Probabilistic & Unsupervised Learning
Approximate Inference

Factored Variational Approximations and Variational Bayes

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Intractabilities

- **Analytic intractability**: non-conjugacy prevents closed-form evaluation of likelihood or free energy

$$\ell(\theta) = \int d\mathcal{Z}\ P(\mathcal{Z}|\theta)P(\mathcal{X}|\mathcal{Z},\theta) \quad \mathcal{F}(\theta, Q) = \int d\mathcal{Z}\ Q(\mathcal{Z}) \log P(\mathcal{X}, \mathcal{Z}|\theta) + \mathbf{H}[Q]$$

- **Computational intractability**: graphical structure prevents simplification of high-dimensional sums or integrals

$$Q(\mathbf{z}_i) \propto \int d\mathbf{z}_1 \ldots d\mathbf{z}_{i-1}\, d\mathbf{z}_{i+1} \ldots d\mathbf{z}_k\ \log P(\mathcal{X}, \mathcal{Z}|\theta)$$

- **Intractable normalisers**: learning requires normaliser gradient

$$\nabla_\theta \mathcal{F}(\theta, Q) = \nabla_\theta \langle E(\mathcal{X}, \mathcal{Z}|\theta) \rangle_Q - \nabla_\theta \log Z(\theta)$$

- **Model selection or weighting** (by marginal likelihood)

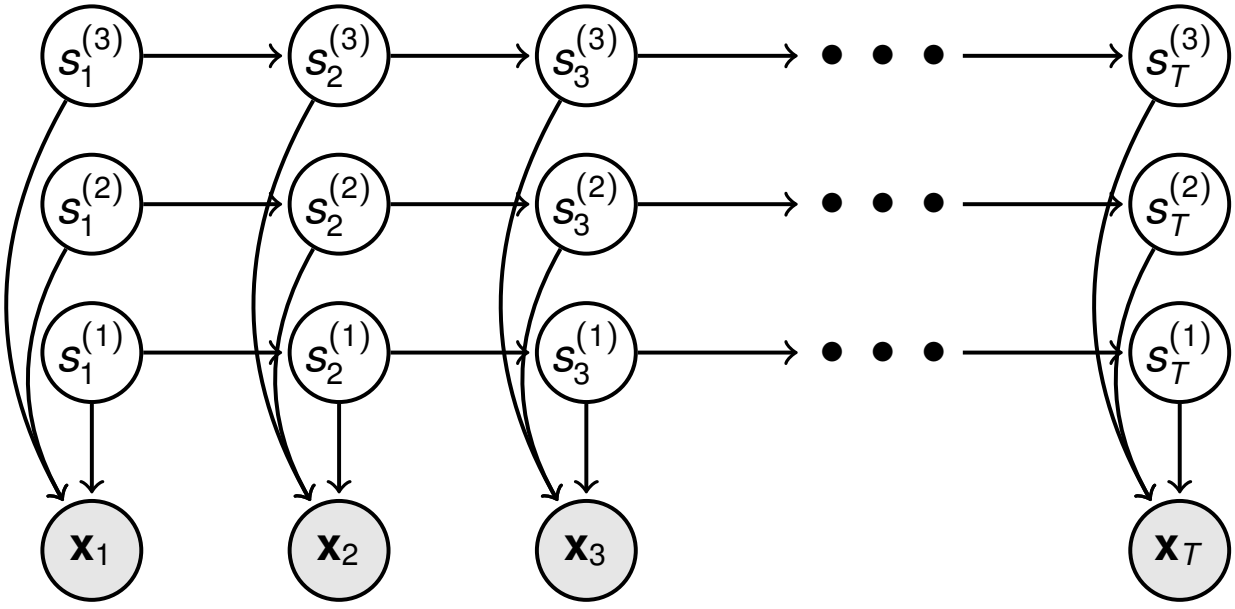
$$p(\mathcal{D}|m) = \int d\theta\ p(\theta|m) p(\mathcal{D}|\theta, m)$$

Intractabilities and approximations

- Inference – computational intractability
  - Factored variational approx
  - Loopy BP/EP/Power EP
  - LP relaxations/ convexified BP
  - Gibbs sampling, other MCMC
- Inference – analytic intractability
  - Laplace approximation (global)
  - Parametric variational approx
  - Message approximations (linearised, sigma-point, Laplace)
  - Assumed-density methods and Expectation-Propagation
  - (Sequential) Monte-Carlo methods
- Learning – intractable partition function
  - Sampling parameters
  - Constrastive divergence
  - Score-matching
- Model selection
  - Laplace approximation / BIC
  - Variational Bayes
  - (Annealed) importance sampling
  - Reversible jump MCMC

Not a complete list!

Computational intractability – distributed models



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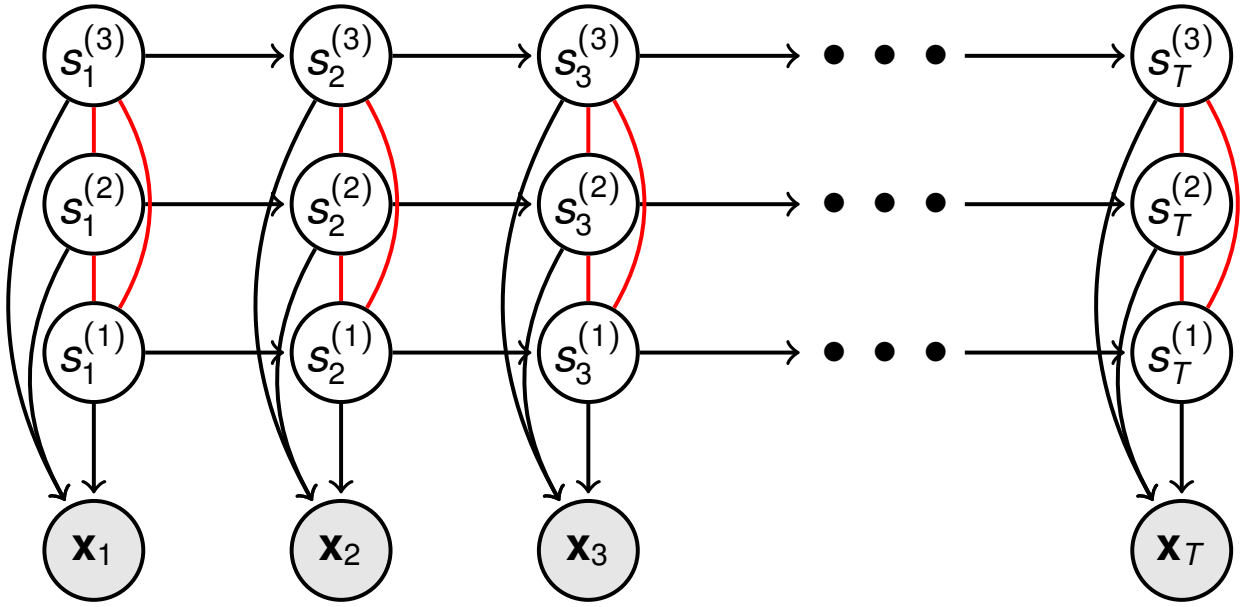

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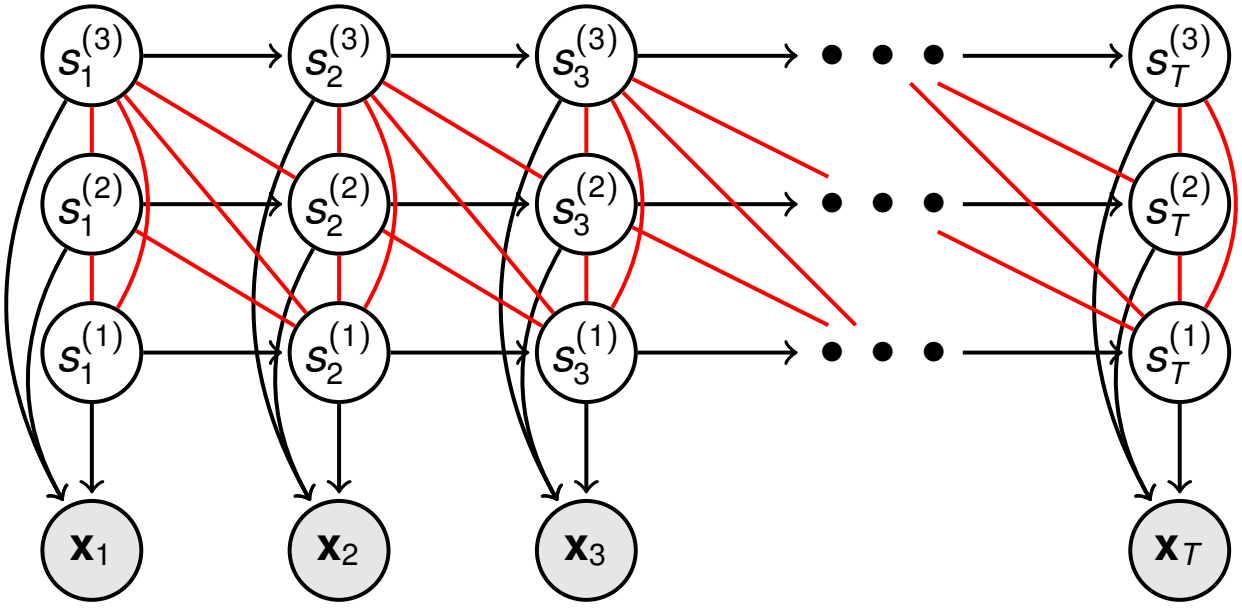

