Assignment 3
Theoretical Neuroscience

TAs:
Kirsty McNaught (kirsty.mcnaught.16@ucl.ac.uk)
Lea Duncker (lea.duncker.11@ucl.ac.uk)
Jorge Menendez (jorge.menendez.15@ucl.ac.uk)

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1. **Stability of equilibria**

Consider Wilson-Cowan equations of the form

$$\tau \dot{\nu}_E = \phi_E(\nu_E, \nu_I) - \nu_E \tag{1a}$$
$$\tau \dot{\nu}_I = \phi_I(\nu_I, \nu_E) - \nu_I \tag{1b}$$

where the gain functions, ϕ_E and ϕ_I, are increasing functions of ν_E and decreasing functions of ν_I (e.g, $\phi_E(\nu_E, \nu_I) \sim 1 + \tanh(W_{EE}\nu_E - W_{EI}\nu_I + \theta_E)$).

Nullclines for Eq. (1) are sketched in the figure below. Show that equilibria A and C are stable, B is unstable, and D may or may not be stable. Give conditions for the stability of equilibrium D in terms of the derivatives of the gain functions evaluated at the equilibrium.

Hint: This problem is relatively hard, in the sense that it requires a somewhat deep understanding of nullclines and their construction, and also strong familiarity with linear stability analysis in two dimensions. On the other hand, the answer doesn't require a huge amount of algebra – only a few lines. The main insight you need is that you can compute the slopes of the nullclines in terms of derivatives of the gain functions. Once you do that, the rest should be easy (ish).

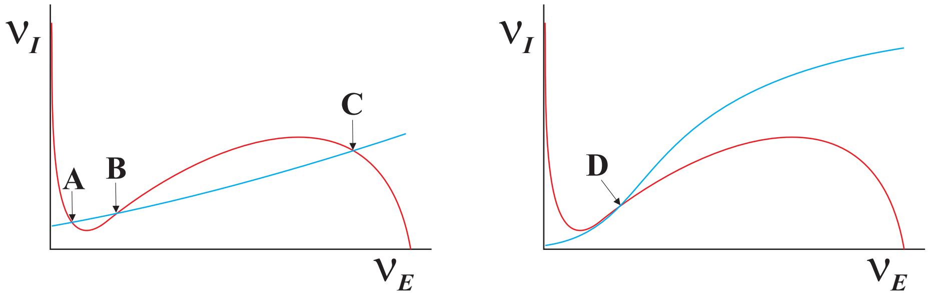

Figure 1: Two possible sets of nullclines. In both figures, the red curve is the excitatory nullcline and the blue curve is the inhibitory one.

2. **Adaptation**

Consider a network of N analog neurons that obey the time-evolution equations

$$\tau \frac{dx_i}{dt} = \phi\left(\sum_{j=1}^{N} W_j x_j - \theta_i\right) - x_i \,. \tag{2}$$

a. Assume that $\theta_i = \theta \;\forall i$. Show that Eq. (2) can be effectively reduced to a one-variable model,

$$\tau \frac{dz}{dt} = \phi\left(Jz - \theta\right) - z \,. \tag{3}$$

Write down expressions for z in terms of the W_i and x_i and J in terms of the W_i.

b. Let's go back to Eq. (2), where θ_i depends on i. Show that Eq. (2) can still be reduced to a one-variable model,

$$\tau \frac{dz}{dt} = \tilde{\phi}\left(Jz\right) - z \tag{4}$$

where J is the same as in part a. Write down an expressions for $\tilde{\phi}(\cdot)$ in terms of $\phi(\cdot)$ and W_i and θ_i.

c. Assume that both W_i and θ_i are correlated random variables with joint distribution $p(W, \theta)$. Assuming $N \to \infty$, write down an expression for $\tilde{\phi}(Jz)$ as an integral over this joint distribution.

d. Let's go back to the case in which $\theta_i = \theta$, so that z evolves according to Eq. (3). Let $\phi(y) = \tanh(y)$ (which isn't realistic because it allows negative firing rates, but it makes the analysis easier). To model spike frequency adaptation, let θ evolve according to

$$\tau_0 \dot{\theta} = -(\theta - \theta_0 z)\,, \tag{5}$$

with $\tau_0 \gg \tau$. Assume that $\theta_0 > J - 1 > 0$. Sketch the nullclines.

e. Show that the system exhibits bursting, and sketch $z(t)$ and $\theta(t)$ versus time. Here "bursting" just means a limit cycle in θ-z space. We call it bursting because $\tau_0 \gg \tau$, so z spends most of its time changing slowly, with only brief periods during which it changes very rapidly from positive to negative or back.

3. **Averages**

Consider the quantity

$$\eta_i = \sum_j W_{ij} \nu_j \,, \tag{6}$$

with weights given by

$$W_{ij} = \frac{1}{K^{1/2}} \begin{cases} W_0 + w_{ij} & \text{with probability } K/N \\ 0 & \text{with probability } 1 - K/N \,. \end{cases} \tag{7}$$

Here K is the average number of connections per neuron and N is the number of neurons (so $K < N$). Assume that the w_{ij} are drawn *iid* from a distribution $p(w)$ with mean 0 and variance σ_w^2,

$$\int dw\ wp(w) = 0 \tag{8a}$$

$$\int dw\ w^2 p(w) = \sigma_w^2\,. \tag{8b}$$

Treat η_i as a random variable with respect to the index i. Assuming W_{ij} and ν_j are independent, compute the mean and variance of η_i.