Graphical Models

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Graphs, independence and factorisation.

The (Markov) independence structure of a latent chain model implied that the joint-data likelihood factorised:

$$P(X, Y) = P(y_1) \prod_{t=1}^{T} P(y_t | y_{t-1}) P(x_t | y_t)$$

We exploited the factored form to obtain local $O(T)$ learning algorithms.

- Learning: requires only local marginals of posterior
- Inference: local marginals found by passing local messages

The independence structure of the model (and the factorisation of its likelihood) is encoded in its graph.

Why the graph?

- Gives an intuitive representation of the relationships amongst many variables, possibly embodying prior beliefs or knowledge about causal relationships. (Examples: inheritance in family trees, noise in electric circuits, neural networks)
- Provides a precise syntax to describe these relationships, and to infer any implied (in)dependencies amongst larger groups of variables.
  
  $A \perp \perp E \mid \{B, C\}$?
  
- Each graphical structure corresponds to a parametric family of distributions that satisfy all the implied (in)dependencies.
- Graph-based manipulations allow us to identify the sufficient statistics of these distributions needed for learning, and to construct general-purpose message-passing algorithms that implement inference efficiently.

Find $P(A | C = c)$ without enumerating all settings of $B, D, E \ldots$
Types of independence

For events or random variables $X, Y, V$:

**Conditional Independence:**

$$X \perp \!\!\!\!\!\!\perp Y \mid V \iff P(X \mid Y, V) = P(X \mid V)$$  \[provided, for events, P(Y, V) > 0\]

Thus,

$$X \perp \!\!\!\!\!\!\perp Y \mid V \iff P(X, Y \mid V) = P(X \mid V)P(Y \mid V)$$

We can generalise to conditional independence between sets of random variables:

$$\mathcal{X} \perp \!\!\!\!\!\!\perp \mathcal{Y} \mid \mathcal{V} \iff \{X \perp \!\!\!\!\!\!\perp Y \mid \mathcal{V}, \ \forall X \in \mathcal{X} \text{ and } \forall Y \in \mathcal{Y}\}$$

**Marginal Independence:**

$$X \perp \!\!\!\!\!\!\perp Y \iff X \perp \!\!\!\!\!\!\perp Y \mid \emptyset \iff P(X, Y) = P(X)P(Y)$$

Factor graphs

A factor graph is a direct graphical representation of the factorised model structure: each square indicates a factor that depends on the linked variables.

$$P(\mathcal{X}) = \frac{1}{Z} \prod_j f_j(\mathcal{X}_j)$$

where $\mathcal{X} = \{X_1, \ldots, X_k\}, \mathcal{X}_j = \{X_i : i \in S\}, j$ indexes the factors, $f_j$ is the factor function (also called the factor potential or clique potential) and $Z$ is a normalisation constant.

Factor graphs: examples

Examples:

(a) $P(A, B, C, D, E) = \frac{1}{Z} f_1(A, C)f_2(B, C, D)f_3(C, D, E)$

(b) $P(A, B, C, D, E) = \frac{1}{Z} f_1(A, C)f_2(B, C)f_3(C, D)f_4(B, D)f_5(C, E)f_6(D, E)$

and [e.g.]:

$$Z_4 = \sum_{a \in A} \sum_{b \in B} \sum_{c \in C} \sum_{d \in D} \sum_{e \in E} f(a, c)f(b, c, d)f(c, d, e)$$

where $A, B, C, D$ and $E$ are the domains of the corresponding random variables.

Factor graphs: conditional independence

**Conditional independence:** $X \perp \!\!\!\!\!\!\perp Y \mid V$ if every path between $X$ and $Y$ contains some $V \in \mathcal{V}$.

