Homework 6
Systems & Theoretical Neuroscience [SWC]

Due: Mon, 12th February

1 Numerically integrating Hodgkin-Huxley equations

One of the goals of analysing dynamical systems is to compute the trajectory of the system (how it evolves over time) from a particular starting point. Sometimes it is possible to derive an exact analytical solution (for example the exponential decay you saw in the lectures). In other cases we can numerically approximate the evolution of the system by iteratively calculating how the system changes over small timesteps. The simplest method - which you actually applied during the tutorial - is called the **Euler method** and simply involves computing the instantaneous time derivative and applying it for a whole time-step.

For instance, if you have a dynamical system of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{1}$$

then you would compute:

$$x_{t+1} = x_t + \Delta t \frac{\mathrm{d}x}{\mathrm{d}t} \tag{2}$$

where Δt is the timestep and $\frac{\mathrm{d}x}{\mathrm{d}t}$ is computed using the value of x at time t, i.e. $f(x_t)$.

If you apply this repeatedly with a small enough Δt, you get a good approximation of the timecourse $x(t)$.

a) Can you suggest why this method isn't entirely accurate? What errors are introduced if Δt is too high? Bonus: read Wikipedia to learn about more accurate ways to numerically integrate ordinary differential equations.

We are going to use the Euler method to analyse the spiking of neurons with the Hodgkin Huxley model, which is defined below. If you missed the lecture on Hodgkin Huxley, please first read one of the following sources[1]:

http://neuronaldynamics.epfl.ch/online/Ch2.S2.html
https://en.wikipedia.org/wiki/Hodgkin%E2%80%93Huxley_model

The Hodgkin-Huxley model:

$$
\begin{aligned}
C\frac{dV}{dt} &= -\overline{g}_{Na}m^3h(V-E_{Na}) - \overline{g}_K n^4(V-E_K) - \overline{g}_L(V-E_L) + I_{stim} && (3)\\
\frac{dm}{dt} &= \alpha_m(1-m) - \beta_m m && (4)\\
\frac{dn}{dt} &= \alpha_n(1-n) - \beta_n n && (5)\\
\frac{dh}{dt} &= \alpha_h(1-h) - \beta_h h && (6)\\
\alpha_n(V) &= 0.01(V+55)/[1-\exp(-(V+55)/10)] && (7)\\
\beta_n(V) &= 0.125\exp(-(V+65)/80) && (8)\\
\alpha_m(V) &= 0.1(V+40)/[1-\exp(-(V+40)/10)] && (9)\\
\beta_m(V) &= 4\exp(-(V+65)/18) && (10)\\
\alpha_h(V) &= 0.07\exp(-(V+65)/20) && (11)\\
\beta_h(V) &= 1/[\exp(-(V+35)/10)+1] && (12)
\end{aligned}
$$

Let $C = 10\,\mathrm{nF/mm^2}, \overline{g}_L = .003\,\mathrm{mS/mm^2}, \overline{g}_K = 0.36\,\mathrm{mS/mm^2}, \overline{g}_{Na} = 1.2\,\mathrm{mS/mm^2}, E_K = -77\,\mathrm{mV}, E_L = -54.387\,\mathrm{mV}$, and $E_{Na} = 50\,\mathrm{mV}$.

[1] Another good resource is this book. You might just be able to find a version online, but you didn't hear that from us.

b) In python or Matlab, run simulations of this model using the Euler method to update each variable. Use an integration timestep of 0.05ms. Run the simulations with $I_{stim} = 200$ nA/mm^2. Plot the membrane potential (V) and gating variables (m, h, and n) versus time.

c) Write down expressions for the equilibrium values of the gating variables (m_∞, h_∞, and n_∞), and plot them versus voltage.

d) Plot the firing rate versus I_{stim}, up to a firing rate of 50 Hz. The firing rate should jump suddenly from zero to a non-zero value. This is called a type II behaviour. Type I behaviour is when the firing rate begins at zero and increases continuously without any jumps.

e) What happens to the plot of firing rate versus I_{stim} as you decrease $\overline{g}_K$?

f) Spikes are initiated at the axon hillock, where the axon meets the soma. This is because $\overline{g}_{Na}$ is very high there. What happens to the plot of firing rate versus I_{stim} as you increase $\overline{g}_{Na}$?

2 Nullclines - graphical analysis

Consider the phase portraits sketched in Figure 1. All four of these represent nullclines of dynamical systems of the form:

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

You might recall from tutorial that without looking at the exact differential equations, quite some information about the dynamical system can be gained from analysing their phase portraits. For all four subfigures in Figure 1:

a) Identify the fixed points of the dynamical system drawn in this phase plane.

b) Fill in the pictures with arrows designating the direction of flow. Two arrows are already drawn - these should be sufficient to infer flow directions everywhere else.

c) For each fixed point, say whether it is stable or unstable.

