

# Probabilistic & Unsupervised Learning

## Back to Bayes: Model selection, Hyperparameter optimisation, and Gaussian Processes

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### Model selection

Models (labelled by  $m$ ) have parameters  $\theta_m$  that specify the probability of data:

$$P(\mathcal{D}|\theta_m, m).$$

If model is known, learning  $\theta_m$  means finding posterior or point estimate (ML, MAP, ...).

What if we need to learn the model too?

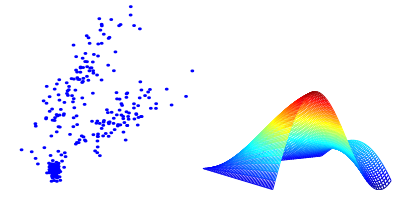
- ▶ Could combine models into a single “supermodel”, with composite parameter  $(m, \theta_m)$ .
  - ▶ ML learning will **overfit**: favours most flexible (nested) model with most parameters, even if the data actually come from a simpler one.
  - ▶ Density function on composite parameter space (union of manifolds of different dimensionalities) difficult to define  $\Rightarrow$  MAP learning ill-posed.
  - ▶ Joint posterior difficult to compute — dimension of composite parameter varies [although Monte-Carlo methods may be able to sample from such a posterior.]

$\Rightarrow$  Separate model selection step:

$$P(\theta_m, m|\mathcal{D}) = \underbrace{P(\theta_m|m, \mathcal{D})}_{\text{model-specific posterior}} \cdot \underbrace{P(m|\mathcal{D})}_{\text{model selection}}$$

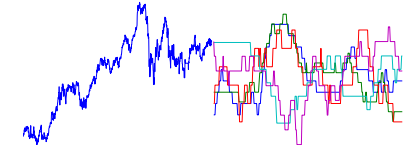
### Learning model structure

How many clusters in the data?



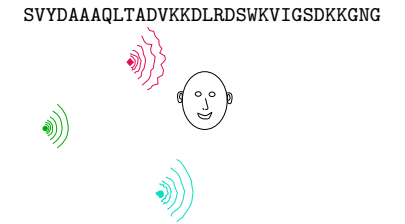
How smooth should the function be?

Is this input relevant to predicting that output?



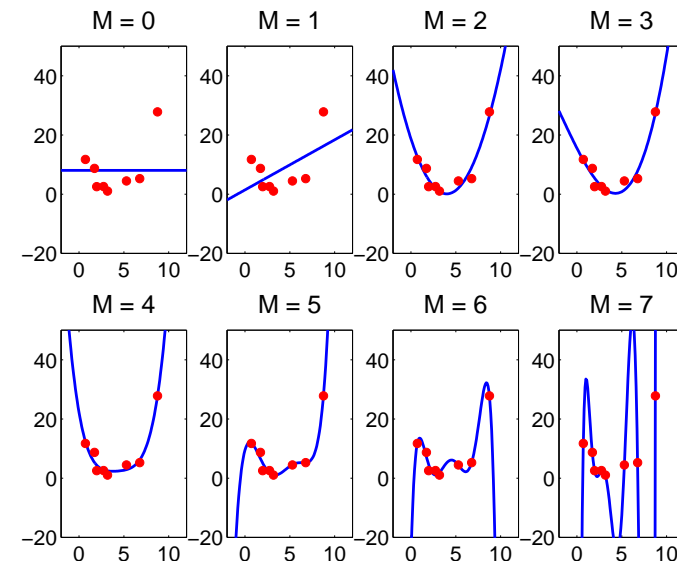
What is the order of a dynamical system?

How many states in a hidden Markov model?



How many auditory sources in the input?

### Model complexity and overfitting: a simple example



## Model selection

Given models labeled by  $m$  with parameters  $\theta_m$ , identify the “correct” model for data  $\mathcal{D}$ .

ML/MAP has no good answer:  $P(\mathcal{D}|\theta_m^{\text{ML}})$  is always larger for more complex (nested) models.

### Neyman-Pearson hypothesis testing

- ▶ For **nested** models. Starting with simplest model ( $m = 1$ ), compare (e.g. by likelihood ratio test) **null hypothesis**  $m$  to **alternative**  $m + 1$ . Continue until  $m + 1$  is rejected.
- ▶ Tests often only exact asymptotically in data number.
- ▶ Conservative (N-P hypothesis tests are asymmetric by design).

### Likelihood validation

- ▶ Partition data into disjoint *training* and *validation* data sets  $\mathcal{D} = \mathcal{D}_{\text{tr}} \cup \mathcal{D}_{\text{vid}}$ . Choose model with greatest  $P(\mathcal{D}_{\text{vid}}|\theta_m^{\text{ML}})$ , with  $\theta_m^{\text{ML}} = \text{argmax} P(\mathcal{D}_{\text{tr}}|\theta)$ . [Or, better, greatest  $P(\mathcal{D}_{\text{vid}}|\mathcal{D}_{\text{tr}}, m)$ .]
- ▶ **Consistent**; and selects most **useful** model, even if all are **incorrect**.
- ▶ May be biased towards simpler models; often high-variance.
- ▶ **Cross-validation** uses multiple partitions and averages likelihoods.

### Bayesian model selection

- ▶ Choose **most likely** model:  $\text{argmax} P(m|\mathcal{D})$ .
- ▶ **Consistent**; probabilistically principled **if true model is in set being considered**, but sensitive to assumed priors etc.
- ▶ Posterior probabilities can **weight** models for combined predictions (avoiding selection).

## The Bayesian Occam’s razor

**Occam’s Razor** is a principle of scientific philosophy: of two explanations adequate to explain the same set of observations, the simpler should always be preferred.

Bayesian inference formalises and *automatically* implements a form of Occam’s Razor.

