Assignment 3

Probabilistic and Unsupervised Learning

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1. **[35 points] Deriving Gibbs Sampling for LDA**.

   In this question we derive two Gibbs sampling algorithms for latent Dirichlet allocation (LDA). LDA is a topic model that defines multiple mixtures of discrete distributions with shared components. The archetypical application is to words in documents. Suppose there are W possible words, D documents and K topics. The LDA model specifies the distribution of the ith word in the dth document, $x_{id} \in \{1 \dots W\}$, in terms of the hyperparameters α and β, by way of latent Dirichlet parameters:

topic distribution for dth document	$\boldsymbol{\theta}_d \vert \alpha \sim \text{Dirichlet}(\alpha, \dots, \alpha)$	(1)
word distribution for kth topic	$\boldsymbol{\phi}_k \vert \beta \sim \text{Dirichlet}(\beta, \dots, \beta)$	(2)
topic for ith word in dth document	$z_{id} \vert \boldsymbol{\theta}_d \sim \text{Discrete}(\boldsymbol{\theta}_d)$	(3)
identity of ith word in dth document	$x_{id} \vert z_{id}, \boldsymbol{\phi}_{z_{id}} \sim \text{Discrete}(\boldsymbol{\phi}_{z_{id}})$	(4)

   Let $A_{dk} = \sum_i \delta(z_{id} = k)$ be the number of z_{id} variables taking on value k in document d, and $B_{kw} = \sum_d \sum_i \delta(x_{id} = w)\delta(z_{id} = k)$ be the number of times word w is assigned to topic k across all the documents. Let N_d be the total number of words in document d and let $M_k = \sum_w B_{kw}$ be the total number of words assigned to topic k.

   (a) Write down the joint probability over the observed data and latent variables, expressing the joint probability in terms of the counts N_d, M_k, A_{dk}, and B_{kw}. [5 points]

   (b) Derive the Gibbs sampling updates for all the latent variables z_{id} and parameters $\boldsymbol{\theta}_d$ and $\boldsymbol{\phi}_k$. [10 points]

   (c) Integrate out all the parameters $\boldsymbol{\theta}_d$ and $\boldsymbol{\phi}_k$ from the joint probability in (a), resulting in a joint probability over only the z_{id} topic assignment variables and x_{id} observed variables. Again this expression should relate to z_{id}'s and x_{id}'s only through the counts N_d, M_k, A_{dk}, and B_{kw}. [5 points]

   (d) Derive the Gibbs sampling updates for the z_{id} with all parameters integrated out. This is called **collapsed Gibbs sampling**. You will need the the following identity of the Gamma function: $\Gamma(1 + x) = x\Gamma(x)$ for $x > 0$. [10 points]

   (e) What hyperpriors would you give to α and β? How would you generate samples of α and β from the appropriate conditionals? [You should suggest an algorithm and justify its feasibility, but do not need to derive the update equations; 5 points]

2. **[65 points] Decrypting Messages with MCMC**. You are given a passage of English text that has been encrypted by remapping each symbol to a (usually) different one. For example,

$$\begin{aligned} a &\to s \\ b &\to ! \\ \langle\text{space}\rangle &\to v \\ &\vdots \end{aligned}$$

Thus a text like 'a boy...' might be encrypted by 'sv!op...'. Assume that the mapping between symbols is one-to-one. The file **symbols.txt** gives the list of symbols, one per line (the second line is ⟨space⟩). The file **message.txt** gives the encrypted message.

Decoding the message by brute force is impossible, since there are 53 symbols and thus 53! possible permutations to try. Instead we will set up a Markov chain Monte Carlo sampler to find modes in the space of permutations.

We model English text, say $s_1 s_2 \cdots s_n$ where s_i are symbols, as a Markov chain, so that each symbol is independent of the preceding text given only the symbol before:

$$p(s_1 s_2 \cdots s_n) = p(s_1) \prod_{i=2}^{n} p(s_i | s_{i-1})$$

(a) Learn the transition statistics of letters and punctuation in English by downloading a large text [say *War and Peace* (in translation!)] from the web and estimating the transition probabilities $p(s_i = \alpha | s_{i-1} = \beta) \equiv \psi(\alpha, \beta)$ as well as the stationary distribution $\lim_{i\to\infty} p(s_i = \gamma) \equiv \phi(\gamma)$. Assume that the first letter of your text (and also that of the encrypted text provided) is itself sampled from the stationary distribution.