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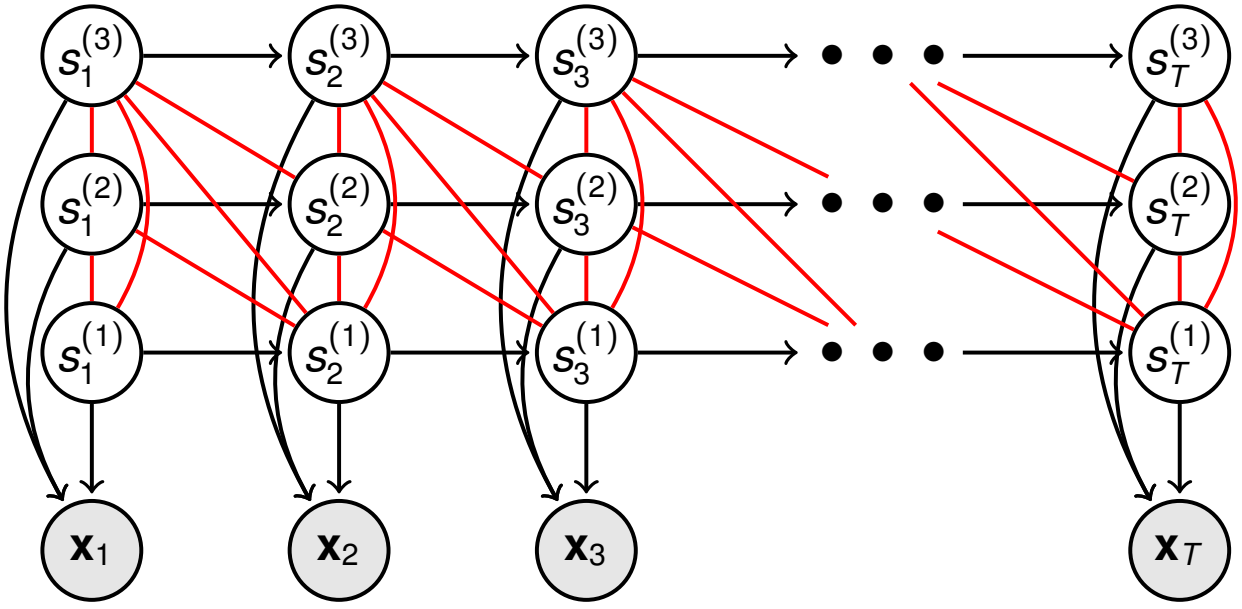

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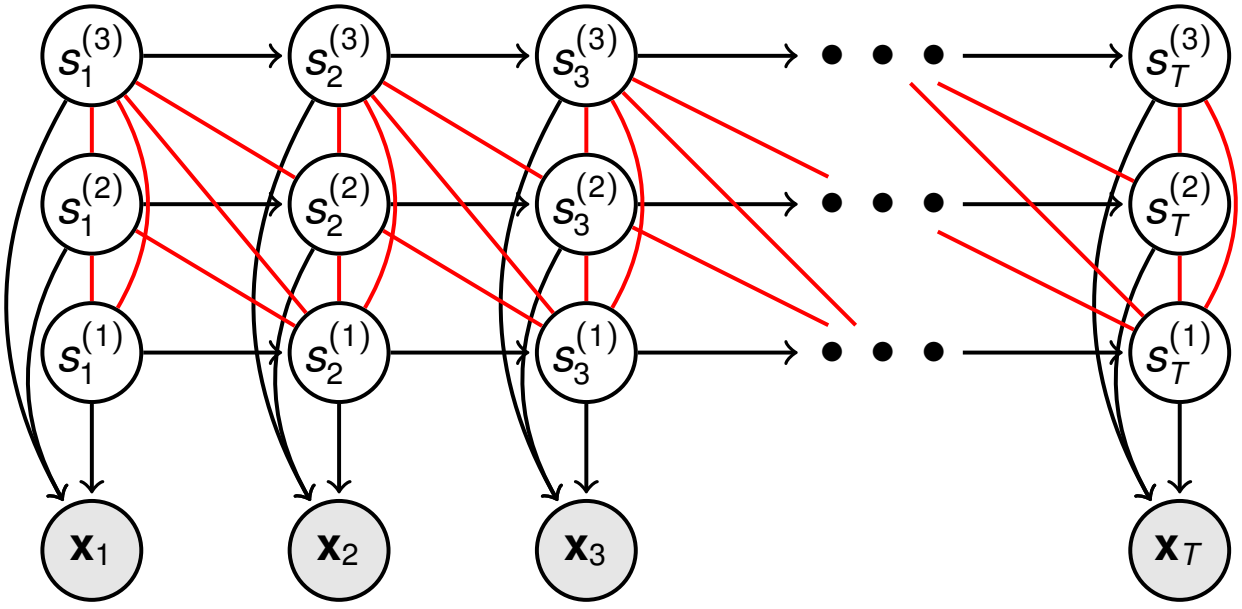

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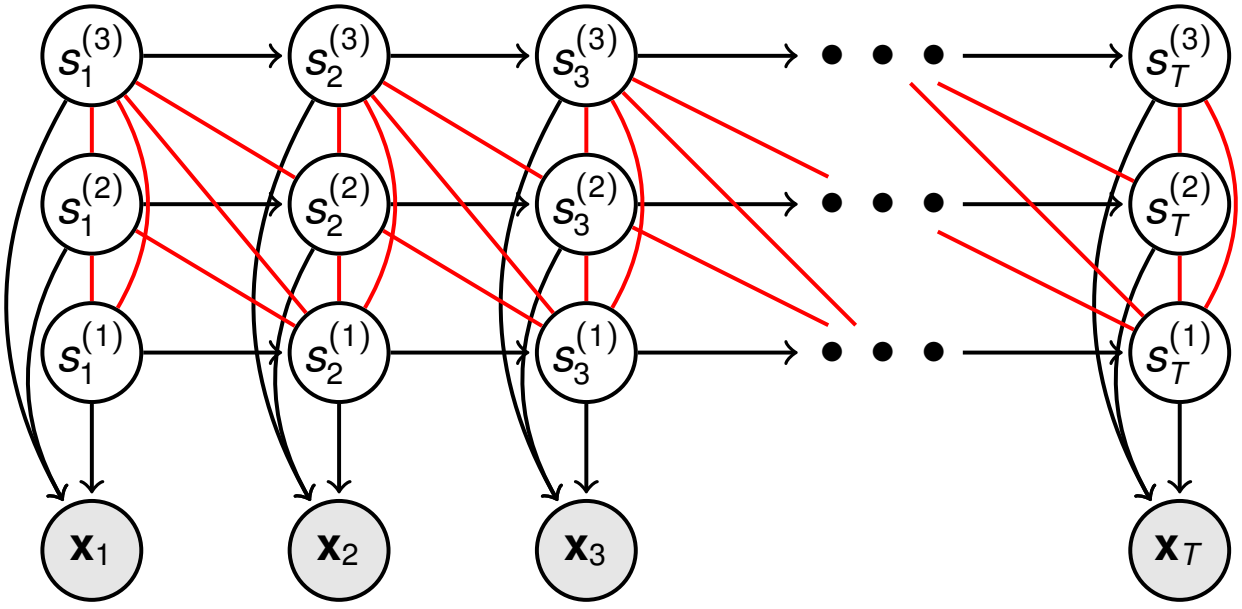

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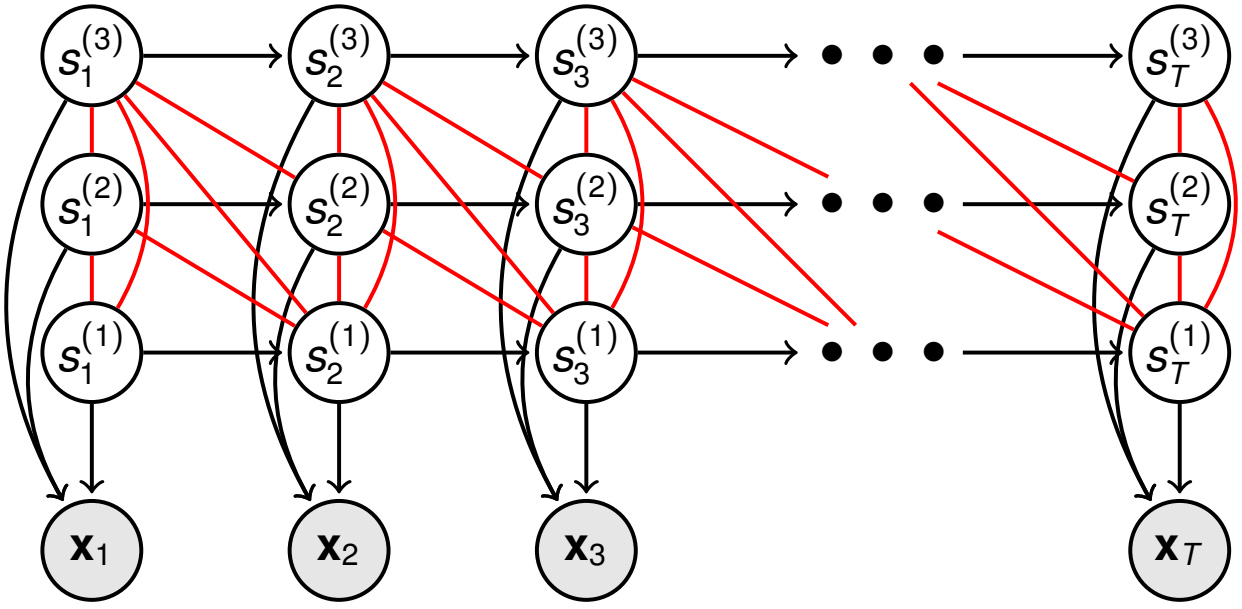

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To see how they work, we need to review the free-energy interpretation of EM.

The Free Energy for a Latent Variable Model

Observed data $\mathcal{X} = \{\mathbf{x}_i\}$; Latent variables $\mathcal{Z} = \{\mathbf{z}_i\}$; Parameters θ.