Compare model classes  $m$  using their posterior probability given the data:

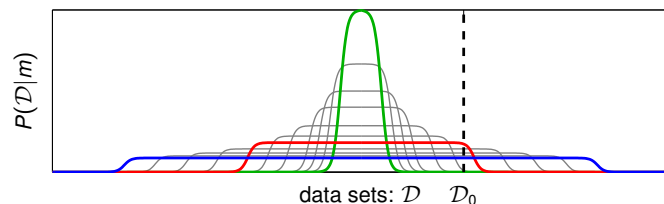
$$P(m|\mathcal{D}) = \frac{P(\mathcal{D}|m)P(m)}{P(\mathcal{D})}, \quad P(\mathcal{D}|m) = \int_{\Theta_m} P(\mathcal{D}|\theta_m, m)P(\theta_m|m) d\theta_m$$

$P(\mathcal{D}|m)$ : The probability that *randomly selected* parameter values from the model class would generate data set  $\mathcal{D}$ .

Model classes that are **too simple** are unlikely to generate the observed data set.

Model classes that are **too complex** can generate many possible data sets, so again, they are unlikely to generate that particular data set at random.

Like Goldilocks, we favour a model that is **just right**.



## Bayesian model selection: some terminology

A **model class**  $m$  is a set of distributions parameterised by  $\theta_m$ , e.g. the set of all possible mixtures of  $m$  Gaussians.

The model implies both a **prior** over the parameters  $P(\theta_m|m)$ , and a **likelihood** of data given parameters (which might require integrating out latent variables)  $P(\mathcal{D}|\theta_m, m)$ .

The **posterior** distribution over parameters is

$$P(\theta_m|\mathcal{D}, m) = \frac{P(\mathcal{D}|\theta_m, m)P(\theta_m|m)}{P(\mathcal{D}|m)}$$

The **marginal probability** of the data under model class  $m$  is:

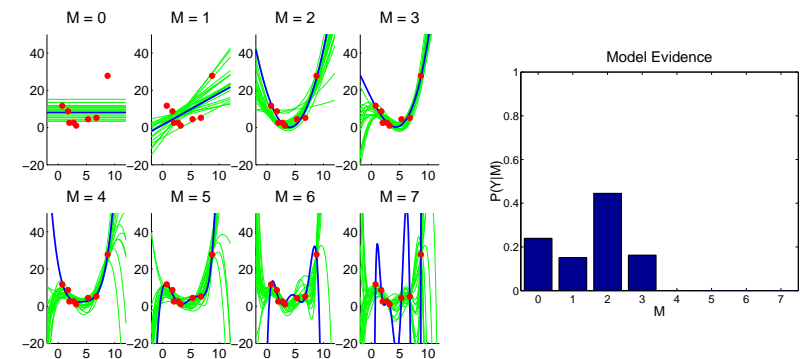
$$P(\mathcal{D}|m) = \int_{\Theta_m} P(\mathcal{D}|\theta_m, m)P(\theta_m|m) d\theta_m$$

(also called the **Bayesian evidence** for model  $m$ ).

The ratio of two marginal probabilities (or sometimes its log) is known as the **Bayes factor**:

$$\frac{P(\mathcal{D}|m)}{P(\mathcal{D}|m')} = \frac{P(m|\mathcal{D}) p(m')}{P(m'|\mathcal{D}) p(m)}$$

## Bayesian model comparison: Occam’s razor at work



## Conjugate-exponential families (recap)

Can we compute  $P(\mathcal{D}|m)$ ? . . . . Sometimes.

Suppose  $P(\mathcal{D}|\theta_m, m)$  is a member of the exponential family:

$$P(\mathcal{D}|\theta_m, m) = \prod_{i=1}^N P(\mathbf{x}_i|\theta_m, m) = \prod_{i=1}^N e^{\mathbf{s}(\mathbf{x}_i)^\top \theta_m - A(\theta_m)}.$$

If our prior on  $\theta_m$  is **conjugate**:

$$P(\theta_m|m) = e^{\mathbf{s}_p^\top \theta_m - n_p A(\theta_m)} / Z(\mathbf{s}_p, n_p)$$

then the joint is in the same family:

$$P(\mathcal{D}, \theta_m|m) = e^{(\sum_i \mathbf{s}(\mathbf{x}_i) + \mathbf{s}_p)^\top \theta_m - (N + n_p)A(\theta_m)} / Z(\mathbf{s}_p, \rho)$$

and so:

$$P(\mathcal{D}|m) = \int d\theta_m P(\mathcal{D}, \theta_m|m) = \frac{Z(\sum_i \mathbf{s}(\mathbf{x}_i) + \mathbf{s}_p, N + n_p)}{Z(\mathbf{s}_p, \rho)} \leftarrow \begin{array}{l} \text{posterior volume} \\ \text{prior volume} \end{array}$$

While the intuition is general, tractability is a special case. Thus, we must approximate . . .

## Laplace approximation

We want to find  $P(\mathcal{D}|m) = \int P(\mathcal{D}, \theta_m|m) d\theta_m$ .

As data size  $N$  grows (relative to parameter count  $d$ ),  $\theta_m$  becomes more constrained  $\Rightarrow P(\mathcal{D}, \theta_m|m) \propto P(\theta_m|\mathcal{D}, m)$  becomes concentrated on posterior mode  $\theta_m^*$ .

**Idea:** approximate  $\log P(\mathcal{D}, \theta_m|m)$  to second-order around  $\theta^*$ .

$$\begin{aligned} \int P(\mathcal{D}, \theta_m|m) d\theta_m &= \int e^{\log P(\mathcal{D}, \theta_m|m)} d\theta_m \\ &= \int e^{\log P(\mathcal{D}, \theta_m^*|m) + \underbrace{\nabla \log P(\mathcal{D}, \theta_m^*|m)}_{=0} \cdot (\theta_m - \theta_m^*) + \frac{1}{2} (\theta_m - \theta_m^*)^\top \underbrace{\nabla^2 \log P(\mathcal{D}, \theta_m^*|m)}_{=-A} (\theta_m - \theta_m^*)} d\theta_m \\ &= \int P(\mathcal{D}, \theta_m^*|m) e^{-\frac{1}{2} (\theta_m - \theta_m^*)^\top A (\theta_m - \theta_m^*)} d\theta_m \\ &= P(\mathcal{D}|\theta_m^*, m) P(\theta_m^*|m) (2\pi)^{\frac{d}{2}} |A|^{-\frac{1}{2}} \end{aligned}$$

$A = -\nabla^2 \log P(\mathcal{D}, \theta_m^*|m)$  is the negative Hessian of  $\log P(\mathcal{D}, \theta|m)$  evaluated at  $\theta_m^*$ .

