximation to the evidence.

# Variational Bayesian Learning . . .

Maximizing this lower bound, $\mathcal{F}$, leads to **EM-like** updates:

$$Q^*_{\mathcal{Y}}(\mathcal{Y}) \;\propto\; \exp\left\langle \log P(\mathcal{Y},\mathcal{X}|\boldsymbol{\theta})\right\rangle_{Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})} \qquad E-like\ step$$

$$Q^*_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \;\propto\; P(\boldsymbol{\theta}) \exp\left\langle \log P(\mathcal{Y},\mathcal{X}|\boldsymbol{\theta})\right\rangle_{Q_{\mathcal{Y}}(\mathcal{Y})} \qquad M-like\ step$$

Maximizing $\mathcal{F}$ is equivalent to minimizing KL-divergence between the *approximate posterior*, $Q(\boldsymbol{\theta})Q(\mathcal{Y})$ and the *true posterior*, $P(\boldsymbol{\theta},\mathcal{Y}|\mathcal{X})$.

$$\begin{aligned} \log P(\mathcal{X}) - \mathcal{F}(Q_{\mathcal{Y}}(\mathcal{Y}), Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}), \mathcal{X}) \;&=\\ \log P(\mathcal{X}) - \int d\mathcal{Y}\, d\boldsymbol{\theta}\; Q_{\mathcal{Y}}(\mathcal{Y})Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \log \frac{P(\mathcal{X},\mathcal{Y},\boldsymbol{\theta})}{Q_{\mathcal{Y}}(\mathcal{Y})Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})} \;&=\\ \int d\mathcal{Y}\, d\boldsymbol{\theta}\; Q_{\mathcal{Y}}(\mathcal{Y})Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \log \frac{Q_{\mathcal{Y}}(\mathcal{Y})Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})}{P(\mathcal{Y},\boldsymbol{\theta}|\mathcal{X})} \;&=\; KL(Q||P) \end{aligned}$$

# Conjugate-Exponential models

Let's focus on *conjugate-exponential* (**CE**) models, which satisfy **(1)** and **(2)**:

- **Condition (1).** The joint probability over *variables* is in the exponential family:

$$P(\mathcal{Y}, \mathcal{X}|\boldsymbol{\theta}) = f(\mathcal{Y}, \mathcal{X})\ g(\boldsymbol{\theta}) \exp\left\{\boldsymbol{\phi}(\boldsymbol{\theta})^\top \mathbf{u}(\mathcal{Y}, \mathcal{X})\right\}$$

  where $\boldsymbol{\phi}(\boldsymbol{\theta})$ is the vector of *natural parameters*, $\mathbf{u}$ are *sufficient statistics*

- **Condition (2).** The prior over *parameters* is conjugate to this joint probability:

$$P(\boldsymbol{\theta}|\eta, \boldsymbol{\nu}) = h(\eta, \boldsymbol{\nu})\ g(\boldsymbol{\theta})^\eta \exp\left\{\boldsymbol{\phi}(\boldsymbol{\theta})^\top \boldsymbol{\nu}\right\}$$

  where $\eta$ and $\boldsymbol{\nu}$ are hyperparameters of the prior.

Conjugate priors are computationally convenient and have an intuitive interpretation:

- $\eta$: number of pseudo-observations
- $\boldsymbol{\nu}$: values of pseudo-observations

# Conjugate-Exponential examples

In the **CE** family:

- Gaussian mixtures
- factor analysis, probabilistic PCA
- hidden Markov models and factorial HMMs
- linear dynamical systems and switching models
- discrete-variable belief networks

Other as yet undreamt-of models can combine Gaussian, Gamma, Poisson, Dirichlet, Wishart, Multinomial and others.

Not in the **CE** family:

- Boltzmann machines, MRFs (no simple conjugacy)
- logistic regression (no simple conjugacy)
- sigmoid belief networks (not exponential)
- independent components analysis (not exponential)

Note: one can often approximate these models with models in the **CE** family.

# A Useful Result

Given an iid data set $\mathcal{X} = (\mathcal{X}_1, \ldots \mathcal{X}_n)$, if the model is **CE** then:

(a) $Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ is also conjugate, *i.e.*

$$Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) = h(\tilde{\eta}, \tilde{\boldsymbol{\nu}}) g(\boldsymbol{\theta})^{\tilde{\eta}} \exp\left\{\boldsymbol{\phi}(\boldsymbol{\theta})^\top \tilde{\boldsymbol{\nu}}\right\}$$

where $\tilde{\eta} = \eta + n$ and $\tilde{\boldsymbol{\nu}} = \boldsymbol{\nu} + \sum_i \overline{\mathbf{u}}(\mathcal{Y}_i, \mathcal{X}_i)$.