Goal: Maximize the log likelihood wrt θ (i.e. ML learning):

$$\ell(\theta) = \log P(\mathcal{X}|\theta) = \log \int P(\mathcal{Z}, \mathcal{X}|\theta) d\mathcal{Z}$$

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Any distribution, $q(\mathcal{Z})$, over the hidden variables can be used to obtain a lower bound on the log likelihood using Jensen's inequality:

$$\ell(\theta) = \log \int q(\mathcal{Z}) \frac{P(\mathcal{Z}, \mathcal{X}|\theta)}{q(\mathcal{Z})} \, d\mathcal{Z} \geq \int q(\mathcal{Z}) \log \frac{P(\mathcal{Z}, \mathcal{X}|\theta)}{q(\mathcal{Z})} \, d\mathcal{Z} \stackrel{\text{def}}{=} \mathcal{F}(q, \theta)$$

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$$\begin{aligned} \int q(\mathcal{Z}) \log \frac{P(\mathcal{Z}, \mathcal{X}|\theta)}{q(\mathcal{Z})} \, d\mathcal{Z} &= \int q(\mathcal{Z}) \log P(\mathcal{Z}, \mathcal{X}|\theta) \, d\mathcal{Z} - \int q(\mathcal{Z}) \log q(\mathcal{Z}) \, d\mathcal{Z} \\ &= \int q(\mathcal{Z}) \log P(\mathcal{Z}, \mathcal{X}|\theta) \, d\mathcal{Z} + \mathbf{H}[q], \end{aligned}$$

where $\mathbf{H}[q]$ is the entropy of $q(\mathcal{Z})$.
So: $\mathcal{F}(q, \theta) = \langle \log P(\mathcal{Z}, \mathcal{X}|\theta) \rangle_{q(\mathcal{Z})} + \mathbf{H}[q]$

The E and M steps of EM

The log likelihood is bounded below by:

$$\mathcal{F}(q,\theta) = \langle \log P(\mathcal{Z},\mathcal{X}|\theta)\rangle_{q(\mathcal{Z})} + \mathbf{H}[q] = \ell(\theta) - \mathbf{KL}[q(\mathcal{Z})\|P(\mathcal{Z}|\mathcal{X},\theta)]$$

EM alternates between:

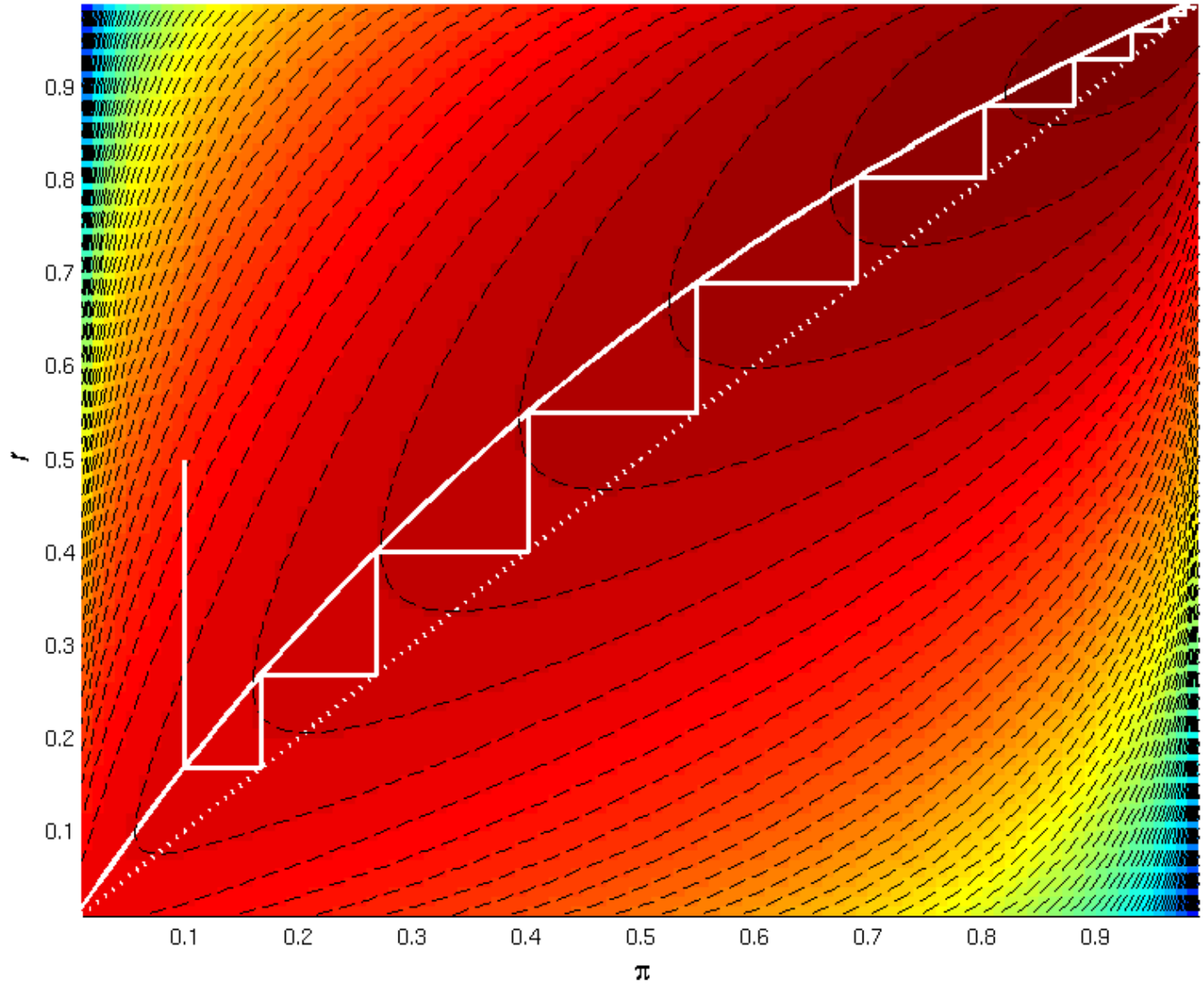
E step: optimise $\mathcal{F}(q,\theta)$ wrt distribution over hidden variables holding parameters fixed:

$$q^{(k)}(\mathcal{Z}) := \underset{q(\mathcal{Z})}{\text{argmax}}\ \mathcal{F}\left(q(\mathcal{Z}),\theta^{(k-1)}\right) = P(\mathcal{Z}|\mathcal{X},\theta^{(k-1)})$$

M step: maximise $\mathcal{F}(q,\theta)$ wrt parameters holding hidden distribution fixed:

$$\theta^{(k)} := \underset{\theta}{\text{argmax}}\ \mathcal{F}\left(q^{(k)}(\mathcal{Z}),\theta\right) = \underset{\theta}{\text{argmax}}\ \langle \log P(\mathcal{Z},\mathcal{X}|\theta)\rangle_{q^{(k)}(\mathcal{Z})}$$

EM as Coordinate Ascent in $\mathcal{F}$



EM Never Decreases the Likelihood

The E and M steps together never decrease the log likelihood:

$$\ell(\theta^{(k-1)}) \underset{\text{E step}}{=} \mathcal{F}(q^{(k)}, \theta^{(k-1)}) \underset{\text{M step}}{\leq} \mathcal{F}(q^{(k)}, \theta^{(k)}) \underset{\text{Jensen}}{\leq} \ell(\theta^{(k)}),$$

- The E step brings the free energy to the likelihood.
- The M-step maximises the free energy wrt θ.
- $\mathcal{F} \leq \ell$ by Jensen – or, equivalently, from the non-negativity of KL

If the M-step is executed so that $\theta^{(k)} \neq \theta^{(k-1)}$ iff $\mathcal{F}$ increases, then the overall EM iteration will step to a new value of θ iff the likelihood increases.

Free-energy-based variational approximation

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For the E-step we could:

- Parameterise $q = q_\rho(\mathcal{Z})$ and take a gradient step in ρ.
- Assume some simplified form for q, usually factored: $q = \prod_i q_i(\mathcal{Z}_i)$ where $\mathcal{Z}_i$ partition $\mathcal{Z}$, and maximise within this form.

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In either case, we choose q from within a limited set $\mathcal{Q}$:

VE step: maximise $\mathcal{F}(q,\theta)$ wrt constrained latent distribution given parameters:

$$q^{(k)}(\mathcal{Z}) := \underset{q(\mathcal{Z})\in\mathcal{Q}\leftarrow\textbf{Constraint}}{\operatorname{argmax}} \; \mathcal{F}\left(q(\mathcal{Z}), \theta^{(k-1)}\right).$$

M step: unchanged

$$\theta^{(k)} := \underset{\theta}{\operatorname{argmax}} \; \mathcal{F}\left(q^{(k)}(\mathcal{Z}), \theta\right) = \underset{\theta}{\operatorname{argmax}} \int q^{(k)}(\mathcal{Z}) \log p(\mathcal{Z},\mathcal{X}|\theta) d\mathcal{Z},$$

Unlike in GEM, the fixed point may not be at an unconstrained optimum of $\mathcal{F}$.

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$$\ell\left(\theta^{(k-1)}\right) \underset{\text{E step}}{\not\geq} \mathcal{F}\left(q^{(k)}, \theta^{(k-1)}\right) \underset{\text{M step}}{\leq} \mathcal{F}\left(q^{(k)}, \theta^{(k)}\right) \underset{\text{Jensen}}{\leq} \ell\left(\theta^{(k)}\right),$$

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[Note that if $P(\mathcal{Z}|\mathcal{X}, \theta^{\mathrm{ML}}) \in \mathcal{Q}$, then θ^{ML} is a fixed point of the variational algorithm.]

KL divergence

Recall that

$$\begin{aligned}\mathcal{F}(q,\theta) &= \langle \log P(\mathcal{X},\mathcal{Z}|\theta)\rangle_{q(\mathcal{Z})} + \mathbf{H}[q] \\ &= \langle \log P(\mathcal{X}|\theta) + \log P(\mathcal{Z}|\mathcal{X},\theta)\rangle_{q(\mathcal{Z})} - \langle \log q(\mathcal{Z})\rangle_{q(\mathcal{Z})} \\ &= \langle \log P(\mathcal{X}|\theta)\rangle_{q(\mathcal{Z})} - \mathbf{KL}[q\|P(\mathcal{Z}|\mathcal{X},\theta)].\end{aligned}$$

Thus,

E step maximise $\mathcal{F}(q,\theta)$ wrt the distribution over latents, given parameters:

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is equivalent to:

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So, in each E step, the algorithm is trying to find the best approximation to $P(\mathcal{Z}|\mathcal{X})$ in $\mathcal{Q}$ in a KL sense. This is related to ideas in *information geometry*. It also suggests generalisations to other distance measures.