This is equivalent to approximating the posterior by a Gaussian: an approximation which is asymptotically correct.

## Practical Bayesian approaches

### Laplace approximation:

- Approximate posterior by a Gaussian centred at the maximum *a posteriori* parameter estimate.

### Bayesian Information Criterion (BIC)

- an asymptotic ( $N \rightarrow \infty$ ) approximation.

### Variational Bayes

- Lower bound on the marginal probability.
- Biased estimate.
- Easy and fast, and often better than Laplace or BIC.

### Monte Carlo methods:

- (Annealed) Importance sampling: estimate evidence using samples  $\theta^{(i)}$  from arbitrary  $f(\theta)$ :

$$\sum_i \frac{P(\mathcal{D}|\theta^{(i)}, m) P(\theta^{(i)}|m)}{f(\theta^{(i)})} \rightarrow \int d\theta f(\theta) \frac{P(\mathcal{D}, \theta|m)}{f(\theta)} = P(\mathcal{D}|m)$$

- "Reversible jump" Markov Chain Monte Carlo: sample from posterior on composite  $(m, \theta_m)$ . # samples for each  $m \propto p(m|\mathcal{D})$ .
- Both exact in the limit of infinite samples, but may have high variance with finite samples.

Not an exhaustive list (Bethe approximations, Expectation propagation, . . .)

We will discuss Laplace and BIC now, leaving the rest for the second half of course.

## Bayesian Information Criterion (BIC)

BIC can be obtained from the Laplace approximation:

$$\log P(\mathcal{D}|m) \approx \log P(\theta_m^*|m) + \log P(\mathcal{D}|\theta_m^*, m) + \frac{d}{2} \log 2\pi - \frac{1}{2} \log |A|$$

We have

$$A = -\nabla^2 \log P(\mathcal{D}, \theta^*|m) = -\nabla^2 \log P(\mathcal{D}|\theta^*, m) - \nabla^2 \log P(\theta^*|m)$$

So as  $N = |\mathcal{D}| \rightarrow \infty$ ,  $A \rightarrow NA_0 + \text{constant}$ , for fixed PD matrix  $A_0 = \langle -\nabla^2 \log P(\mathbf{x}|\theta^*, m) \rangle$ .  $\Rightarrow \log |A| \rightarrow \log |NA_0| = \log(N^d |A_0|) = d \log N + \log |A_0|$ .

Retaining only terms that grow with  $N$  we get:

$$\log P(\mathcal{D}|m) \approx \log P(\mathcal{D}|\theta_m^*, m) - \frac{d}{2} \log N$$

Properties:

- Quick and easy to compute.
- Does not depend on prior.
- We can use the ML estimate of  $\theta$  instead of the MAP estimate (= as  $N \rightarrow \infty$ ).
- Related to the "Minimum Description Length" (MDL) criterion (Asst 2).
- Assumes that in the large sample limit, all the parameters are well-determined (i.e. the model is **identifiable**; otherwise,  $d$  should be the number of **well-determined** parameters).
- Neglects multiple modes (e.g. permutations in a MoG).

## Hyperparameters and Evidence optimisation

In some cases, we need to choose between a family of continuously parameterised models.

$$P(\mathcal{D}|\eta) = \int P(\mathcal{D}|\theta)P(\theta|\eta) d\theta$$

↑  
hyperparameters

This choice can be made by ascending the gradient in:

- ▶ the exact evidence (if tractable).
- ▶ the approximated evidence (Laplace, EP, Bethe, ...)
- ▶ a free-energy bound on the evidence (Variational Bayes)

or by placing a **hyperprior** on the hyperparameters  $\eta$ , and sampling from the posterior

$$P(\eta|\mathcal{D}) = \frac{P(\mathcal{D}|\eta)P(\eta)}{P(\mathcal{D})}$$

using Markov chain Monte Carlo sampling.

## The evidence for linear regression

- ▶ The posterior on  $\mathbf{w}$  is normal:  $\Sigma_{\mathbf{w}} = \left(\frac{XX^T}{\sigma^2} + C^{-1}\right)^{-1}$ ;  $\bar{\mathbf{w}} = \Sigma_{\mathbf{w}} \frac{XY^T}{\sigma^2}$ .

Note:  $X$  is a matrix where columns are input vectors, and  $Y$  is a row vector of corresponding predicted outputs.

- ▶ The evidence,  $\mathcal{E}(C, \sigma^2) = \int P(Y|X, \mathbf{w}, \sigma^2)P(\mathbf{w}|C) d\mathbf{w}$ , is given by:

$$\mathcal{E}(C, \sigma^2) = \sqrt{\frac{|2\pi\Sigma_{\mathbf{w}}|}{|2\pi\sigma^2 I| |2\pi C|}} \exp\left(-\frac{1}{2} Y \left(\frac{I}{\sigma^2} - \frac{X^T \Sigma_{\mathbf{w}} X}{\sigma^4}\right) Y^T\right)$$