In both graphs:

- $A \perp D \mid C$
- $B \perp E \mid C$
- $B \perp E \mid \{C, D\}$
Factorisation and conditional independence

Every path between $X$ and $Y$ contains some $V \in V$ $\Rightarrow$ there exists a factorisation:

$$P(X, Y, V, \ldots) = \frac{1}{Z} g_X(X, V, Q_X) g_Y(Y, V, Q_Y) g_R(Q_R, V_R)$$

where $V_X, V_Y, V_R \subseteq V$ and the sets of remaining variables $Q_X, Q_Y$ and $Q_R$ are disjoint.

$$\Rightarrow P(X|Y, V, \ldots) = \frac{P(X, Y, V, \ldots)}{P(Y, V, \ldots)} = \frac{\frac{1}{Z} g_X(X, V, Q_X) g_Y(Y, V, Q_Y) g_R(Q_R, V_R)}{\sum_{X'} \frac{1}{Z} g_X(X', V, Q_X) g_Y(Y, V, Q_Y) g_R(Q_R, V_R)}$$

$$= \frac{g_X(X, V, Q_X)}{\sum_{X'} g_X(X', V, Q_X)}$$

Since the RHS does not depend on $Y$, it follows that $X \perp \perp Y|V$.

Undirected graphical models: Markov networks

An undirected graphical model is a direct representation of conditional independence structure. Nodes are connected if they are conditionally dependent given all others.

$\Rightarrow$ neighbours (connected nodes) in a Markov net share a factor.
$\Rightarrow$ non-neighbours (disconnected nodes) in a Markov net cannot share a factor.
$\Rightarrow$ the joint probability factors over the maximal cliques $C_i$ of the graph:

$$P(X) = \frac{1}{Z} \prod_i \iota_i(V_c)$$

It may also factor more finely (as we will see in a moment).

[Cliques are fully connected subgraphs, maximal cliques are cliques not contained in other cliques.]

Factor graphs: neighbourhoods and Markov boundaries

Variables are neighbours if they share a common factor; the neighbourhood $\text{ne}(X)$ is the set of all neighbours of $X$.

Each variable $X$ is conditionally independent of all non-neighbours given its neighbours: $X \perp Y|\text{ne}(X), \forall Y /\in \{X \cup \text{ne}(X)\}$ $\Rightarrow \text{ne}(X)$ is a Markov blanket for $X$.

In fact, the neighbourhood is the minimal such set: the Markov boundary.

Undirected graphs: Markov boundaries

$X \perp Y|V$ if every path between $X$ and $Y$ contains some node $V \in V$

$\Rightarrow$ Each variable $X$ is conditionally independent of all non-neighbours given its neighbours: $X \perp Y|\text{ne}(X), \forall Y /\in \{X \cup \text{ne}(X)\}$

$V$ is a Markov blanket for $X$ if $X \perp Y|V$ for all $Y /\in \{X \cup V\}$.

Markov boundary: minimal Markov blanket. For undirected graphs (like factor graphs) this is the set of neighbours of $X$. 
Undirected graphs and factor graphs

(a) (b) (c)

▶ Each node has the same neighbours in each graph, so (a), (b) and (c) represent exactly
the same conditional independence relationships.
▶ The implied maximal factorisations differ: (b) has two three-way factors; (c) has only
pairwise factors; (a) cannot distinguish between these (so we have to adopt factorisation
(b) to be safe).
▶ Suppose all variables are discrete and can take on $K$ possible values. Then the
functions in (a) and (b) are tables with $O(K^3)$ cells, whereas in (c) they are $O(K^2)$.
▶ Factor graphs have richer expressive power than undirected graphical models.
▶ Factors cannot be determined solely by testing for conditional independence.

Limitations of undirected and factor graphs

Undirected and factor graphs fail to capture some useful independencies—a pair of variables
may be connected merely because some other variable depends on them:

The classic example (due to Pearl):

▶ Most sprinklers switch on come rain or shine; and certainly the weather pays no heed to
the state of the sprinklers.
▶ Explaining away: Damp ground suggests that it has rained; but if we also see a running
sprinkler this explains away the damp, returning our belief about rain to the prior.
▶ $R \perp \perp S \mid \emptyset$ but $R \not\perp \perp S \mid G$.

This highlights the difference between marginal and conditional independence.

Directed acyclic graphical models

A directed acyclic graphical (DAG) model represents a factorization of the joint probability
distribution in terms of conditionals:

$$P(X) = \frac{1}{Z} \exp \left\{ \sum_j \lambda_j g_j(X) \right\}$$

In general:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^n P(X_i | \text{pa}(i))$$

where pa($i$) are the parents of node $i$.

DAG models are also known as Bayesian networks or Bayes nets.

Some examples of undirected graphical models

▶ Markov random fields (used in computer vision)

▶ Maximum entropy language models (used in speech and language modelling)

▶ Conditional random fields are undirected graphical models (conditioned on the input
variables).