Factored Variational E-step

The most common form of variational approximation partitions $\mathcal{Z}$ into disjoint sets $\mathcal{Z}_i$ with

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- q_i updates iterated to convergence to "complete" VE-step.
- In fact, every (VE)$_i$-step separately increases $\mathcal{F}$, so any schedule of (VE)$_i$- and M-steps will converge. Choice can be dictated by practical issues (rarely efficient to fully converge E-step before updating parameters).

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$$\frac{\delta}{\delta q_i}\left(\mathcal{F} + \lambda\left(\int q_i - 1\right)\right) = \Big\langle \log P(\mathcal{X}, \mathcal{Z}|\theta^{(k-1)})\Big\rangle_{\prod_{j\neq i} q_j(\mathcal{Z}_j)} - \log q_i(\mathcal{Z}_i) - \frac{q_i(\mathcal{Z}_i)}{q_i(\mathcal{Z}_i)} + \lambda$$

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Factored Variational E-step

The Factored Variational E-step has a general form.

The free energy is:

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In general, this depends only on the expected sufficient statistics under q_j. Thus, again, we don't actually need the *entire* distributions, just the relevant expectations (now for approximate inference as well as learning).

Mean-field approximations

If $\mathcal{Z}_i = z_i$ (*i.e.*, q is factored over all variables) then the variational technique is often called a "mean field" approximation.

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$$P(\mathcal{X}, \mathcal{Z}) = \frac{1}{Z} \exp\left(\sum_{ij} W_{ij} s_i s_j + \sum_i b_i s_i \right)$$

  with some $s_i \in \mathcal{Z}$ and others observed.

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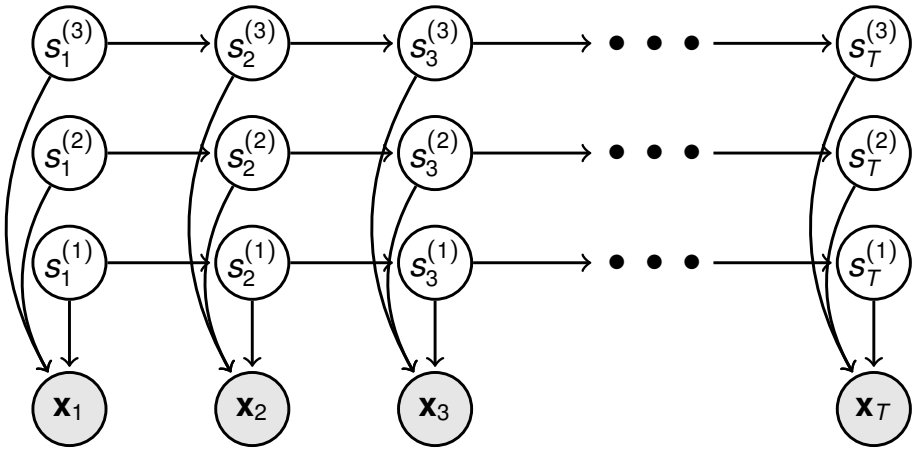
- Each variable sees the mean field imposed by its neighbours, and we update these fields until they all agree.

Mean-field FHMM



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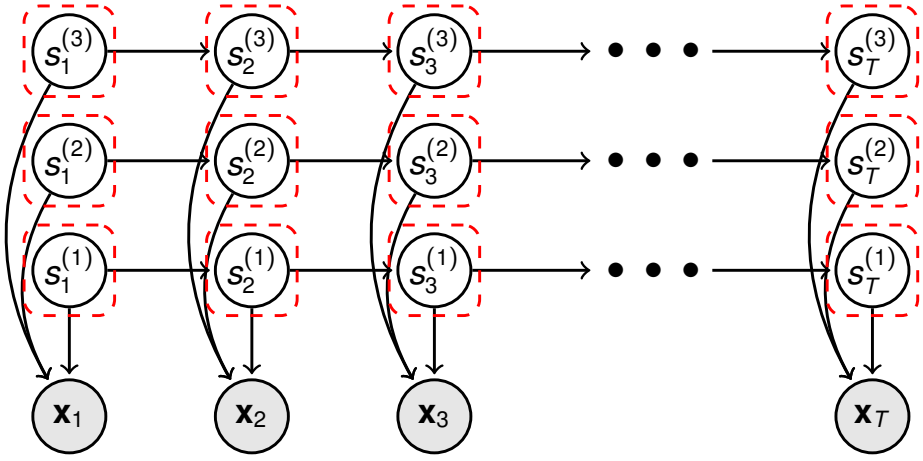

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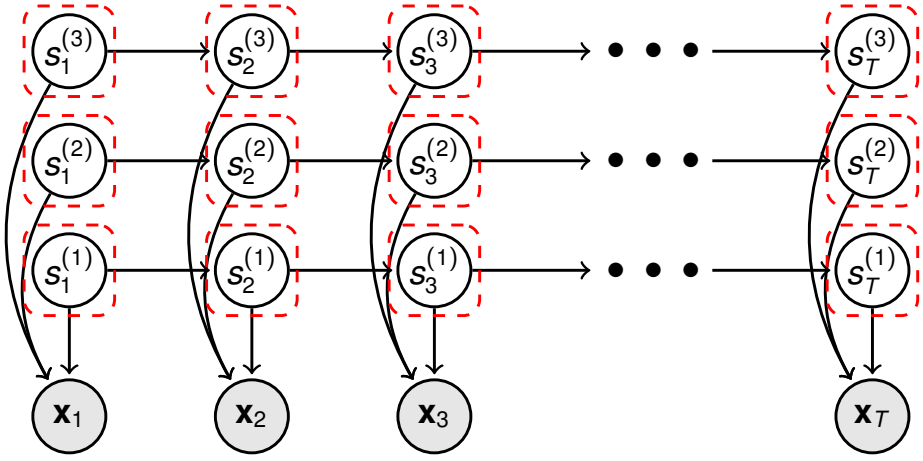

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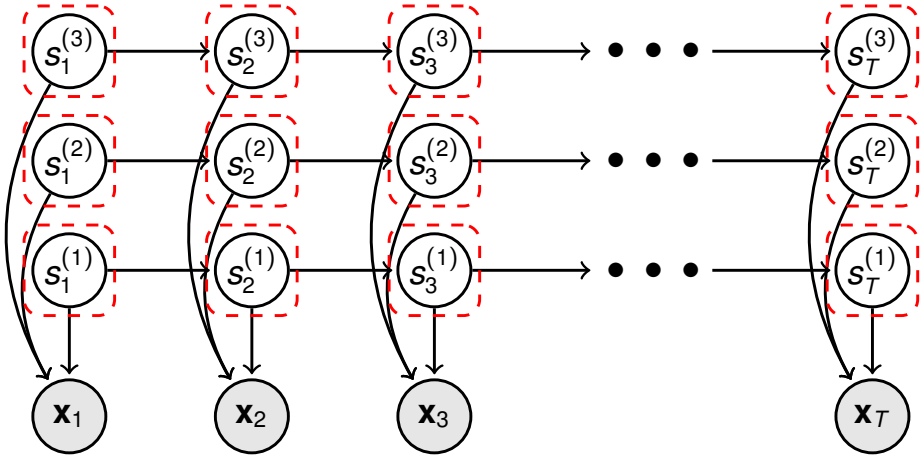

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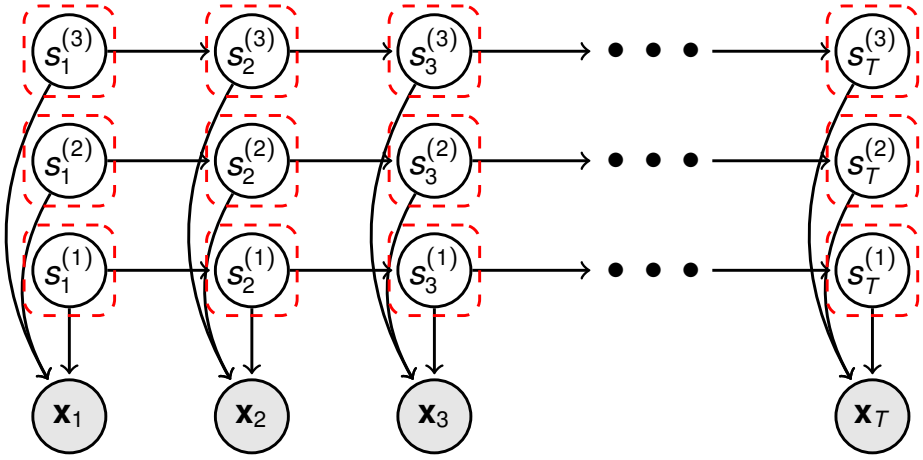

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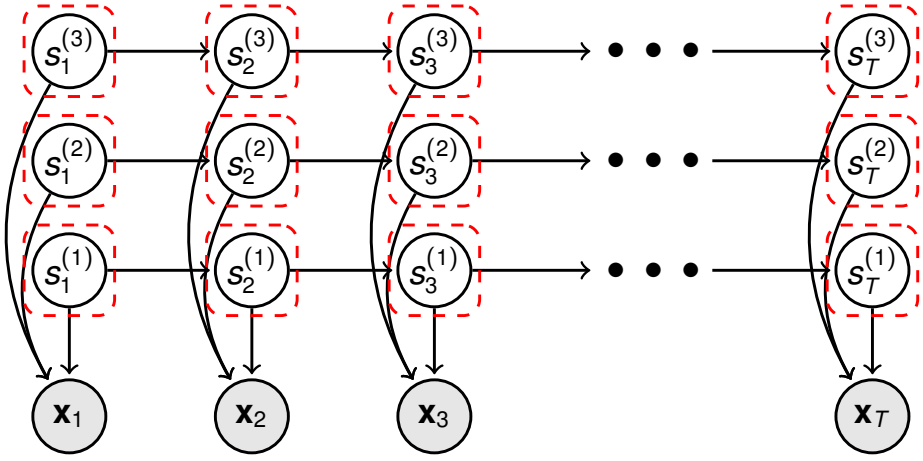