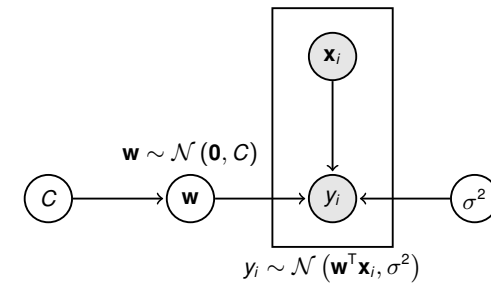
- ▶ For optimization, general forms for the gradients are available. If  $\theta$  is a parameter in  $C$ :

$$\frac{\partial}{\partial \theta} \log \mathcal{E}(C, \sigma^2) = \frac{1}{2} \text{Tr} \left[ (C - \Sigma_{\mathbf{w}} - \bar{\mathbf{w}}\bar{\mathbf{w}}^T) \frac{\partial}{\partial \theta} C^{-1} \right]$$

$$\frac{\partial}{\partial \sigma^2} \log \mathcal{E}(C, \sigma^2) = \frac{1}{\sigma^2} \left( -N + \text{Tr} [I - \Sigma_{\mathbf{w}} C^{-1}] + \frac{1}{\sigma^2} (Y - \bar{\mathbf{w}}^T X)(Y - \bar{\mathbf{w}}^T X)^T \right)$$

## Evidence optimisation in linear regression

Consider simple linear regression:



- ▶ Maximize

$$P(y_1 \dots y_N | \mathbf{x}_1 \dots \mathbf{x}_N, C, \sigma^2) = \int P(y_1 \dots y_N | \mathbf{x}_1 \dots \mathbf{x}_N, \mathbf{w}, \sigma^2) P(\mathbf{w}|C) d\mathbf{w}$$

to find optimal values of  $C, \sigma$ .

- ▶ Compute the posterior  $P(\mathbf{w}|y_1 \dots y_N, \mathbf{x}_1 \dots \mathbf{x}_N, C, \sigma^2)$  given these optimal values.

## Automatic Relevance Determination

The most common form of evidence optimization for regression (due to MacKay and Neal) takes  $C^{-1} = \text{diag}(\alpha)$  (i.e.  $w_i \sim \mathcal{N}(0, \alpha_i^{-1})$ ) and then optimizes the precisions  $\{\alpha_i\}$ .

Setting the gradients to 0 and solving gives

$$\alpha_i^{\text{new}} = \frac{1 - \alpha_i [\Sigma_{\mathbf{w}}]_{ii}}{\bar{\mathbf{w}}_i^2}$$

$$(\sigma^2)^{\text{new}} = \frac{(Y - \bar{\mathbf{w}}^T X)(Y - \bar{\mathbf{w}}^T X)^T}{N - \sum_i (1 - [\Sigma_{\mathbf{w}}]_{ii} \alpha_i)}$$

During optimization the  $\alpha_i$ s meet one of two fates

$$\begin{aligned} \alpha_i \rightarrow \infty &\Rightarrow w_i = 0 && \text{irrelevant input } x_i \\ \alpha_i \text{ finite} &\Rightarrow w_i = \text{argmax } P(w_i | X, Y, \alpha_i) && \text{relevant input } x_i \end{aligned}$$

This procedure, **Automatic Relevance Determination** (ARD), yields **sparse** solutions that improve on ML regression. (cf. L<sub>1</sub>-regression or LASSO).

Evidence optimisation is also called **maximum marginal likelihood** or **ML-2** (Type 2 maximum likelihood).

## Prediction averaging

Sometimes, our goal is not to learn the model structure or parameters (or their posteriors); but rather to predict the (conditional) density at a new data point.

The Bayesian approach in this case should **integrate out the parameters**:

- Density (unsupervised learning):  $\mathcal{D} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$

$$p(\mathbf{x}|\mathcal{D}, m) = \int d\theta p(\mathbf{x}|\theta, m)p(\theta|\mathcal{D}, m)$$

- Predictions (supervised learning):  $\mathcal{D} = \{(\mathbf{x}_1, y_1)(\mathbf{x}_2, y_2) \dots (\mathbf{x}_n, y_n)\}$

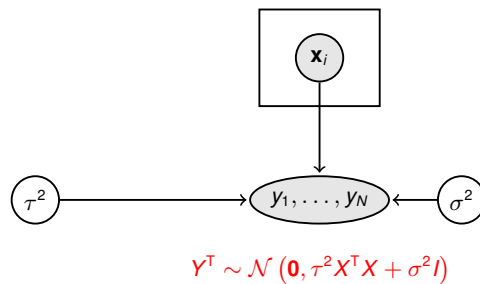
$$p(\mathbf{y}|\mathbf{x}, \mathcal{D}, m) = \int d\theta p(\mathbf{y}|\mathbf{x}, \theta, m)p(\theta|\mathcal{D}, m)$$

The integral naturally favours predictions associated with large volumes in the posterior, and so incorporates an Occam-like factor.

In principle, predictions may resist overfitting even with an infinitely complex model [or, put another way, the marginalised model has **no** parameters]  $\Rightarrow$  **Bayesian nonparametrics**.

Taking this approach to (non)linear regression leads to a powerful supervised learning method called **Gaussian process regression**.

## Marginalised linear regression



Integrate out  $\mathbf{w}$  in the model: the **joint** distribution of  $y_1, \dots, y_N$  given  $\mathbf{x}_1, \dots, \mathbf{x}_N$  is Gaussian. The means and covariances are:

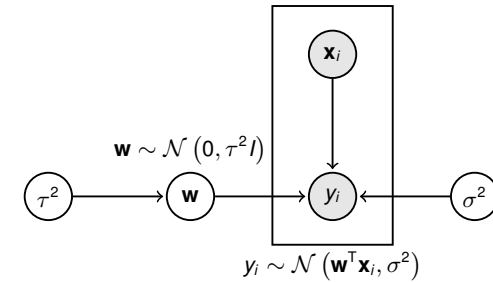
$$E[y_i] = E[\mathbf{w}^T \mathbf{x}_i] = \mathbf{0}^T \mathbf{x}_i = 0$$

$$E[(y_i - \bar{y}_i)^2] = E[(\mathbf{x}_i^T \mathbf{w})(\mathbf{w}^T \mathbf{x}_i)] + \sigma^2 = \tau^2 \mathbf{x}_i^T \mathbf{x}_i + \sigma^2$$

$$E[(y_i - \bar{y}_i)(y_j - \bar{y}_j)] = E[(\mathbf{x}_i^T \mathbf{w})(\mathbf{w}^T \mathbf{x}_j)] = \tau^2 \mathbf{x}_i^T \mathbf{x}_j$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \Bigg| \mathbf{x}_1, \dots, \mathbf{x}_N \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} \tau^2 \mathbf{x}_1^T \mathbf{x}_1 + \sigma^2 & \tau^2 \mathbf{x}_1^T \mathbf{x}_2 & \dots & \tau^2 \mathbf{x}_1^T \mathbf{x}_N \\ \tau^2 \mathbf{x}_2^T \mathbf{x}_1 & \tau^2 \mathbf{x}_2^T \mathbf{x}_2 + \sigma^2 & & \tau^2 \mathbf{x}_2^T \mathbf{x}_N \\ \vdots & & \ddots & \vdots \\ \tau^2 \mathbf{x}_N^T \mathbf{x}_1 & \tau^2 \mathbf{x}_N^T \mathbf{x}_2 & \dots & \tau^2 \mathbf{x}_N^T \mathbf{x}_N + \sigma^2 \end{bmatrix} \right)$$

## Prediction averaging in linear regression



Let  $X = [\mathbf{x}_1 \dots \mathbf{x}_N]$ ,  $Y = [y_1 \dots y_N]$ . Then (as we've seen)

$$\mathbf{w}|\mathcal{D} \sim \mathcal{N}(\bar{\mathbf{w}}, \Sigma_{\mathbf{w}})$$

where  $\Sigma_{\mathbf{w}} = (\frac{1}{\sigma^2} X X^T + \frac{1}{\tau^2} I)^{-1}$  and  $\bar{\mathbf{w}} = \frac{1}{\sigma^2} \Sigma_{\mathbf{w}} X Y^T$

Thus, given a new input vector  $\mathbf{x}$ , the predicted output  $y$  (integrating out  $\mathbf{w}$ ) is:

$$y|\mathbf{x} \sim \mathcal{N}(\bar{\mathbf{w}}^T \mathbf{x}, \mathbf{x}^T \Sigma_{\mathbf{w}} \mathbf{x} + \sigma^2).$$

Added variance  $\mathbf{x}^T \Sigma_{\mathbf{w}} \mathbf{x}$  comes from posterior uncertainty in  $\mathbf{w}$  (cf Factor Analysis).

## Predictions with marginalised regression

Now, include the test input vector  $\mathbf{x}$  and test output  $y$ :

$$\begin{bmatrix} Y^T \\ y \end{bmatrix} \Bigg| X, \mathbf{x} \sim \mathcal{N} \left( \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}, \begin{bmatrix} \tau^2 X^T X + \sigma^2 I & \tau^2 X^T \mathbf{x} \\ \tau^2 \mathbf{x}^T X & \tau^2 \mathbf{x}^T \mathbf{x} + \sigma^2 \end{bmatrix} \right) = \mathcal{N} \left( \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}, \begin{bmatrix} \tilde{K}_{XX} & \tilde{K}_{Xx} \\ \tilde{K}_{xX} & \tilde{K}_{xx} \end{bmatrix} \right)$$

We can find  $y|Y$  by the standard multivariate Gaussian result:

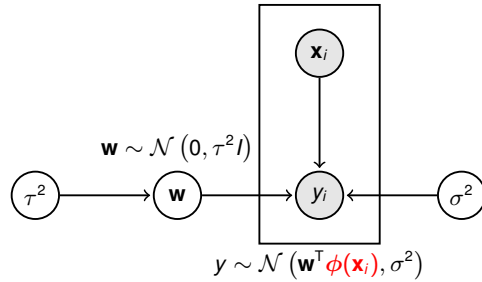
$$\begin{bmatrix} \mathbf{a} \\ b \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}, \begin{bmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{bmatrix} \right) \Rightarrow b|\mathbf{a} \sim \mathcal{N}(K_{BA} K_{AA}^{-1} \mathbf{a}, K_{BB} - K_{BA} K_{AA}^{-1} K_{AB})$$

So

$$\begin{aligned} y|Y, X, \mathbf{x} &\sim \mathcal{N} \left( \tilde{K}_{xx} \tilde{K}_{XX}^{-1} Y^T, \tilde{K}_{xx} - \tilde{K}_{xx} \tilde{K}_{XX}^{-1} \tilde{K}_{xx} \right) \\ &\sim \mathcal{N} \left( \tau^2 \mathbf{x}^T X (\tau^2 X^T X + \sigma^2 I)^{-1} Y^T, \tau^2 \mathbf{x}^T \mathbf{x} + \sigma^2 - \tau^2 \mathbf{x}^T X (\tau^2 X^T X + \sigma^2 I)^{-1} \tau^2 X^T \mathbf{x} \right) \\ &\sim \mathcal{N} \left( \underbrace{\mathbf{x}^T \frac{1}{\sigma^2} \Sigma X Y^T}_{\bar{w}}, \underbrace{\mathbf{x}^T \Sigma \mathbf{x} + \sigma^2}_{\Sigma_{\mathbf{w}}} \right) \quad \Sigma = \left( \frac{1}{\sigma^2} X X^T + \frac{1}{\tau^2} I \right)^{-1} \quad [\text{Matrix Inv. Lem.}] \end{aligned}$$

- Same answer as obtained by integrating wrt **posterior** over  $\mathbf{w}$ .
- Evidence  $P(Y|X)$  is just joint Gaussian probability; reduces to previous expression.
- Thus, Bayesian linear regression can be derived from a **joint, parameter-free** distribution on all the outputs conditioned on all the inputs.