3 Nullclines with math

The following model exhibits interesting behaviour that you are likely to see again, where the dynamics change depending on certain hyperparameters.

$$\tau_x \frac{dx}{dt} = -x + \tanh\left(\beta(x - y)\right)$$
$$\tau_y \frac{dy}{dt} = -y + \alpha x.$$

a) Draw the y nullcline for a few different values of $\alpha > 0$. Add arrows to show the y-flow.

b) Draw the x nullcline for a few different values of $\beta > 1$. Add arrows to show the x-flow. This nullcline is harder to draw by hand - you may wish to use a computer.

c) Find a setting of α and β such that there is only one fixed point and draw the full phase diagram. Is the fixed point stable? Draw trajectories starting **near** $x = y = 0$.

d) Find a setting of α and β such that there are 3 fixed points. Draw the full phase diagram. Which fixed points are stable/unstable? Draw trajectories starting **near** $x = y = 0$.

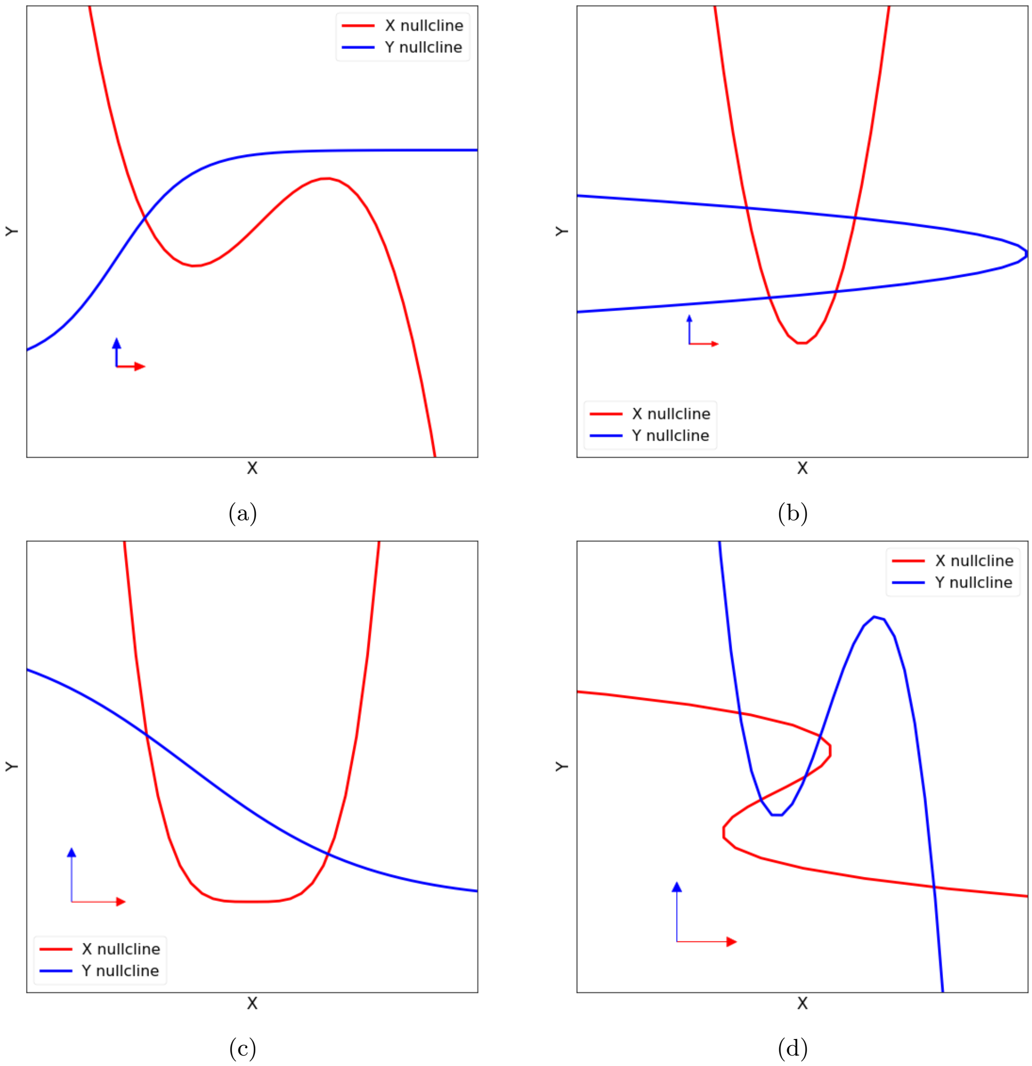

Figure 1: Phase planes for four different dynamical systems.