Give formulae for the ML estimates of these probabilities as functions of the counts of numbers of occurrences of symbols and pairs of symbols.

Compute the estimated probabilities. You may report the values using a table, Hinton diagram or other method. [6 marks]

(b) The state variable for our MCMC sampler will be the symbol permutation. Let $\sigma(s)$ be the symbol that stands for symbol s in the encrypted text, e.g., $\sigma(a) = s$ and $\sigma(b) =!$ above. Assume a uniform prior distribution over permutations.

Are the latent variables $\sigma(s)$ for different symbols s independent?

Let $e_1 e_2 \cdots e_n$ be an encrypted English text. Write down the joint probability of $e_1 e_2 \cdots e_n$ given σ. [6 marks]

(c) We use a Metropolis-Hastings (MH) chain, with the proposal given by choosing two symbols s and s' at random and swapping the corresponding encrypted symbols $\sigma(s)$ and $\sigma(s')$.

How does the proposal probability $S(\sigma \to \sigma')$ depend on the permutations σ and σ'? What is the MH acceptance probability for a given proposal? [10 marks]

(d) Implement the MH sampler, and run it on the provided encrypted text. Report the current decryption of the first 60 symbols after every 100 iterations. Your Markov chain should converge to give you a fairly sensible message. (Hint: it may help to initialize your chain intelligently and to try multiple times; in any case, please describe what you did). [30 marks]

(e) Note that some $\psi(\alpha, \beta)$ values may be zero. Does this affect the ergodicity of the chain? If the chain remains ergodic, give a proof; if not, explain and describe how you can restore ergodicity. [5 marks]

(f) Analyse this approach to decoding. For instance, would symbol probabilities alone (rather than transitions) be sufficient? If we used a second order Markov chain for English text, what problems might we encounter? Will it work if the encryption scheme allows two symbols to be mapped to the same encrypted value? Would it work for Chinese with > 10000 symbols? [8 marks]

3. **[Bonus 60 points] Implementing Gibbs sampling for LDA**. Take a look at the accompanying code, which sets up a framework in which you will implement both the standard and collapsed Gibbs sampling inference for LDA. Read the README which lays out the MATLAB variables used.

   (a) Implement both standard and collapsed Gibbs sampline updates, and the log joint probabilities in question 1(a), 1(c) above. The files you need to edit are stdgibbs_logjoint, stdgibbs_update, colgibbs_logjoint,colgibbs_update. Debug your code by running toyexample. Show sample plots produced by toyexample, and attach and document the MATLAB code that you wrote. [20 points]

   (b) Based upon the plots of log predictive and joint probabilities produced by toyexample, how many iterations do you think are required for burn-in? Discarding the burn-in iterations, compute and plot the autocorrelations of the log predictive and joint probabilities for both Gibbs samplers. You will need to run toyexample for a larger number of iterations to reduce the noise in the autocorrelation. Based upon the autocorrelations how many samples do you think will be need to have a representative set of samples from the posterior? Describe what you did and justify your answers with one or two sentences. [10 points]

   (c) Based on the computed autocorrelations, which of the two Gibbs samplers do you think converge faster, or do they converge at about the same rate? If they differ, why do you think this might be the case? Justify your answers. [5 points]

   (d) Try varying α, β and K. What effects do these have on the posterior and predictive performance of the model? Justify your answers. [5 points]

   Topic modelling of NIPS papers. Now that we have code for LDA, we can try our hands on finding the topics at a major machine learning conference (NIPS). In the provided code there is a file nips.data which contains preprocessed data. The vocabulary is given in nips.vocab.

   (e) The data in nips.data is probably too big so that our MATLAB implementation will be too slow. We will try to reduce the data set to a more tractable size, by removing words from the vocabulary. Come up with a metric for how informative/relevant/topical a vocabulary word is. You may want to experiment and try multiple metrics, and make sure that keywords like "Bayesian", "graphical", "Gaussian", "support", "vector", "kernel", "representation", "regression", "classification" etc have high metric. Report on your experiences, and use your metric to prune the data set to just the top few hundred words (say 500, or lower if the implementation is still too slow). You may find it useful to read up on `tf-idf` on wikipedia. [10 points]

   (f) Now run LDA on the reduced NIPS data, using one of the Gibbs samplers you have just written. You will need to experiment with various settings of α, β and K until the topics discovered looks "reasonable". Describe the topics you found. How do the topics change (qualitatively) as α, β and K are varied? [10 points]