## Nonlinear regression



Introduce nonlinear mapping  $\mathbf{x} \mapsto \phi(\mathbf{x})$ . Each element of  $\phi(\mathbf{x})$  is a (nonlinear) **feature** extracted from  $\mathbf{x}$ . May be many more features than elements in  $\mathbf{x}$ .

The regression function  $f(\mathbf{x}) = \mathbf{w}^T \phi(\mathbf{x})$  is nonlinear, but outputs  $Y$  are still jointly Gaussian:

$$Y^T | X \sim \mathcal{N}(0_N, \tau^2 \Phi^T \Phi + \sigma^2 I_N)$$

where the  $i^{\text{th}}$  column of matrix  $\Phi$  is  $\phi(\mathbf{x}_i)$ .

Proceeding as before, the predictive distribution over  $y$  for a test input  $\mathbf{x}$  is:

$$y | \mathbf{x}, Y, X \sim \mathcal{N}(\tilde{K}_{\mathbf{x}X} \tilde{K}_{XX}^{-1} Y^T, \tilde{K}_{\mathbf{x}\mathbf{x}} - \tilde{K}_{\mathbf{x}X} \tilde{K}_{XX}^{-1} \tilde{K}_{X\mathbf{x}})$$

where, now  $\tilde{K}_{XX} = \tau^2 \Phi^T \Phi + \sigma^2 I$ ;  $\tilde{K}_{\mathbf{x}X} = \tau^2 \phi(\mathbf{x})^T \Phi$  and  $\tilde{K}_{\mathbf{x}\mathbf{x}} = \tau^2 \phi(\mathbf{x})^T \phi(\mathbf{x}) + \sigma^2$ .

## Regression using the covariance kernel

For Bayesian regression, prediction depends on  $\tilde{K}(\mathbf{x}, \mathbf{x})$  rather than explicitly on  $\phi(\mathbf{x})$ .

So we can define the joint in terms of  $\tilde{K}$  *implicitly* using a (potentially infinite-dimensional) feature map  $\phi(\mathbf{x})$ .

$$Y | X, \tilde{K} \sim \mathcal{N}(0_N, \tilde{K}_{XX})$$

where the  $i, j$  entry in the covariance matrix  $\tilde{K}_{XX}$  is  $\tilde{K}(\mathbf{x}_i, \mathbf{x}_j)$ .

This is the **kernel trick**.

**Prediction:** compute the predictive distribution of  $y$  conditioned on  $Y$  as before:

$$y | \mathbf{x}, X, Y, \tilde{K} \sim \mathcal{N}(\tilde{K}_{\mathbf{x}X}^T \tilde{K}_{XX}^{-1} Y^T, \tilde{K}_{\mathbf{x}\mathbf{x}} - \tilde{K}_{\mathbf{x}X}^T \tilde{K}_{XX}^{-1} \tilde{K}_{X\mathbf{x}})$$

where now  $[\tilde{K}_{XX}]_{ij} = \tilde{K}(\mathbf{x}_i, \mathbf{x}_j)$ ;  $[\tilde{K}_{\mathbf{x}X}]_i = \tilde{K}(\mathbf{x}_i, \mathbf{x})$  and  $\tilde{K}_{\mathbf{x}\mathbf{x}} = \tilde{K}(\mathbf{x}, \mathbf{x})$ .

**Evidence:** given by the Gaussian likelihood:

$$P(Y | X, \tilde{K}) = |2\pi \tilde{K}_{XX}|^{-\frac{1}{2}} e^{-\frac{1}{2} Y^T \tilde{K}_{XX}^{-1} Y}$$

**Evidence optimisation:** the covariance kernel  $\tilde{K}$  often has (hyper)parameters, and these can be optimized by gradient ascent in  $\log P(Y | X, \tilde{K})$ .

## The covariance kernel

$$Y^T | X \sim \mathcal{N}(\mathbf{0}_N, \tau^2 \Phi^T \Phi + \sigma^2 I_N)$$

The covariance of the output vector  $Y$  plays a central role in the development of the theory of Gaussian processes.

Define a **covariance kernel** function  $\tilde{K} : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}$  such that if  $\mathbf{x}, \mathbf{x}' \in \mathbb{X}$  are two input vectors with corresponding outputs  $y, y'$ , then

$$\tilde{K}(\mathbf{x}, \mathbf{x}') = \text{Cov}[y, y'] = E[yy'] - E[y]E[y']$$

In the nonlinear regression example we have  $\tilde{K}(\mathbf{x}, \mathbf{x}') = \tau^2 \phi(\mathbf{x})^T \phi(\mathbf{x}') + \sigma^2 \delta_{\mathbf{x}=\mathbf{x}'}$ .

Any covariance kernel  $K$  has two properties:

- ▶ **Symmetric:**  $K(\mathbf{x}, \mathbf{x}') = K(\mathbf{x}', \mathbf{x})$  for all  $\mathbf{x}, \mathbf{x}'$ .
- ▶ **Positive semidefinite:** the matrix  $[K(\mathbf{x}_i, \mathbf{x}_j)]$  formed by any finite set of input vectors  $\mathbf{x}_1, \dots, \mathbf{x}_N$  is positive semidefinite.

**Theorem:** A covariance kernel  $K : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}$  is symmetric and positive semidefinite if and only if there is a feature map  $\phi : \mathbb{X} \mapsto \mathbb{H}$  such that

$$K(\mathbf{x}, \mathbf{x}') = \phi(\mathbf{x})^T \phi(\mathbf{x}')$$

The feature space  $\mathbb{H}$  might be an infinite-dimensional Hilbert space.

## The Gaussian process

A covariance kernel  $K(\mathbf{x}, \mathbf{x}')$  (and mean function  $m(\mathbf{x})$ ) defined on a domain  $\mathbb{X}$  defines a **Gaussian process** (GP): a stochastic process (ie collection of random variables) on  $\mathbb{R}$  indexed by  $\mathbf{x} \in \mathbb{X}$ , any finite subset of which have (consistent) Gaussian distributions.