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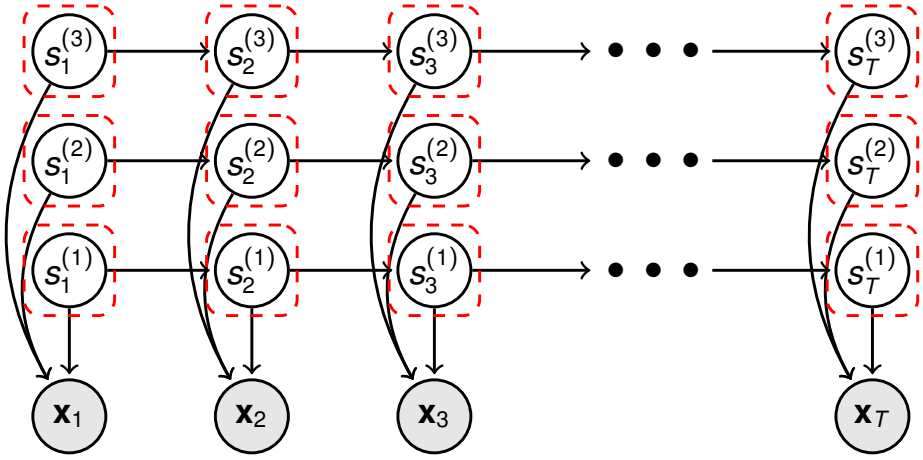

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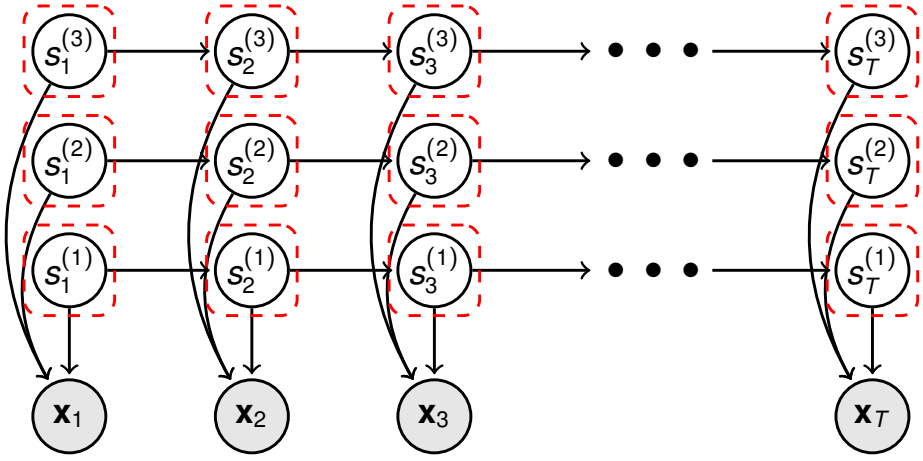

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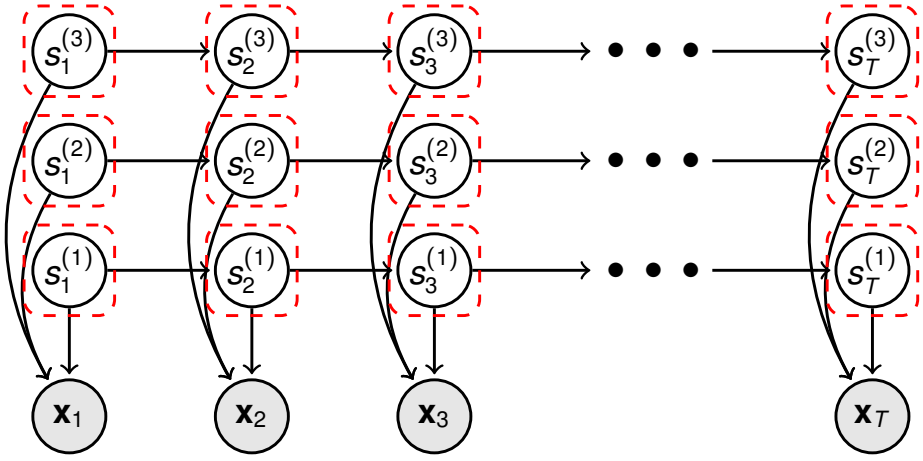

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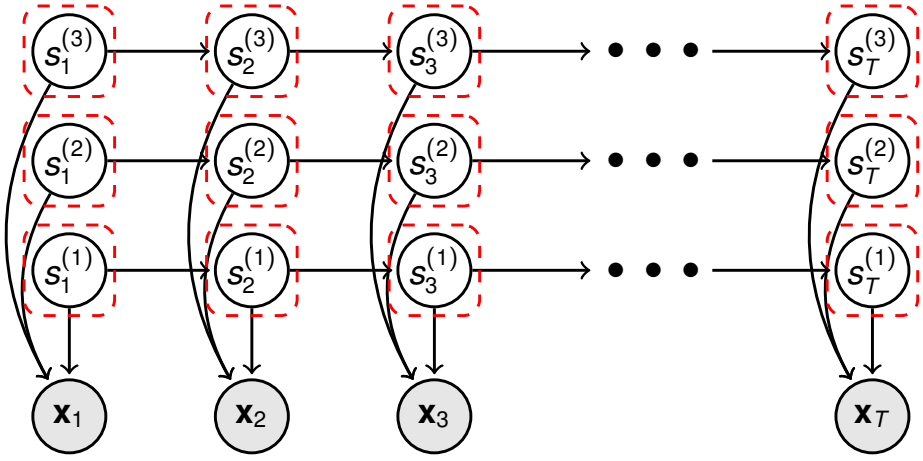

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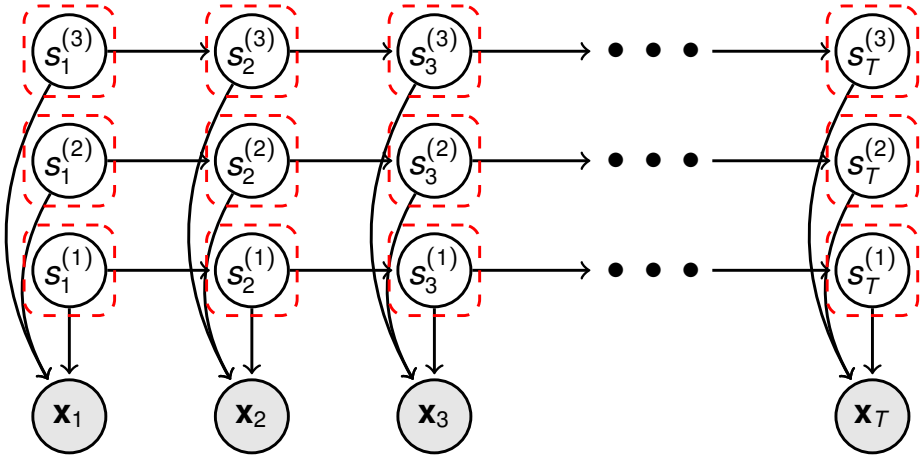

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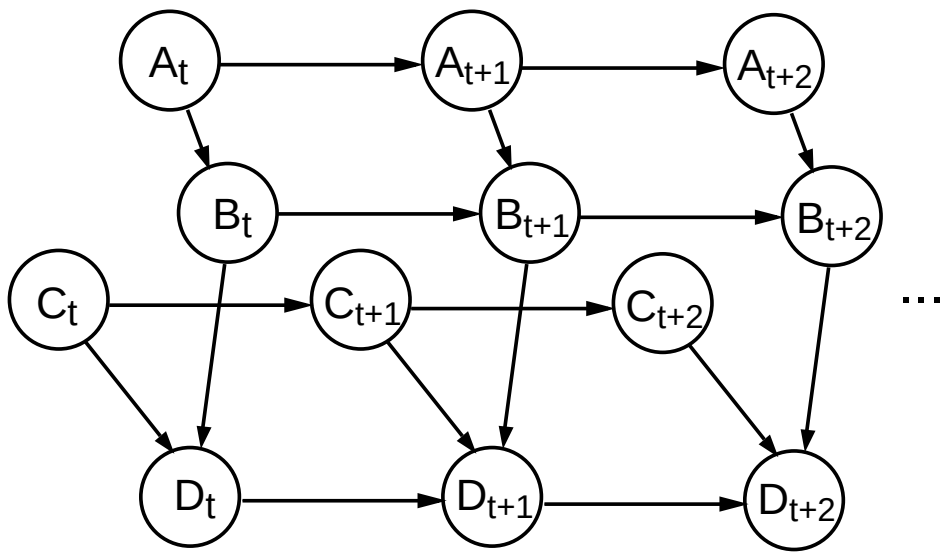
- Evidence does not appear explicitly in backward message (cf Kalman smoothing)

Structured variational approximation

- $q(\mathcal{Z})$ need not be completely factorized.