▶ Boltzmann machines (a kind of neural network/Ising model)
Conditional independence in DAGs

Reading conditional independence from DAGs is more complicated than in undirected graphs.
- $A \perp \perp E \mid \{B, C\}$: conditioning nodes block paths
- $A \perp \perp B \mid \varnothing$: other nodes block reflected paths
- $A \perp \perp E \mid C$: conditioning node creates a reflected path by explaining away
- $A \perp \perp E \mid \{C, D\}$: the created path extends to $E$ via $D$
- $A \perp \perp B \mid \emptyset$: other nodes block reflected paths
- $A \perp \perp E \mid \{C, D\}$: but is blocked by observing $D$

So conditioning on (i.e. observing) nodes can both create and remove dependencies.

The Bayes-ball algorithm

Game: can you get a ball from $X$ to $Y$ without being blocked by $V$? If so, $X \perp \perp Y \mid V$.

Rules: ball follow edges, and are passed on or bounced back from nodes according to:
- Nodes $V \notin V$ pass balls down or up chains: $\rightarrow V \rightarrow$ or $\leftarrow V \leftarrow$.
- Nodes $V \notin V$, bounce balls from children to children.
- Nodes $V \in V$, bounce balls from parents to parents (including returning the ball whence it came).

Otherwise the ball is blocked. (So $V \in V$ blocks all balls from children, and stops balls from parents reaching children.)

Expressive power of directed and undirected graphs

No DAG can represent these and only these independencies

No matter how we direct the arrows there will always be two non-adjacent parents sharing a common child $\Rightarrow$ dependence in DAG but independence in undirected graph.

One three-way factor, but this does not encode marginal independence.
Graphs, conditional independencies, and families of distributions

Each graph $G$ implies a set of conditional independence statements $\mathcal{C}(G) = \{X_i \perp \perp Y_i | V_i\}$.

Each such set $\mathcal{C}$ defines a family of distributions that satisfy all the statements in $\mathcal{C}$:

$$\mathcal{P}_{\mathcal{C}(G)} = \{ P(X) : P(X_i, Y_i | V_i) = P(X_i | V_i)P(Y_i | V_i) \text{ for all } X_i \perp \perp Y_i | V_i \text{ in } \mathcal{C} \}$$

$G$ may also encode a family of distributions by their functional form, e.g. for a factor graph

$$\mathcal{P}_G = \{ P(X) : P(X) = Z^{-1} \prod_i f_i(X_{C_i}) \text{ for some non-negative functions } f_i \}$$

▶ For directed graphs, $\mathcal{P}_G = \mathcal{P}_{\mathcal{C}(G)}$.
▶ For undirected graphs, $\mathcal{P}_G = \mathcal{P}_{\mathcal{C}(G)}$ if all distributions are positive, i.e. $P(X) > 0$ for all values of $X$ (Hammersley-Clifford Theorem).
▶ There are factor graphs for which $\mathcal{P}_G \neq \mathcal{P}_{\mathcal{C}(G)}$.
▶ Factor graphs are more expressive than undirected graphs: for every undirected graph $G$, there is a factor graph $G_2$ with $\mathcal{P}_G = \mathcal{P}_{G_2}$ but not vice versa.
▶ Adding edges to graph $\Rightarrow$ removing conditional independency statements $\Rightarrow$ enlarging the family of distributions (converse true for removing edges).

Tree-structured graphical models

Rooted directed tree

Directed polytree

Undirected tree

Tree-structured factor graph

These are all tree-structured or “singly-connected” graphs.

Polytrees to tree-structured factor graphs

Polytrees are tree-structured DAGs that may have more than one root.

$$P(X) = \prod_i P(X_i | X_{pa(i)})$$

$$= \prod_i f_i(X_{C_i})$$

where $C_i = i \cup pa(i)$ and $f_i(X_{C_i}) = P(X_i | X_{pa(i)})$.

