

Assignment 1

Unsupervised & Probabilistic Learning

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Due: Tuesday 22 Oct, 2019; 12:00

Assignments are due at noon on the date specified. The standard UCL late policy applies. To ensure legibility *typeset* your answers using LaTeX or a similar package and upload them as a single pdf document to the course moodle page. Show your working.

1. **[28 marks] Statistics and Distributions.** In the coming weeks we will be making extensive use of the following distributions, all of which belong to the exponential family. For each of these distributions, find:

- The standard exponential form, identifying the natural parameters in terms of the conventional parameters given in the table (i.e. the function $\phi(\theta)$), and the sufficient statistic (i.e. $\mathbf{T}(x)$).
- The expected value of the sufficient statistics in terms of the natural or conventional parameters (i.e. $\langle \mathbf{T}(x) \rangle_{p(x|\theta)}$). These expectations are often called the “mean” or “moment” parameters of the distribution. [Note: show your derivation of the expectations; don’t just look them up.]

The distributions to consider are:

Name	Domain	Symbol	Density or Probability fn
Multivariate Normal	\mathbb{R}^D	$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$	$ 2\pi\Sigma ^{-1/2} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}$
Binomial	\mathbb{Z}_{0-N}	$x \sim \text{Binom}(p)$	$\binom{N}{x} p^x (1-p)^{N-x}$
Multinomial	$[\mathbb{Z}_{0-N}]^D$	$\mathbf{x} \sim \text{Multinom}(\mathbf{p})$	$\frac{N!}{x_1! x_2! \dots x_D!} \prod_{d=1}^D p_d^{x_d}$
Poisson	\mathbb{Z}_{0+}	$x \sim \text{Poisson}(\mu)$	$\mu^x e^{-\mu} / x!$
Beta	$[0, 1]$	$x \sim \text{Beta}(\alpha, \beta)$	$\frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$
Gamma	\mathbb{R}_+	$x \sim \text{Gamma}(\alpha, \beta)$	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$
Dirichlet	$[0, 1]^D$	$\mathbf{x} \sim \text{Dirichlet}(\boldsymbol{\alpha})$	$\frac{\Gamma(\sum_{d=1}^D \alpha_d)}{\prod_{d=1}^D \Gamma(\alpha_d)} \prod_{d=1}^D x_d^{\alpha_d-1}$

[4 marks each]

2. **[7 marks] ML in the exponential family.**

Express the maximum-likelihood value of the *mean* parameters (as defined in the question above) of the general exponential family distribution

$$p(\mathbf{x}|\theta) = g(\theta) f(\mathbf{x}) e^{\theta^\top \mathbf{T}(\mathbf{x})}$$

as a function of a data set of iid observations $\mathcal{D} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$.

3. [25 marks] **Models for binary vectors.** Consider a data set of binary (black and white) images. Each image is arranged into a vector of pixels by concatenating the columns of pixels in the image. The data set has N images $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)}\}$ and each image has D pixels, where D is (number of rows \times number of columns) in the image. For example, image $\mathbf{x}^{(n)}$ is a vector $(x_1^{(n)}, \dots, x_D^{(n)})$ where $x_d^{(n)} \in \{0, 1\}$ for all $n \in \{1, \dots, N\}$ and $d \in \{1, \dots, D\}$.

(a) Explain why a multivariate Gaussian would not be an appropriate model for this data set of images. [5 marks]

Assume that the images were modelled as independently and identically distributed samples from a D -dimensional **multivariate Bernoulli distribution** with parameter vector $\mathbf{p} = (p_1, \dots, p_D)$, which has the form

$$P(\mathbf{x}|\mathbf{p}) = \prod_{d=1}^D p_d^{x_d} (1 - p_d)^{(1-x_d)}$$

where both \mathbf{x} and \mathbf{p} are D -dimensional vectors

(b) What is the equation for the maximum likelihood (ML) estimate of \mathbf{p} ? Note that you can solve for \mathbf{p} directly. [5 marks]

(c) Assuming independent Beta priors on the parameters p_d

$$P(p_d) = \frac{1}{B(\alpha, \beta)} p_d^{\alpha-1} (1 - p_d)^{\beta-1}$$

and $P(\mathbf{p}) = \prod_d P(p_d)$ What is the maximum a posteriori (MAP) estimate of \mathbf{p} ? Hint: maximise the log posterior with respect to \mathbf{p} . [5 marks]

Download the data set [binarydigits.txt](#) from the course website, which contains $N = 100$ images with $D = 64$ pixels each, in an $N \times D$ matrix. These pixels can be displayed as 8×8 images by rearranging them. View them in Matlab by running [bindigit.m](#) or in Python by running [bindigit.py](#).

(d) Write code to learn the ML parameters of a multivariate Bernoulli from this data set and display these parameters as an 8×8 image. Include a listing of your code within your submission, and a visualisation of the learned parameter vector as an image. (You may use Matlab, Octave or Python) [5 marks]

(e) Modify your code to learn MAP parameters with $\alpha = \beta = 3$. Show the new learned parameter vector for this data set as an image. Explain why this might be better or worse than the ML estimate. [5 marks]

4. [15 marks] **Model selection.** In the binary data model above, find the expressions needed to calculate the (relative) probability of the three different models:

(a) all D components are generated from a Bernoulli distribution with $p_d = 0.5$

(b) all D components are generated from Bernoulli distributions with unknown, but identical, p_d

(c) each component is Bernoulli distributed with separate, unknown p_d

Assume that all three models are equally likely *a priori*, and take the prior distributions for any unknown probabilities to be uniform. Calculate the posterior probabilities of each of the three models having generated the data in [binarydigits.txt](#).