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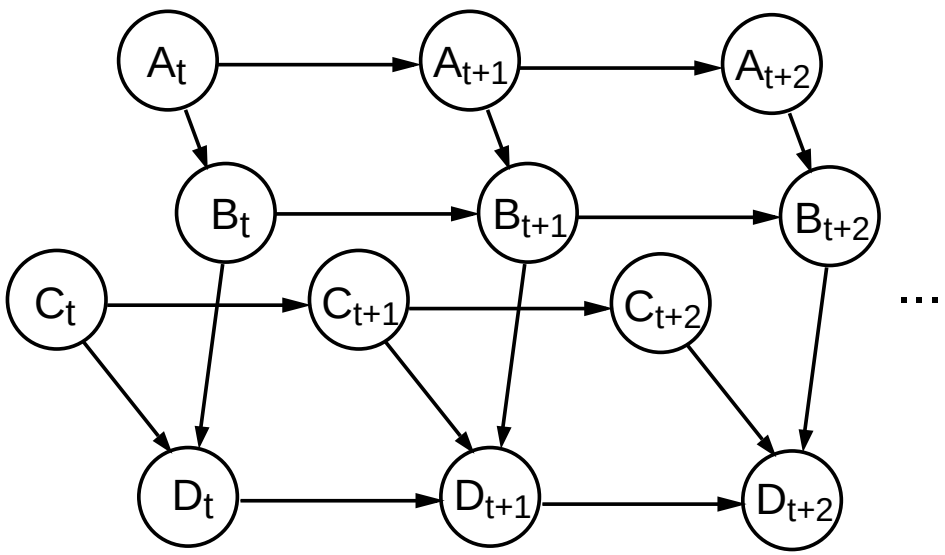
$\Rightarrow$ Then the factored approximation $q(\mathcal{Z}) = q(\mathcal{Z}_1)q(\mathcal{Z}_2)$ is tractable.



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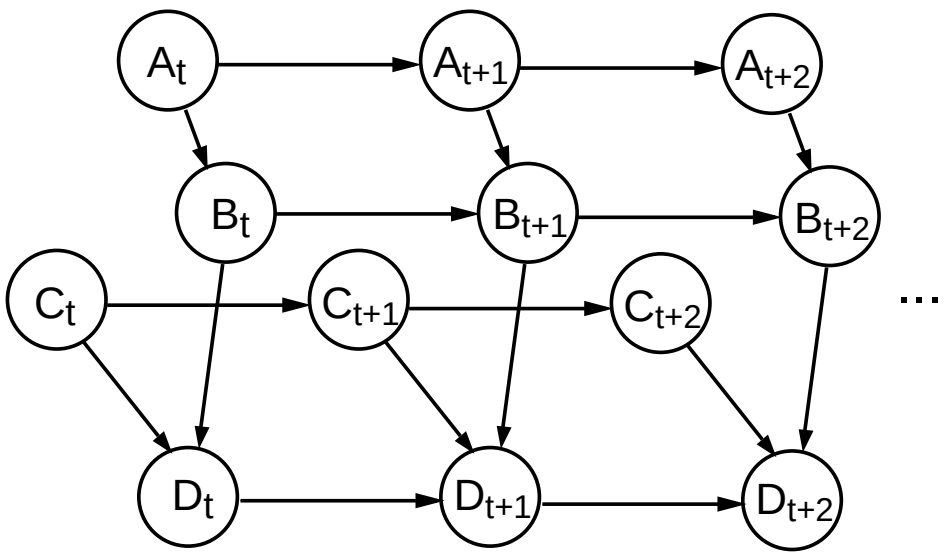
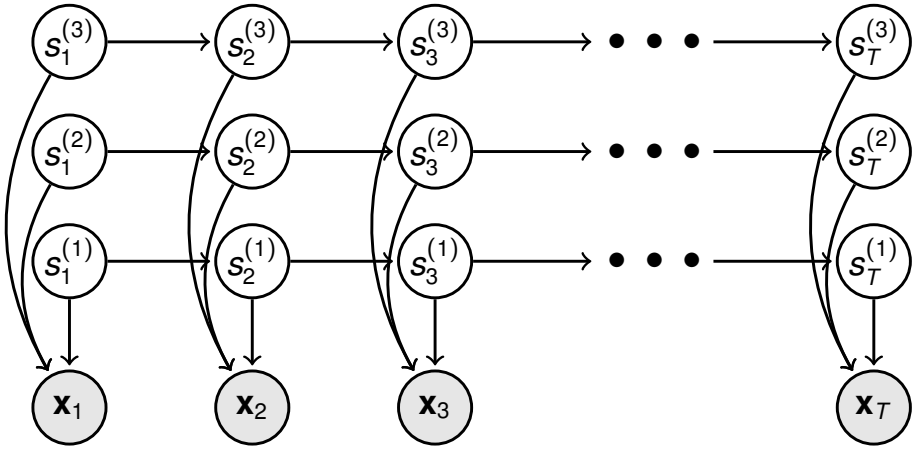
- In particular, any factorisation of $q(\mathcal{Z})$ into a product of distributions on trees, yields a tractable approximation.



Stuctured FHMM



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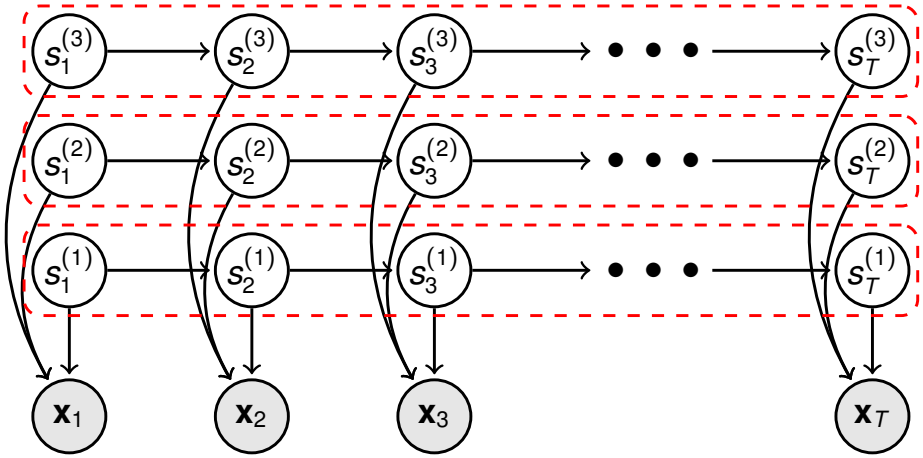

For the FHMM we can factor the chains:

$$q(s_{1:T}^{1:M}) = \prod_m q^m(s_{1:T}^m)$$

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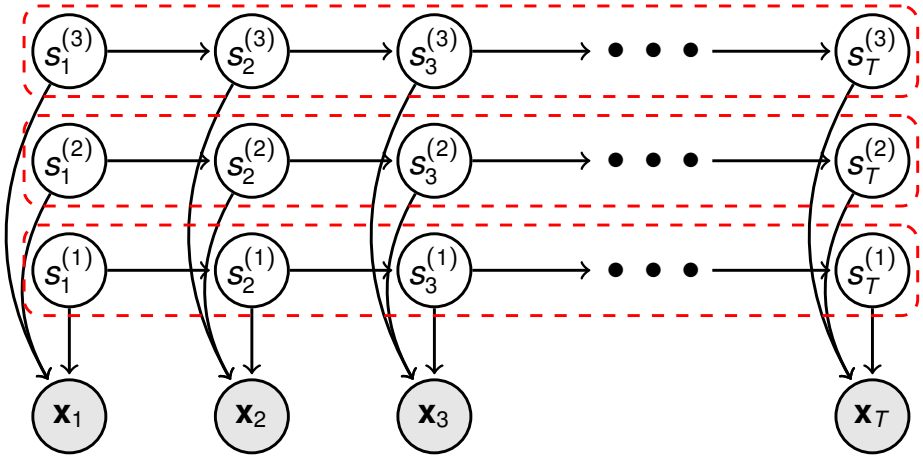

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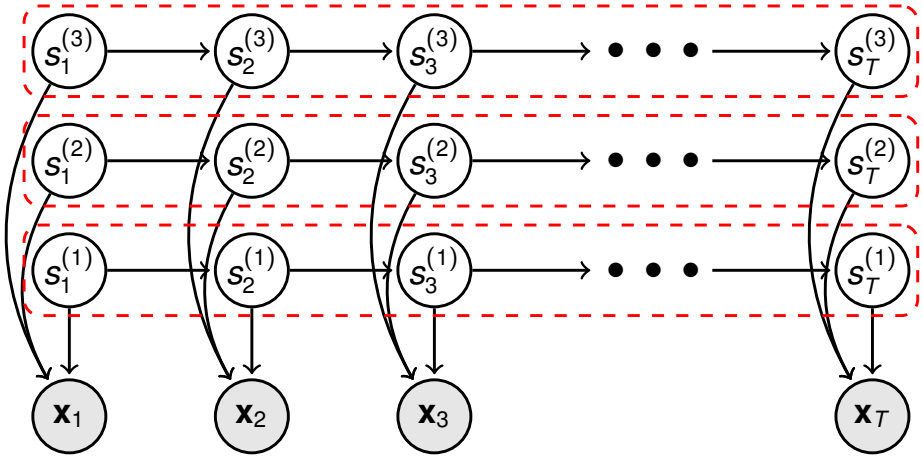
$$= \prod_t P(s_t^m | s_{t-1}^m) \prod_t e^{\left\langle \log P(\mathbf{x}_t | s_t^{1:M}) \right\rangle_{\prod_{\neg m} q^{m'}(s_t^{m'})}}$$

Stuctured FHMM



For the FHMM we can factor the chains:

$$q(s_{1:T}^{1:M}) = \prod_m q^m(s_{1:T}^m)$$

$$q^m(s_{1:T}^m) \propto \exp \left\langle \log P(\mathbf{s}_{1:T}^{1:M}, \mathbf{x}_{1:T}) \right\rangle_{\prod_{\neg m} q^{m'}(s_{1:T}^{m'})}$$

$$= \exp \left\langle \sum_\mu \sum_t \log P(s_t^\mu | s_{t-1}^\mu) + \sum_t \log P(\mathbf{x}_t | s_t^{1:M}) \right\rangle_{\prod_{\neg m} q^{m'}}$$

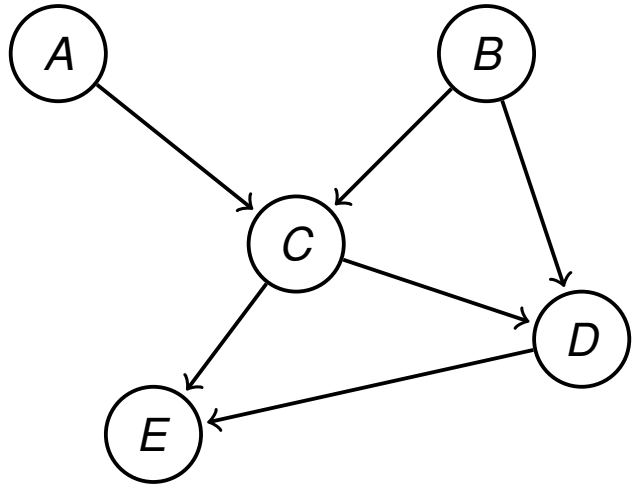
$$\propto \exp \left[\sum_t \log P(s_t^m | s_{t-1}^m) + \sum_t \left\langle \log P(\mathbf{x}_t | s_t^{1:M}) \right\rangle_{\prod_{\neg m} q^{m'}(s_t^{m'})} \right]$$

$$= \prod_t P(s_t^m | s_{t-1}^m) \prod_t e^{\left\langle \log P(\mathbf{x}_t | s_t^{1:M}) \right\rangle_{\prod_{\neg m} q^{m'}(s_t^{m'})}}$$

This looks like a standard HMM joint, with a modified likelihood term $\Rightarrow$ cycle through multiple forward-backward passes, updating likelihood terms each time.