(b) $Q_{\mathcal{Y}}(\mathcal{Y}) = \prod_{i=1}^n Q_{\mathcal{Y}_i}(\mathcal{Y}_i)$ is of the same form as in the E step of regular EM, but using pseudo parameters computed by averaging over $Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})$

$$Q_{\mathcal{Y}_i}(\mathcal{Y}_i) \propto f(\mathcal{Y}_i, \mathcal{X}_i) \exp\left\{\overline{\boldsymbol{\phi}}(\boldsymbol{\theta})^\top \mathbf{u}(\mathcal{Y}_i, \mathcal{X}_i)\right\} = P(\mathcal{Y}_i | \mathcal{X}_i, \overline{\boldsymbol{\phi}}(\boldsymbol{\theta}))$$

**KEY points**:

(a) the approximate parameter posterior is of the same form as the prior, so it is easily summarized in terms of two sets of hyperparameters, $\tilde{\eta}$ and $\tilde{\boldsymbol{\nu}}$;

(b) the approximate hidden variable posterior, *averaging over all parameters*, is of the same form as the hidden variable posterior for a *single setting of the parameters*, so again, it is easily computed using the usual methods.

# The Variational Bayesian EM algorithm

## EM for MAP estimation

Goal: maximize $p(\boldsymbol{\theta}|\mathcal{X}, m)$ w.r.t. $\boldsymbol{\theta}$

**E Step:** compute

$$q_{\mathcal{Y}}^{(t+1)}(\mathcal{Y}) = p(\mathcal{Y}|\mathcal{X}, \boldsymbol{\theta}^{(t)})$$

**M Step:**

$$\boldsymbol{\theta}^{(t+1)} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}} \int q_{\mathcal{Y}}^{(t+1)}(\mathcal{Y}) \log p(\mathcal{Y}, \mathcal{X}, \boldsymbol{\theta})\, d\mathcal{Y}$$

## Variational Bayesian EM

Goal: lower bound $p(\mathcal{X}|m)$

**VB-E Step:** compute

$$q_{\mathcal{Y}}^{(t+1)}(\mathcal{Y}) = p(\mathcal{Y}|\mathcal{X}, \bar{\boldsymbol{\phi}}^{(t)})$$

**VB-M Step:**

$$q_{\boldsymbol{\theta}}^{(t+1)}(\boldsymbol{\theta}) = \exp\left[\int q_{\mathcal{Y}}^{(t+1)}(\mathcal{Y}) \log p(\mathcal{Y}, \mathcal{X}, \boldsymbol{\theta})\, d\mathcal{Y}\right]$$

**Properties:**

- Reduces to the EM algorithm if $q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) = \delta(\boldsymbol{\theta} - \boldsymbol{\theta}^*)$.
- $\mathcal{F}_m$ increases monotonically, and incorporates the model complexity penalty.
- Analytical parameter distributions (but not constrained to be Gaussian).
- VB-E step has same complexity as corresponding E step.
- We can use the junction tree, belief propagation, Kalman filter, etc, algorithms in the VB-E step of VB-EM, but **using expected natural parameters**, $\bar{\boldsymbol{\phi}}$.

## Variational Bayes: History of Models Treated

- multilayer perceptrons (Hinton & van Camp, 1993)
- mixture of experts (Waterhouse, MacKay & Robinson, 1996)
- hidden Markov models (MacKay, 1995)
- other work by Jaakkola, Jordan, Barber, Bishop, Tipping, etc

## Examples of Variational Learning of Model Structure

- mixtures of factor analysers (Ghahramani & Beal, 1999)
- mixtures of Gaussians (Attias, 1999)
- independent components analysis (Attias, 1999; Miskin & MacKay, 2000; Valpola 2000)
- principal components analysis (Bishop, 1999)
- linear dynamical systems (Ghahramani & Beal, 2000)
- mixture of experts (Ueda & Ghahramani, 2000)
- discrete graphical models (Beal & Ghahramani, 2002)
- VIBES software for conjugate-exponential graphs (Winn, 2003)

# ARD for unsupervised learning

A idea similar to supervised ARD can be used with Variational Bayesian methods to learn the dimensionality of a latent space. Consider factor analysis:

$$\mathbf{x} \sim \mathcal{N}(\Lambda \mathbf{y}, \Psi) \qquad \mathbf{y} \sim \mathcal{N}(0, I)$$

with a prior

$$\Lambda_i \sim \mathcal{N}\left(0, \alpha_i^{-1} I\right)$$

The VB free energy is a function of the data, $Q_{\mathcal{Y}}(\mathcal{Y})$, $Q_\Lambda(\Lambda)$ and $\boldsymbol{\alpha}$:

$$\mathcal{F}(Q_{\mathcal{Y}}(\mathcal{Y}), Q_\Lambda(\Lambda), \mathcal{X}, \boldsymbol{\alpha}) = \left\langle \log P(\mathcal{X}, \mathcal{Y}|\Lambda, \Psi) + \log P(\Lambda|\boldsymbol{\alpha}) + \log P(\Psi) \right\rangle_{Q_{\mathcal{Y}} Q_\Lambda} + \mathbf{H}[Q_{\mathcal{Y}}] + \mathbf{H}[Q_\Lambda]$$

Optimising this wrt the distributions and $\boldsymbol{\alpha}$ in turn (like EM) causes some $\alpha_i$ to diverge, restricting the effective dimensionality of $\mathbf{y}$.