5. [10 marks] **Latent Variable Models.**

- (a) Describe a real-world data set which you believe could be modelled using factor analysis. Argue why factor analysis is a sensible model for this data. What do you expect the factors to represent? How many factors do you think there would be? Are the linearity and Gaussianity assumptions reasonable, and if not, how would you modify the model? [5 marks]
- (b) Describe a real-world data set which you believe could be modelled using a mixture model. Argue why a mixture model is a sensible model for your real world data set. What do you expect the mixture components to represent? How many components (or clusters) do you think there would be? What parametric form would each component have? [5 marks]

6. [15 marks] **Principal Components Analysis.**

The conventional latent variable model for Probabilistic Principal Components Analysis has a standard normal latent \mathbf{z} and an arbitrary *loading matrix* Λ .

$$p(\mathbf{z}) = \mathcal{N}(\mathbf{0}, I)$$
$$p(\mathbf{x}|\mathbf{z}) = \mathcal{N}(\Lambda\mathbf{z}, \psi I).$$

An alternative model would be to draw \mathbf{z} from a normal with diagonal covariance (say Υ), and then restrict Λ to have orthonormal columns:

$$p(\mathbf{z}) = \mathcal{N}(\mathbf{0}, \Upsilon); \quad \Upsilon_{ij} = 0 \text{ for } i \neq j$$
$$p(\mathbf{x}|\mathbf{z}) = \mathcal{N}(\Lambda\mathbf{z}, \psi I); \quad \Lambda^T \Lambda = I.$$

- (a) Show that this alternative model is equivalent to the standard one in the sense that in the sense that it can model exactly the same set of possible marginal distributions. [5 marks]
- (b) Derive the mean and covariance of $p(\mathbf{z}|\mathbf{x})$ within the alternative model in the non-probabilistic PCA limit, $\psi \rightarrow 0$. [10 marks]

BONUS QUESTIONS: you must attempt the questions above before answering those below.

7. [Bonus: 10 marks] **Consistent beliefs.** A friend (perhaps not for much longer) reports that he is willing to bet on the following beliefs about two events A and B .

$$b(A) = 0.5$$
$$b(A \cap B) = 0.5$$
$$b(B) = 0.6$$
$$b(A \cup B) = 0.7$$

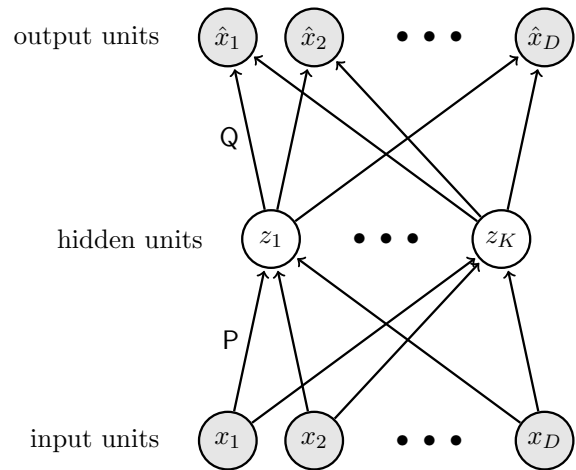
- (a) Show that these beliefs are inconsistent. [3 marks]
- (b) Construct a Dutch Book, which he would accept according to his beliefs, but which will guarantee that he will lose money. [7 marks]
8. [Bonus: 20 marks] **Linear Autoencoders.** A linear autoencoder is a “neural network” implementation of PCA. The network has three “layers” of “units” — each unit represents a single scalar variable, and so each layer represents a vector:

\hat{x}_j $j = 1 \dots D$ output units
 z_k $k = 1 \dots K < D$ hidden units
 x_i $i = 1 \dots D$ input units

The mappings from input to hidden, and hidden to output layers, are linear:

$$z_k = \sum_i P_{ki} x_i$$

$$\hat{x}_j = \sum_k Q_{jk} z_k$$



Given a set of N observed “input” vectors $\{\mathbf{x}_n\}$, the weight matrices P and Q are set to minimise the “autoencoding error” $\sum_n \|\hat{\mathbf{x}}_n - \mathbf{x}_n\|^2$, often using iterative gradient descent. Assume the input vector distribution has zero mean.

- (a) Show that after the weights have converged, they obey the identities:

$$P = (Q^T Q)^{-1} Q^T$$

$$Q = \Sigma_X P^T (P \Sigma_X P^T)^{-1}$$

where $\Sigma_X = \sum_n \mathbf{x}_n \mathbf{x}_n^T$ and we assume that Q is rank K . [5 marks]

- (b) It is clear that if P and Q minimize the error, then, for any invertible $K \times K$ matrix C the matrices $P_* = CP$ and $Q_* = QC^{-1}$ also achieve the same minimum. Show that you can always find a matrix C such that $Q_*^T Q_* = I$ and so $P_* = Q_*^T$. [2 marks]
- (c) Show that the minimisation of the error is equivalent to the maximisation of

$$\text{Tr} [Q_* Q_*^T \Sigma_X]$$

and that this maximum is achieved by choosing the columns of Q_* proportional to the first K eigenvectors of Σ_X . (There are many ways to do this, but one option is to start from the argument presented in lecture showing that projection on the leading eigenvector maximises the projected variance.) [7 marks]

This demonstrates the assertion made in lecture that the linear autoencoder will find the subspace of the leading K principal components.

- (d) How would you adapt the algorithm to implement factor analysis in the case that the uniquenesses (another term for the output noise variances Ψ_{dd}) are known? [3 marks]
- (e) Is it possible to use an autoencoder for FA in the case of unknown uniquenesses? How, if so; or why not, if not? [3 marks]