Messages on an arbitrary graph

Consider a DAG:



$$P(\mathcal{X}, \mathcal{Z}) = \prod_k P(V_k | \text{pa}(V_k))$$

and let $q(\mathcal{Z}) = \prod_i q_i(\mathcal{Z}_i)$ for disjoint sets $\{\mathcal{Z}_i\}$.

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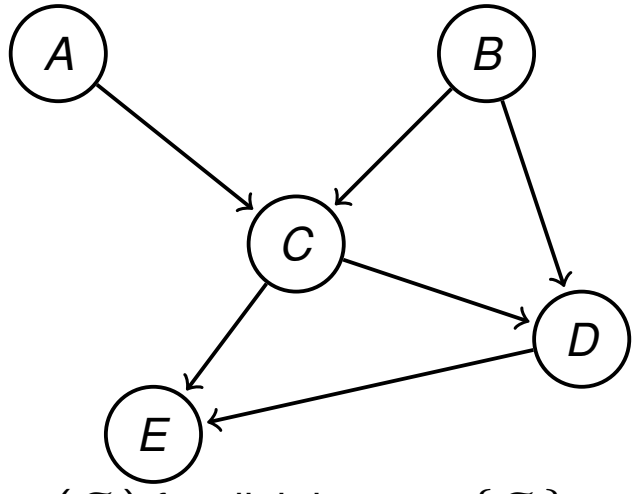

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We have that the VE update for q_i is given by $q_i^*(\mathcal{Z}_i) \propto \exp \langle \log p(\mathcal{Z}, \mathcal{X}) \rangle_{q_{\neg i}(\mathcal{Z})}$ where $\langle \cdot \rangle_{q_{\neg i}(\mathcal{Z})}$ denotes averaging with respect to $q_j(\mathcal{Z}_j)$ for all $j \neq i$

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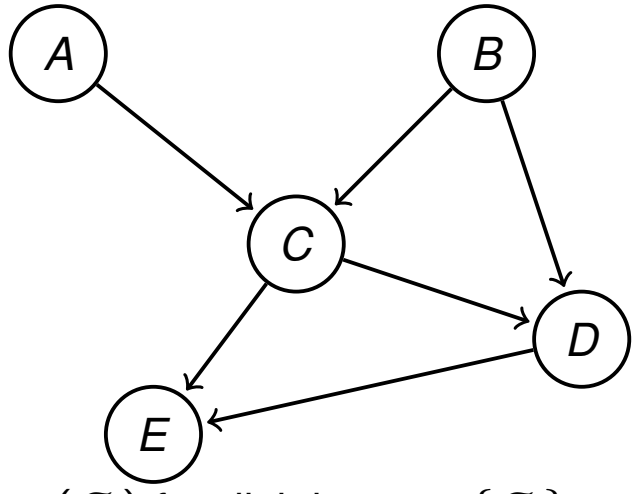

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Then:

$$\begin{aligned} \log q_i^*(\mathcal{Z}_i) &= \left\langle \sum_k \log P(V_k | \text{pa}(V_k)) \right\rangle_{q_{\neg i}(\mathcal{Z})} + \text{const} \\ &= \sum_{j \in \mathcal{Z}_i} \langle \log P(Z_j | \text{pa}(Z_j)) \rangle_{q_{\neg i}(\mathcal{Z})} + \sum_{j \in \text{ch}(\mathcal{Z}_i)} \langle \log P(V_j | \text{pa}(V_j)) \rangle_{q_{\neg i}(\mathcal{Z})} + \text{const} \end{aligned}$$

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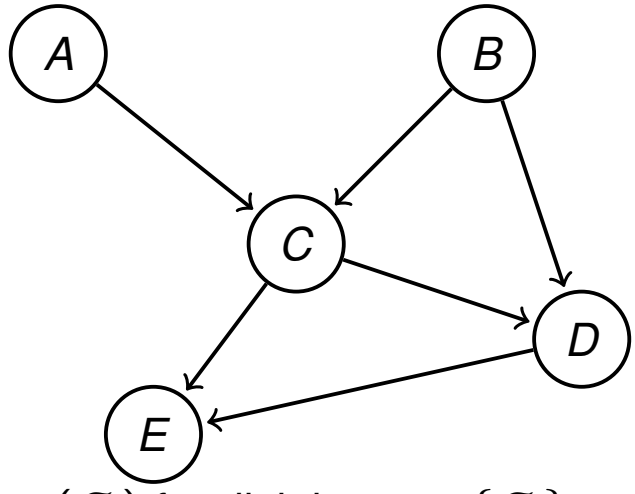

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This defines messages that are passed between nodes in the graph. Each node receives messages from its Markov boundary: parents, children and parents of children (all neighbours in the corresponding factor graph).

Non-factored variational methods

The term variational approximation is used whenever a bound on the likelihood (or on another estimation cost function) is optimised, but does not necessarily become tight.

Many further variational approximations have been developed, including:

- parametric forms (e.g. Gaussian) for non-linear models (later lecture)
  - closed form updates in special cases
  - numerical or sampling-based computation of expectations
  - 'recognition networks' or amortisation to estimate variational parameters
- non-free-energy-based bounds (both upper and lower) on the likelihood.

We can also see MAP- or zero-temperature EM and recognition models as parametric forms of variational inference.

Variational Bayes

So far, we have applied Jensen's bound and factorisations to help with integrals over latent variables.

Variational Bayes

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We can do the same for integrals over parameters in order to bound the log marginal likelihood or evidence.

$$\begin{aligned}\log P(\mathcal{X}|\mathcal{M}) &= \log \iint d\mathcal{Z}\, d\boldsymbol{\theta}\;\; P(\mathcal{X},\mathcal{Z}|\boldsymbol{\theta},\mathcal{M})P(\boldsymbol{\theta}|\mathcal{M}) \\ &= \max_{Q} \iint d\mathcal{Z}\, d\boldsymbol{\theta}\;\; Q(\mathcal{Z},\boldsymbol{\theta}) \log \frac{P(\mathcal{X},\mathcal{Z},\boldsymbol{\theta}|\mathcal{M})}{Q(\mathcal{Z},\boldsymbol{\theta})}\end{aligned}$$

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The constraint that the distribution Q must factor into the product $Q_y(\mathcal{Z})Q_\theta(\boldsymbol{\theta})$ leads to the **variational Bayesian EM algorithm** or just **"Variational Bayes"**.

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The constraint that the distribution Q must factor into the product $Q_y(\mathcal{Z})Q_\theta(\boldsymbol{\theta})$ leads to the **variational Bayesian EM algorithm** or just **"Variational Bayes"**.

Some call this the "Evidence Lower Bound" (ELBO). I'm not fond of that term.

Variational Bayesian EM . . .

Coordinate maximization of the VB free-energy lower bound

$$\mathcal{F}(Q_{\mathcal{Z}}, Q_{\boldsymbol{\theta}}) = \iint d\mathcal{Z}\, d\boldsymbol{\theta}\;\; Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\theta}(\boldsymbol{\theta}) \log \frac{p(\mathcal{X}, \mathcal{Z}, \boldsymbol{\theta}|\mathcal{M})}{Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})}$$

leads to **EM-like** updates:

Variational Bayesian EM ...

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leads to **EM-like** updates:

$$Q^*_{\mathcal{Z}}(\mathcal{Z}) \propto \exp \left\langle \log P(\mathcal{Z}, \mathcal{X}|\boldsymbol{\theta}) \right\rangle_{Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})} \qquad \textit{E-like step}$$

$$Q^*_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \propto P(\boldsymbol{\theta}) \exp \left\langle \log P(\mathcal{Z}, \mathcal{X}|\boldsymbol{\theta}) \right\rangle_{Q_{\mathcal{Z}}(\mathcal{Z})} \qquad \textit{M-like step}$$

Variational Bayesian EM ...

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Maximizing $\mathcal{F}$ is equivalent to minimizing KL-divergence between the *approximate posterior*, $Q(\boldsymbol{\theta})Q(\mathcal{Z})$ and the *true posterior*, $P(\boldsymbol{\theta}, \mathcal{Z}|\mathcal{X})$.

$$\log P(\mathcal{X}) - \mathcal{F}(Q_{\mathcal{Z}}, Q_{\boldsymbol{\theta}}) = \log P(\mathcal{X}) - \iint d\mathcal{Z}\, d\boldsymbol{\theta}\;\; Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \log \frac{P(\mathcal{X}, \mathcal{Z}, \boldsymbol{\theta})}{Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})}$$

$$= \iint d\mathcal{Z}\, d\boldsymbol{\theta}\;\; Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \log \frac{Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})}{P(\mathcal{Z}, \boldsymbol{\theta}|\mathcal{X})} = KL(Q||P)$$

Conjugate-Exponential models

Let's focus on *conjugate-exponential* (**CE**) latent-variable models:

- **Condition (1)**. The joint probability over *variables* is in the exponential family:

$$P(\mathcal{Z}, \mathcal{X}|\boldsymbol{\theta}) = f(\mathcal{Z}, \mathcal{X})\ g(\boldsymbol{\theta}) \exp\left\{\boldsymbol{\phi}(\boldsymbol{\theta})^{\mathsf{T}} \mathsf{T}(\mathcal{Z}, \mathcal{X})\right\}$$

  where $\boldsymbol{\phi}(\boldsymbol{\theta})$ is the vector of *natural parameters*, T are *sufficient statistics*

- **Condition (2)**. The prior over *parameters* is conjugate to this joint probability:

$$P(\boldsymbol{\theta}|\nu, \tau) = h(\nu, \tau)\ g(\boldsymbol{\theta})^{\nu} \exp\left\{\boldsymbol{\phi}(\boldsymbol{\theta})^{\mathsf{T}} \tau\right\}$$

  where ν and τ are hyperparameters of the prior.