Marginal distribution on roots $P(X_r)$ absorbed into an adjacent factor.
Undirected trees and factor graphs

In an undirected tree all maximal cliques are of size 2, and so the equivalent factor graph has only pairwise factors.

\[ P(X) = \frac{1}{Z} \prod_{\text{edges } (i)} f_{ij}(X_i, X_j) \]

Rooted directed trees to undirected trees

The distribution for a single-rooted directed tree can be written as a product of pairwise factors \( \Rightarrow \) undirected tree.

\[ P(X) = P(X_r) \prod_{i \neq r} P(X_i | X_{pa(i)}) = \prod_{\text{edges } (i)} f_{ij}(X_i, X_j) \]

Undirected trees to rooted directed trees

This direction is slightly trickier:
- Choose an arbitrary node \( X_r \) to be the root and point all the arrows away from it
- Compute the marginal distributions on single nodes \( P(X_i) \) and on edges \( P(X_i, X_j) \) implied by the undirected graph.
- Compute the conditionals in the DAG:

\[ P(X) = P(X_r) \prod_{i \neq r} P(X_i | X_{pa(i)}) = \prod_{\text{nodes } i, j \neq r} \frac{P(X_i, X_j)}{P(X_r)} \]

Finding marginals in undirected trees

Undirected tree \( \Rightarrow \) pairwise factored joint distribution: \( P(X) = \frac{1}{Z} \prod_{(i) \in E_T} f_{ij}(X_i, X_j) \)

Each neighbour \( X_i \) of \( X_r \) defines a disjoint subtree \( T_{j\rightarrow i} \). So we can split up the product:

\[ P(X) = \sum_{X \setminus \{X_r\}} P(X) \times \sum_{X \setminus \{X_r\}} \prod_{(i) \in E_T} f_{ij}(X_i, X_j) \]

\[ = \sum_{X \setminus \{X_r\}} \prod_{(i) \in E_T} f_{ij}(X_i, X_j) \prod_{(f) \in E_{T_{j\rightarrow i}}} f_{ij}(X_j, X_f) \]

\[ = \prod_{X \setminus \{X_r\}} \left( \sum_{X_{T_{j\rightarrow i}}} f_{ij}(X_j, X_f) M_{j\rightarrow i}(X) \right) \]

How do we compute \( P(X) \) and \( P(X_r, X_i) \)? \( \Rightarrow \) Belief propagation.
Recall: BP on undirected trees

\[ M_{i \to j}(X) = \sum_{X_{T_{i \to j}}} \prod_{(i', j') \in E_{T_{i \to j}}} f_{i', j'}(X_{i'}, X_{j'}) \]

\[ = \sum_{X_j} f_{i}(X_j, X_i) \prod_{X_{T_{i \to j}} \setminus X_j} \prod_{(i', j') \in E_{T_{i \to j}}} f_{i', j'}(X_{i'}, X_{j'}) \times P_{T_{i \to j}}(X_j) \prod_{X \in E_{T_{i \to j}}} M_{k \to j}(X) \]

\[ = \sum_{X_j} f_{i}(X_j, X_i) \prod_{X \in \mathcal{X}(X_j) \setminus X_i} M_{k \to j}(X) \]

BP for inference

Messages from observed leaf nodes are conditioned rather than marginalised:

To compute \( P(X) \):

\[ M_{a \to i} = \sum_{X_a} f_a(X_a, X_i) \]

To compute \( P(X|X_a = a) \):

\[ M_{a \to i} = f_a(X_a = a, X_i) \]

Observed internal nodes partition the graph, and so messages propagate independently.

\[ M_{b \to a} = f_b(X_b = b, X_i) \]
\[ M_{b \to k} = f_b(X_b = b, X_k) \]

Messages \( M_{i \to j} \) are proportional to the likelihood based on any observed variables \( \mathcal{O} \) within the messages subtree \( T_{i \to j} \), possibly scaled by a prior factor (depending on factorisation)

\[ M_{i \to j}(X) \propto P(X_{T_{i \to j}} \cap \mathcal{O}(X)) P(X) \]

BP for pairwise marginals in undirected trees

\[ P(X_i, X_j) = \sum_{X_i \setminus \{X_i, X_j\}} P(X_i) \times \sum_{X_j \setminus \{X_i, X_j\}} \prod_{(i', j') \in E \setminus \{(i', j')\}} f_{i', j'}(X_{i'}, X_{j'}) \]

\[ = \sum_{X_i \setminus \{X_i, X_j\}} f_i(X_i, X_j) \prod_{X \setminus \{X_i, X_j\}} \prod_{(i', j') \in E \setminus \{(i', j')\}} f_{i', j'}(X_{i'}, X_{j'}) \times P_{T_{i \to j}}(X_j) \prod_{X \in E \setminus \{(i', j')\}} M_{k \to j}(X) \]

\[ = \sum_{X_i \setminus \{X_i, X_j\}} f_i(X_i, X_j) \prod_{X \setminus \{X_i, X_j\}} M_{k \to j}(X) \]

BP for latent chain models

A latent chain model is a rooted directed tree \( \Rightarrow \) an undirected tree.