Let  $f(\mathbf{x})$  be the random variable indexed by  $\mathbf{x}$ . Then a draw from the whole GP (ie for all  $\mathbf{x}$ ) is a random **function**  $f : \mathbb{X} \mapsto \mathbb{R}$ . We write

$$f(\cdot) \sim \mathcal{GP}(m(\cdot), K(\cdot, \cdot))$$

where the  $(\cdot)$ s emphasise that these are all functions.

The GP is defined such that, given a finite list of points  $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ , the joint distribution of the function values  $\mathbf{f} = [f(\mathbf{x}_1), \dots, f(\mathbf{x}_N)]^T$  is:

$$\mathbf{f} | X, K \sim \mathcal{N}(\mathbf{m}, K_{XX})$$

where, as usual,  $[K_{XX}]_{ij} = K(\mathbf{x}_i, \mathbf{x}_j)$  and  $[\mathbf{m}]_i = m(\mathbf{x}_i)$ . If we enlarge or reduce the set of  $\mathbf{x}$ s then the means and covariance matrices produced by fixed functions marginalise correctly.

For nonlinear regression,  $f(\cdot)$  could instead be defined by drawing the weight vector  $\mathbf{w} \in \mathbb{H}$  from the prior. But  $\mathbb{H}$  may be infinite dimensional  $\Rightarrow$  need an infinite-size object to make even a single prediction. In the GP view, each  $f(\mathbf{x})$  can be drawn separately.

## Regression with Gaussian processes

We seek to learn a function that maps inputs  $\mathbf{x}_1, \dots, \mathbf{x}_N$  to outputs  $y_1, \dots, y_N$ .

Instead of assuming a specific form, consider a random function drawn from a GP prior:

$$f(\cdot) \sim \mathcal{GP}(0, K(\cdot, \cdot)).$$

Any function is possible (no restriction on support) but some are (much) more likely *a priori*.

Observations  $y_i$  are taken to be noisy versions of the (almost surely continuous) latent  $f(\mathbf{x}_i)$ :

$$y_i | \mathbf{x}_i, f(\cdot) \sim \mathcal{N}(f(\mathbf{x}_i), \sigma^2) \quad [\text{so } Y \sim \mathcal{N}(0, \tilde{K}_{XX}) \text{ with } \tilde{K}_{XX} = K_{XX} + \sigma^2 I]$$

**Evidence:** given by the multivariate Gaussian likelihood:

$$P(Y|X) = |2\pi(K_{XX} + \sigma^2 I)|^{-\frac{1}{2}} e^{-\frac{1}{2} Y(K_{XX} + \sigma^2 I)^{-1} Y^T}$$

**Posterior:** on latent  $f$  is also a GP:

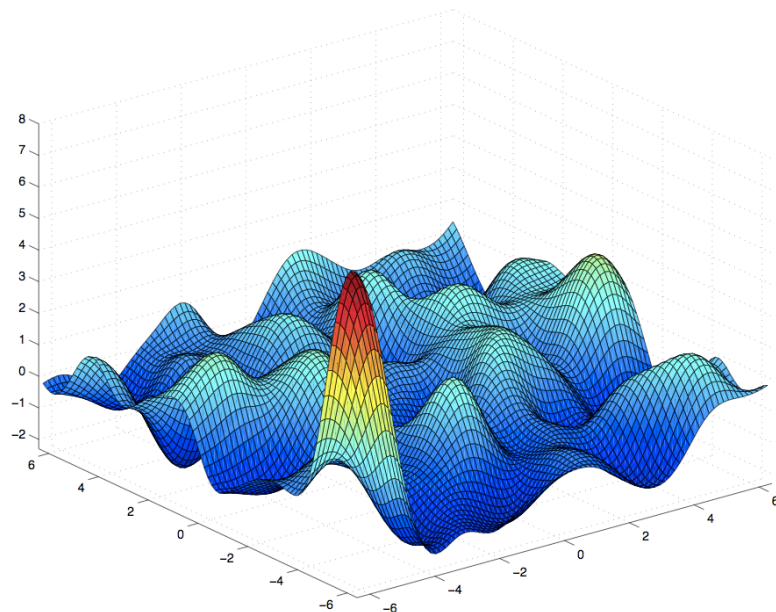
$$f(\cdot) | X, Y \sim \mathcal{GP}(K_{.X}(K_{XX} + \sigma^2 I)^{-1} Y^T, K(\cdot, \cdot) - K_{.X}(K_{XX} + \sigma^2 I)^{-1} K_{X.})$$

**Predictions:** posterior on  $f$ , plus observation noise:

$$y | X, Y, \mathbf{x} \sim \mathcal{N}(E[f(\mathbf{x}) | X, Y], \text{Var}[f(\mathbf{x}) | X, Y] + \sigma^2) = \mathcal{N}(K_{XX} \tilde{K}_{XX}^{-1} Y, K_{XX} \tilde{K}_{XX}^{-1} K_{XX} + \sigma^2)$$

**Evidence Optimisation:** gradient ascent in  $\log P(Y|X)$ .

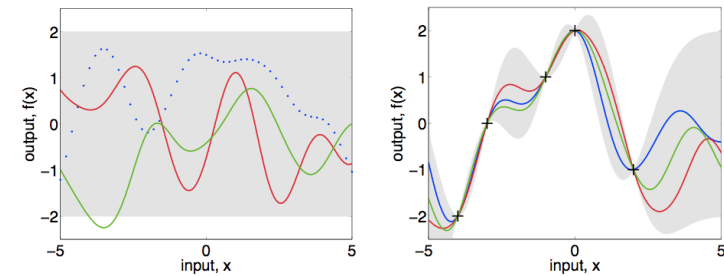
## Sample from a 2D Gaussian process



## Samples from a Gaussian process

We can draw sample functions from a GP by fixing a set of input vectors  $\mathbf{x}_1, \dots, \mathbf{x}_N$ , and drawing a sample  $f(\mathbf{x}_1), \dots, f(\mathbf{x}_N)$  from the corresponding multivariate Gaussian.