Conjugate priors are computationally convenient and have an intuitive interpretation:

- ν: number of pseudo-observations
- τ: values of pseudo-observations

Conjugate-Exponential examples

In the **CE** family:

- Gaussian mixtures
- factor analysis, probabilistic PCA
- hidden Markov models and factorial HMMs
- linear dynamical systems and switching models
- discrete-variable belief networks

Other as yet undreamt-of models combinations of Gaussian, Gamma, Poisson, Dirichlet, Wishart, Multinomial and others.

Not in the **CE** family:

- Boltzmann machines, MRFs (no simple conjugacy)
- logistic regression (no simple conjugacy)
- sigmoid belief networks (not exponential)
- independent components analysis (not exponential)

Note: one can often approximate such models with a suitable choice from the **CE** family.

Conjugate-exponential VB

Given an iid data set $\mathcal{D} = (\mathbf{x}_1, \ldots \mathbf{x}_n)$, if the model is **CE** then:

Conjugate-exponential VB

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- $Q_\theta(\boldsymbol{\theta})$ is also conjugate, *i.e.*

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\begin{aligned}
Q_\theta(\boldsymbol{\theta}) &\propto P(\boldsymbol{\theta}) \quad \exp\left\langle \sum_i \log P(\mathbf{z}_i, \mathbf{x}_i|\boldsymbol{\theta}) \right\rangle_{Q_\mathcal{Z}} \\
&= h(\nu, \boldsymbol{\tau}) g(\boldsymbol{\theta})^\nu e^{\boldsymbol{\phi}(\boldsymbol{\theta})^\mathsf{T} \boldsymbol{\tau}} \quad g(\boldsymbol{\theta})^n e^{\left\langle \log f(\mathcal{Z}, \mathcal{X}) \right\rangle_{Q_\mathcal{Z}}} e^{\boldsymbol{\phi}(\boldsymbol{\theta})^\mathsf{T} \left\langle \sum_i \mathsf{T}(\mathbf{z}_i, \mathbf{x}_i) \right\rangle_{Q_\mathcal{Z}}} \\
&\propto h(\tilde{\nu}, \tilde{\boldsymbol{\tau}}) g(\boldsymbol{\theta})^{\tilde{\nu}} e^{\boldsymbol{\phi}(\boldsymbol{\theta})^\mathsf{T} \tilde{\boldsymbol{\tau}}}
\end{aligned}
$$

with $\tilde{\nu} = \nu + n$ and $\tilde{\boldsymbol{\tau}} = \tau + \sum_i \langle \mathsf{T}(\mathbf{z}_i, \mathbf{x}_i) \rangle_{Q_\mathcal{Z}}$

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The Variational Bayesian EM algorithm

EM for MAP estimation

Goal: maximize $P(\boldsymbol{\theta}|\mathcal{X}, m)$ wrt $\boldsymbol{\theta}$

E Step: compute

$Q_{\mathcal{Z}}(\mathcal{Z}) \leftarrow p(\mathcal{Z}|\mathcal{X}, \boldsymbol{\theta})$

M Step:

$$\boldsymbol{\theta} \leftarrow \operatorname*{argmax}_{\boldsymbol{\theta}} \int d\mathcal{Z}\, Q_{\mathcal{Z}}(\mathcal{Z}) \log P(\mathcal{Z}, \mathcal{X}, \boldsymbol{\theta})$$

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- Reduces to the EM algorithm if $Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) = \delta(\boldsymbol{\theta} - \boldsymbol{\theta}^*)$.

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- $\mathcal{F}_m$ increases monotonically, and incorporates the model complexity penalty.
- Analytical parameter distributions (but not constrained to be Gaussian).

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$$Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \leftarrow \exp \int d\mathcal{Z}\, Q_{\mathcal{Z}}(\mathcal{Z}) \log P(\mathcal{Z}, \mathcal{X}, \boldsymbol{\theta})$$

Properties:

- Reduces to the EM algorithm if $Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) = \delta(\boldsymbol{\theta} - \boldsymbol{\theta}^*)$.
- $\mathcal{F}_m$ increases monotonically, and incorporates the model complexity penalty.
- Analytical parameter distributions (but not constrained to be Gaussian).
- VB-E step has same complexity as corresponding E step.

The Variational Bayesian EM algorithm

EM for MAP estimation

Goal: maximize $P(\boldsymbol{\theta}|\mathcal{X}, m)$ wrt $\boldsymbol{\theta}$

E Step: compute

$Q_{\mathcal{Z}}(\mathcal{Z}) \leftarrow p(\mathcal{Z}|\mathcal{X}, \boldsymbol{\theta})$

M Step:

$$\boldsymbol{\theta} \leftarrow \operatorname*{argmax}_{\boldsymbol{\theta}} \int d\mathcal{Z}\, Q_{\mathcal{Z}}(\mathcal{Z}) \log P(\mathcal{Z}, \mathcal{X}, \boldsymbol{\theta})$$

Variational Bayesian EM

Goal: maximise bound on $P(\mathcal{X}|m)$ wrt $Q_{\boldsymbol{\theta}}$

VB-E Step: compute

$Q_{\mathcal{Z}}(\mathcal{Z}) \leftarrow p(\mathcal{Z}|\mathcal{X}, \bar{\boldsymbol{\phi}})$

VB-M Step:

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Properties:

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- $\mathcal{F}_m$ increases monotonically, and incorporates the model complexity penalty.
- Analytical parameter distributions (but not constrained to be Gaussian).
- VB-E step has same complexity as corresponding E step.
- We can use the junction tree, belief propagation, Kalman filter, etc, algorithms in the VB-E step of VB-EM, but **using expected natural parameters**, $\bar{\boldsymbol{\phi}}$.

VB and model selection

- Variational Bayesian EM yields an approximate posterior $Q_{\boldsymbol{\theta}}$ over model parameters.

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- These lower bounds can be compared amongst models to learn the right (structure, connectivity . . . of the) model
- If a continuous domain of models is specified by a hyperparameter $\boldsymbol{\eta}$, then the VB free energy depends on that parameter:

$$\mathcal{F}(Q_{\mathcal{Z}}, Q_{\boldsymbol{\theta}}, \boldsymbol{\eta}) = \iint d\mathcal{Z}\, d\boldsymbol{\theta}\;\; Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \log \frac{P(\mathcal{X}, \mathcal{Z}, \boldsymbol{\theta}|\boldsymbol{\eta})}{Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta})} \leq P(\mathcal{X}|\eta)$$

A hyper-M step maximises the current bound wrt η:

$$\boldsymbol{\eta} \leftarrow \underset{\boldsymbol{\eta}}{\operatorname{argmax}} \iint d\mathcal{Z}\, d\boldsymbol{\theta}\;\; Q_{\mathcal{Z}}(\mathcal{Z}) Q_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \log P(\mathcal{X}, \mathcal{Z}, \boldsymbol{\theta}|\boldsymbol{\eta})$$

ARD for unsupervised learning

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- In this case, these parameters select "relevant" latent dimensions, effectively learning the dimensionality of $\mathbf{z}$.

Augmented Variational Methods

In our examples so far, the approximate variational distribution has been over the “natural” latent variables (and parameters) of the generative model.

Sometimes it may be useful to introduce additional latent variables, solely to achieve computational tractability.

Two examples are GP regression and the GPLVM.

Sparse GP approximations

GP predictions:

$$y'|X, Y, \mathbf{x}' \sim \mathcal{N}\left(K_{\mathbf{x}'X}(K_{XX} + \sigma^2 I)^{-1} Y, K_{\mathbf{x}'\mathbf{x}'} - K_{\mathbf{x}'X}(K^{-1}_{XX+\sigma^2 I)} K_{X\mathbf{x}'} + \sigma^2\right)$$

Evidence (for learning kernel hyperparameters):

$$\log P(Y|X) = -\frac{1}{2}\log|2\pi(K_{XX} + \sigma^2 I)| - \frac{1}{2} Y(K_{XX} + \sigma^2 I)^{-1} Y^{\mathsf{T}}$$

Computing either form requires inverting the $N \times N$ matrix K_{XX}, in $\mathcal{O}(N^3)$ time.

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One proposal to make this more efficient is to find (or select) a smaller set of possibly fictitious measurements U at inputs Z such that

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What values should U and Z take?

Variational Sparse GP approximations

Write F for the (smooth) GP function values that underlie Y (so $Y \sim \mathcal{N}\left(F, \sigma^2 I\right)$).

Introduce latent measurements U at inputs Z (and integrate over U).

The likelihood can be written

$$P(Y|X) = \iint dF\, dU\, P(Y, F, U|X, Z) = \iint dF\, dU\, P(Y|F)P(F|U, X, Z)P(U|Z)$$

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$$\mathcal{F}(q(F, U), \theta) = \left\langle \log \frac{P(Y|F)P(F|U, X, Z)P(U|Z)}{q(F, U)} \right\rangle_{q(F,U)}$$

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A similar approach can be used to learn X if they are unobserved (*i.e.* in the GPLVM). Assume $q(X,F,U) = q(X)P(F|X,U)q(U)$. Then $\mathcal{F} = \langle \log P(Y,F,U|X) \log P(X)\rangle_{q(U)q(X)}$ which simplifies into tractable components in much the same way as above.

A few references

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