The forward-backward algorithm is just BP on this graph.

\[ \alpha_t(i) \leftrightarrow M_{s_{t-1} \to s_t}(s_{t-1}) \propto P(s_{t-1} : s_t) \]
\[ \beta_t(i) \leftrightarrow M_{s_{t+1} \to s_t}(s_{t+1}) \propto P(s_{t+1} : s_t) \]

\[ \alpha_t(i) \beta_t(i) = \prod_{j \in \mathcal{O}(a)} M_{j \to a}(s_{t-1}) \propto P(s_{t-1}|\mathcal{O}) \]

Algorithms like BP extend the power of graphical models beyond just encoding of independence and factorisation. A single derivation serves for a wide array of models.
BP in non-trees?

Can we find \( P(D) \) easily?
- Neighbours do not belong to disjoint subtrees, so influence of other nodes cannot be separated into messages.
- Observed nodes may break loops and make subtrees independent, but may not resolve all loops.

Possible strategies:
- Propagate local messages anyway, and hope for the best
  - loopy belief propagation — actually an approximation which we will study later.
- Group variables together into multivariate nodes until the resulting graph is a tree.
- Junction tree

Graph transformations

For exact inference in arbitrary graphical models we need to transform the given graph into one that will be easier to handle (specifically a tree: the junction or join tree).

The original graph \( G \) encoded a distribution \( P(\mathcal{X}) \) with a certain factorisation or independence structure.
- Transformation from \( G \) to an easy-to-handle \( G' \) will only be valid if \( P(\mathcal{X}) \) can also be represented by \( G' \).
- This can be ensured if every step of the graph transformation only removes conditional independencies, never adds them.
- Thus the family of possible encoded distributions grows or stays the same at each step, and \( P(\mathcal{X}) \) will be in the family of distributions represented by \( G' \).
- The factor potentials on the new graph \( G' \) are built from those given on \( G \), so as to encode the same distribution.
- Then inference on \( G' \) with the appropriate potentials acts on \( P(\mathcal{X}) \).

The Junction Tree algorithm

Factors are simply the conditional distributions in the DAG.
\[
P(\mathcal{X}) = \prod_i P(X_i | \text{pa}(i))
= \prod_i f_i(\mathcal{C}_i)
\]
where \( \mathcal{C}_i = i \cup \text{pa}(i) \) and \( f_i(\mathcal{C}_i) = P(X_i | \text{pa}(i)) \).
Marginal distribution on root(s) \( P(X_r) \) absorbed into an adjacent factor.
Inference usually targets a posterior marginal given a set of observed values \( P(X|O) \) e.g. \( P(A|D=\text{wet}, C=\text{rain}) \).

Formally, we can either modify the factors linked to observed nodes, or add singleton factors adjacent to the observed nodes, e.g.

\[
\begin{align*}
  f_D(D) &= \begin{cases} 
    1 & \text{if } D=\text{wet}; \\
    0 & \text{otherwise}. 
  \end{cases} \\
  f_C(C) &= \begin{cases} 
    1 & \text{if } C=\text{rain}; \\
    0 & \text{otherwise}. 
  \end{cases}
\end{align*}
\]

The next step (triangulation) will depend on an undirected graph. Every factor from the DAG must be contained within a maximal clique of the undirected graph.

- Replace each factor by an undirected clique (i.e. place edge between every pair of nodes in the factor).
- Construct the potentials on each maximal clique by multiplying together factor potentials that fall within it; ensuring each factor potential only appears once.