Example prior and posterior GPs:



Another approach is to

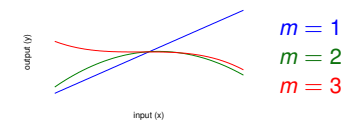
- ▶ sample  $f(\mathbf{x}_1)$  first,
- ▶ then  $f(\mathbf{x}_2) | f(\mathbf{x}_1)$ ,
- ▶ and generally  $f(\mathbf{x}_n) | f(\mathbf{x}_1), \dots, f(\mathbf{x}_{n-1})$  for  $n = 1, 2, \dots$

## Examples of covariance kernels

- Polynomial:

$$K(\mathbf{x}, \mathbf{x}') = (1 + \mathbf{x}^T \mathbf{x}')^m \quad m = 1, 2, \dots$$

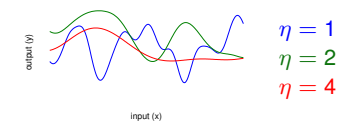
$f$  is inhomogeneous polynomial of degree  $m$



- Squared-exponential (or exponentiated-quadratic):

$$K(\mathbf{x}, \mathbf{x}') = \theta^2 e^{-\frac{\|\mathbf{x} - \mathbf{x}'\|^2}{2\eta^2}}$$

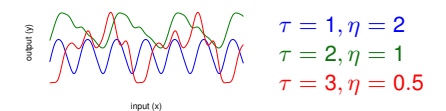
$f$  is smooth ( $C^\infty$  almost surely) on length scale  $\eta$



- Periodic (exp-sine):

$$K(x, x') = \theta^2 e^{-\frac{2 \sin^2(\pi(x-x')/\tau)}{\eta^2}}$$

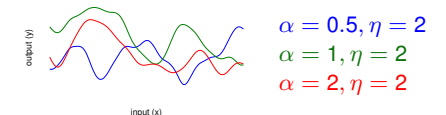
$f$  is smooth and periodic



- Rational Quadratic:

$$K(\mathbf{x}, \mathbf{x}') = \left(1 + \frac{\|\mathbf{x} - \mathbf{x}'\|^2}{2\alpha\eta^2}\right)^{-\alpha} \quad \alpha > 0$$

$f$  is smooth over multiple scales





## Forms of kernels

If  $K_1$  and  $K_2$  are covariance kernels, then so are:

- ▶ Rescaling:  $\alpha K_1$  for  $\alpha > 0$ .
- ▶ Addition:  $K_1 + K_2$
- ▶ Elementwise product:  $K_1 K_2$
- ▶ Mapping:  $K_1(\phi(\mathbf{x}), \phi(\mathbf{x}'))$  for some function  $\phi$ .

A covariance kernel is [translation-invariant](#) if

$$K(\mathbf{x}, \mathbf{x}') = h(\mathbf{x} - \mathbf{x}')$$

A GP with a translation-invariant covariance kernel is stationary: if  $f(\cdot) \sim \mathcal{GP}(0, K)$ , then so is  $f(\cdot - \mathbf{x}) \sim \mathcal{GP}(0, K)$  for each  $\mathbf{x}$ .

A covariance kernel is [radial](#) or [radially symmetric](#) if

$$K(\mathbf{x}, \mathbf{x}') = h(\|\mathbf{x} - \mathbf{x}'\|)$$

A GP with a radial covariance kernel is stationary with respect to translations, rotations, and reflections of the input space.

## Nonparametric Bayesian Models and Occam's Razor

Overparameterised models can [overfit](#). In the GP, the "parameter" is the function  $f(\mathbf{x})$  (or "weights" in non-linear feature space) which can be infinite-dimensional.

However, the Bayesian treatment integrates over these parameters: we never identify a single "best fit"  $f$ , just a posterior (and posterior mean). So  $f$  cannot be adjusted to overfit the data.

The GP is an example of the larger class of [nonparametric Bayesian models](#).

- ▶ Infinite number of parameters.
- ▶ Often constructed as the infinite limit of a nested family of finite models (sometimes equivalent to infinite model averaging).
- ▶ Parameters integrated out, so effective number of parameters to overfit is zero or small (hyperparameters).
- ▶ No need for model selection. Bayesian posterior on parameters will concentrate on "submodel" with largest integral automatically.
- ▶ No explicit need for Occam's razor, validation or added regularisation penalty.
- ▶ Examples include the Dirichlet process (infinite mixtures), Infinite Binary Prior (infinite binary factor models), Infinite HMM ...

## GP methods

- ▶ With suitable kernels, combinations of kernels, and hyperparameter learning, GPs can identify a wide range of functional dependence. (The "automated statistician" project starts with GPs).
- ▶ With approximation, the mapping from  $f$  to  $y$  may be taken to be non-Gaussian, allowing GP classification, ordinal regression, domain-specific noise and more.
- ▶ Functions in more complex hierarchical models may be drawn from GP priors:
  - ▶ GP latent variable model (GPLVM)
  - ▶ Stacked GPs
  - ▶ Deep GP networks
- ▶ Inference and learning require inversion of  $K_{XX}$ : scales as  $N^3$ . [Sparse](#) approximate methods reduce this to order  $N$ .
- ▶ State-of-the-art approach, particularly when data are limited.

## The end of part 1 (GI18)

When we resume:

- ▶ Non-conjugate models.
- ▶ Variational approximation.
- ▶ Expectation propagation.
- ▶ BP in loopy graphs.
- ▶ Convex relaxations.
- ▶ Sampling (Monte-Carlo) methods.
- ▶ [possibly autoencoders]