The transformation from DAG ⇒ undirected graph is called moralization:
- “marry” all parents of each node by adding an edge to connect them
- drop arrows on all the edges

Imagine marginalising the distribution one variable at a time (eliminating each from the graph). Let the order of elimination be \( X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)} \):

\[
P(X_{\sigma(n)}) = \sum_{X_{\sigma(n-1)}} \cdots \sum_{X_{\sigma(1)}} P(X') = \frac{1}{Z} \sum_{X_{\sigma(n-1)}} \cdots \sum_{X_{\sigma(2)}} \sum_{X_{\sigma(1)}} \prod_{i} f_i(X_{C_i})
\]

\[
= \frac{1}{Z} \sum_{X_{\sigma(n-1)}} \cdots \sum_{X_{\sigma(2)}} \prod_{i} f_i(X_{C_i}) \sum_{X_{\sigma(1)}} \prod_{i \notin \sigma(1)} f_i(X_{C_i})
\]

where \( C_{\text{new}} = \text{ne}(X_{\sigma(1)}) \), and edges are added to the graph connecting all nodes in \( C_{\text{new}} \).
**Variable elimination**

**Theorem:** A graph including all edges that would be induced by elimination is chordal.

Finding a good triangulation depends on finding a good order of elimination $\sigma(1), \ldots, \sigma(n)$. This is also NP-complete.

Heuristics: pick next variable to eliminate by
- Minimum deficiency search: choose variable that induces the fewest new edges.
- Maximum cardinality search: choose variable with most previously visited neighbours.

Minimum deficiency search seems (empirically) to be better.

**Triangulation may not be obvious by inspection**

Has this graph been triangulated?

No. Chords must be direct connections — they cannot step through an intermediate node.

Detecting unchorded loops by inspection rapidly becomes difficult in large graphs, necessitating automated algorithms such as variable elimination.

**Chordal graph to the junction tree**

A junction tree (or join tree) is a tree whose nodes and edges are labelled with sets of variables.

Each node represents a clique; edges are labelled by the intersections of cliques, called separators.

- Cliques contain all adjacent separators.
- **Running intersection property:** if two cliques contain variable $X$, all cliques and separators on the path between the two cliques contain $X$.

The running intersection property is required for consistency.
Message passing on junction trees

Maximal cliques in the chordal graph are nodes of the junction tree. Thus, the joint distribution factors over the JT nodes:

\[ P(\mathcal{X}) = \frac{1}{Z} \prod_i f_i(\mathcal{X}_i) = \ldots f_{ABC}(A, B, C)f_{BCD}(B, C, D) \ldots \]

This appears to violate the usual undirected tree semantics of a factor per edge.

**However:** the appearance of the same variables in multiple nodes introduces dependencies:

- Introduce copy variables on each side of the separator.
- Factors on nodes no longer overlap.
- New delta-function factors on separators enforce consistency amongst copies:

\[ P(\mathcal{X}) = \ldots f_{ABC}(A, B^{(1)}, C^{(1)}) \delta(B^{(1)} - B^{(2)}) \delta(C^{(1)} - C^{(2)}) f_{BCD}(B^{(2)}, C^{(2)}, D) \ldots \]

Recall: BP on undirected trees

We can use this view to define BP messages on the junction tree:

- Copy and partition clique variables \( \mathcal{X}_C \):
  - Unshared variables: \( \mathcal{X}_C^{(1)} = \mathcal{X}_C \cup \mathcal{X}_s \)
  - Variables in incoming separators: \( \mathcal{X}_C^{(2)} \) (matching variables \( \mathcal{X}_C^{(1)} \) in \( k \in \text{ne}(i) \)).
  - Variables in outgoing separator: \( \mathcal{X}_C^{(2)} \) (matching variables \( \mathcal{X}_C^{(1)} \) in clique \( j \)).
  - (Variables that appear in more than one separator will need additional copies.)

\[ M_{i \rightarrow j}(\mathcal{X}) = \sum_{x_{T_{j \rightarrow i}}} \prod_{(i'j') \in E_{T_{j \rightarrow i}}} \prod_{(i'j') \in E_{T_{j \rightarrow i}}} f_{ij'}(X_{i'}, X_{j'}) \]

\[ \propto P_{T_{j \rightarrow i}}(\mathcal{X}_j) \propto \prod_{x_i \in \text{ne}(\mathcal{X}_j) \setminus x_i} M_{i \rightarrow j}(\mathcal{X}_i) \]

\[ = \sum_{x_j} \prod_{x_i \in \text{ne}(\mathcal{X}_j) \setminus x_j} M_{i \rightarrow j}(\mathcal{X}_i) \]

We’ve now completed the transformation from a general model to a tree-structured graph.

- Belief propagation on the junction tree should allow us to efficiently compute posterior marginals for inference and learning.

Message passing on junction trees
Then the following is also a distribution (non-negative and sums to one) such that:

\[
q_{ij}(X_{ij}) = \sum_{X_i \setminus X_j} P(X) = \sum_{X_j \setminus X_i} P(X)
\]

This is called Shafer-Shenoy propagation.

### Consistency

The running intersection property and tree structure of the junction trees implies that local consistency between cliques and separator marginals guarantees global consistency. If \(q(X_C), r_i(X_S_i)\) are distributions such that

\[
\sum_{X_C \setminus X_S_i} q(X_C) = r_i(X_S_i)
\]

Then the following

\[
P(X) = \prod_{\text{cliques } i} q(X_C)\prod_{\text{separators } (i)} r_i(X_S_i)
\]

is also a distribution (non-negative and sums to one) such that:

\[
q(X_C) = \sum_{X \setminus X_C} P(X) \quad r_i(X_S_i) = \sum_{X \setminus X_S_i} P(X)
\]

Reparameterisation for message passing

Hugin propagation is a different (but equivalent) message passing algorithm. It is based upon the idea of reparameterisation. Initialize:

\[
q(X_C) \propto f(X_C) \quad r_i(X_S_i) \propto 1
\]

Then our probability distribution is initially

\[
P(X) \propto \prod_{\text{cliques } i} q(X_C)\prod_{\text{separators } (i)} r_i(X_S_i)
\]

A Hugin propagation update for \(i \rightarrow j\) is:

\[
q_i^{\text{new}}(X_S_i) = \sum_{X_C \setminus X_S_i} q(X_C) \quad q_i^{\text{new}}(X_C) = q(X_C) \frac{r_i^{\text{new}}(X_S_i)}{r_i(X_S_i)}
\]
Hugin propagation

Some properties of Hugin propagation:
- The defined distribution $P(X)$ is unchanged by the updates.
- Each update introduces a local consistency constraint:
  $$\sum_{X_C \setminus S_i} q(X_C) = r(X_{S_i})$$
- If each update $i \to j$ is carried out only after incoming updates $k \to i$ have been carried out, then each update needs only be carried out once.
- Each Hugin update is equivalent to the corresponding Shafer-Shenoy update.

Computational Costs of the Junction Tree Algorithm

- Most of the computational cost of the junction tree algorithm is incurred during the message passing phase.
- The running and memory costs of the message passing phase are both $O(\sum_i |X_C|)$.
  This can be significantly (exponentially) more efficient than brute force.
- The variable elimination ordering heuristic can have very significant impact on the message passing costs.
- For certain classes of graphical models (e.g. 2D lattice Markov random field) it is possible to hand-craft an efficient ordering.

Other Inference Algorithms

There are other approaches to inference in graphical models which may be more efficient under specific conditions:

**Cutset conditioning:** or “reasoning by assumptions”. Find a small set of variables which, if they were given (i.e. known) would render the remaining graph “simpler”. For each value of these variables run some inference algorithm on the simpler graph, and average the resulting beliefs with the appropriate weights.

**Loopy belief propagation:** just use belief propagation even though there are loops. No guarantee of convergence, but often works well in practice. Some (weak) guarantees about the nature of the answer if the message passing does converge.

Learning in Graphical Models

We have discussed inference at length — what about learning? The factored structure implied by the graph also makes learning easy.

Consider data points comprising observations of a subset of variables. ML learning => adjust parameters to maximise:

$$\mathcal{L} = P(X_{\text{obs}}|\theta) = \sum_{X_{\text{unobs}}} P(X_{\text{obs}}, X_{\text{unobs}}|\theta)$$

by EM, need to maximise

$$\mathcal{F}(q, \theta) = \langle \log P(X_{\text{obs}}, X_{\text{unobs}}|\theta) \rangle_{q(X_{\text{unobs}})} - \log q(X_{\text{unobs}}) + H(q)$$

So learning only requires posterior marginals on cliques (obtained by messaging passing) and updates on cliques; c.f. the Baum-Welch procedure for HMMs.