Background material for the Kernels course

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September 2017

1 Spaces

Space

A *space* is a *set* with some extra structure or conditions applied.

Vector space

A *vector space* (also called a *linear space*) is composed of three things: a *set* of vectors, and two *operations*: vector addition and scalar multiplication. The result of addition or multiplication must also lie in the same space.

The objects in the set are referred to as vectors precisely because they follow the appropriate rules for vector operations, even if they aren't what you normally call vectors **- for instance, they might be functions**.

Any set of vectors can form a vector space if it satisfies two conditions:

1. If a vector x is in the set, then so is Ax for any complex scalar A.
2. If two vectors x and y are in the set, then so is their sum $x + y$.

Euclidean space

The *euclidean space* is the vector space of all n-tuples of real numbers $\mathbb{R}^n$. It also has the associated euclidean distance metric, which makes it a *metric space*.

Normed vector space

A *normed vector space* comprises a vector space plus an associated *norm*, i.e. a function that assigns a strictly positive length or size to each vector in the space (except zero).

Metric space

A *metric space* is a set with a global distance function (the *metric*) that, for every two points in the space, gives the distance between them as a non-negative real number. The distance function must satisfy a certain geometric constraints, such as symmetry and the triangle inequality.

A norm induces a metric, therefore all normed spaces are metric spaces but the reverse does not always hold.

Complete space

Banach space

A *Banach space* is a normed vector space which is *complete*.

Inner product space

An *inner product space* is a *metric space* in which the metric is an inner product. If the inner product defines a *complete* metric, then the inner product space is called a *Hilbert space*.

Examples

Examples of finite-dimensional Hilbert spaces include:

1. The real numbers $\mathbb{R}^n$ with $\langle u, v \rangle$ the vector dot product of u and v.

2. The complex numbers $\mathbb{C}^n$ with $\langle u, v \rangle$ the vector dot product of u and the complex conjugate of v.

An example of an infinite-dimensional Hilbert space is L^2, the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that the integral of f^2 over the whole real line is finite. In this case, the inner product is $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$.

References

Much of the material above came from the following wikipedia articles, CC BY-SA:
https://en.wikipedia.org/wiki/Space_(mathematics)
https://en.wikipedia.org/wiki/Banach_space
https://en.wikipedia.org/wiki/Metric_space
https://en.wikipedia.org/wiki/Hilbert_space
https://en.wikipedia.org/wiki/Vector_space

2 Vector spaces

2.1 Definition of a vector sapce

A vector space V is a set of elements that is endowed with two operations: addition between vectors and multiplication by scalars:

- for all vectors x and y the addition $x + y$ defines a new vector $z = x + y$ that belongs to V.

- for all real numbers λ and vectors x in V, λx is also a vector

The addition operation and multiplication by a scalar need to satisfy these following properties:

- stability: for all vectors x and y in V the addition $+$ defines a new vector $z = x+y$ that belongs to V.

- commutativity: for all vectors x and y in V: $x + y = y + x$

- associativity: for all vectors x, y and z in V : $(x + y) + z = x + (y + z)$

- neutral element: there exists a vector e in V such that for all x in V : $x + e = x$. (We often denote e by 0.)

- opposite element: every vector x in V has an opposite vector y such that $x+y = e$. the sum of x and its opposite give the neutral element e. We denote the opposite vector of x by $-x$.

The multiplication operation by real numbers needs to satisfy the following:

- stability: for all vectors x in V and all real numbers λ the operation λx defines a new vector $z = \lambda x$.
- compatibility: for all λ and μ real number and x a vetor in V: $\lambda(\mu x) = (\lambda\mu)x$
- multiplication by 1: $1x = x$
- Distributivity of scalar multiplication with respect to vector addition:$\lambda(x + y) = \lambda x + \lambda y$
- Distributivity of scalar multiplication with respect to scalar addition: $(\lambda + \mu)x = \lambda x + \mu x$

These properties allow to make sure the addition and multiplication are similar to the operation we encounter in "real life":

2.2 Examples

- The set of real numbers R is a vector space with the usual addition and multiplication
- the set of vectors in the plane R^2 is also a vector space for the following addition and multiplication: $x + y = (x_1 + y_1, x_2 + y_2)$ and $\lambda x = (\lambda x_1, \lambda x_2)$
- the set of sequences $x = (x_1, x_2, ..., x_n, ...)$that take real values is a vector space for the following operations: for any two sequence $x = (x_1, x_2, ..., x_n, ...)$ and $y = (y_1, y_2, ..., y_n, ...)$
  $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n, ...)$ $\lambda x = (\lambda x_1, \lambda x_2, ..., \lambda x_n, ...)$
- the set of functions f defined on the real numbers R and that real values define a vector space with the following addition and multiplication operation: for any functions f and g the addition operation defined a new function h that satisfies: $h(x) = f(x) + g(x)$ for all real numbers x and the multiplication of f by a scalar λ defines a new function F that satisfies: $F(x) = \lambda f(x)$ for all real numbers x

2.3 Linearly independent families and generating families

- A family F is just a fancy name for a set of vectors in V $F = v_1, v_2, .., v_p$, it could be finite of infinite. It can be countable (can be indexed by integers) or uncountable (can't be indexed by integers). For now we consider a finite family of the form $F = \{v_1, v_2, .., v_p\}$
- A family F is said to be linearly independent if one can't express any vector of this family as a linear combination of the remaining vectors in F. In other words, for any scalars λ_1, λ_2 ,..., λ_p, if we have that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + ... + \lambda_p v_p = 0$ then the only possible way to achieve this is by having all scalars λ_1, λ_2, $...\lambda_n$ equal to 0.
- Example: for $V = R^3$ the family of vectors $\{x = (1, 0, 0), y = (0, 1, 0), z = (0, 0, 1)\}$ is linearly independent, while the family $\{x = (1, 1, 1), y = (1, 1, 0), z = (0, 0, 1)\}$ is not.
- A family F is said to be generating the space V any vector x in V can be expressed as a linear combination of elements in the family F: for all x in V we can find λ_1, λ_2 ,..., λ_p such that $x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + ... + \lambda_p v_p$. Some of the scalars can be 0

- Example: for $V = R^3$, the family of vectors $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is generating V: Indeed any vector x in V is of the form $x = (x_1, x_2, x_3)$ which can further be written as: $x = x_1 e_1 + x_2 e_2 + x_3 e_3$
- A basis is a family F that is linearly independent and generating the space V at the same time. In this case any vector x has a unique decomposition as a linear combination of elements of the family: $x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + ... + \lambda_p v_p$. This decomposition exists, since the family F generates the space V and it is unique because its elements are linearly independent (no redundancy)

Some remarks:

- When a vector space V has a basis that contains a finite number of elements we say that V has a finite dimension and its dimension is equal to the number of elements in the basis.
- Note that a vector space has an infinity of possible basis, but when it is of finite dimension, all these basis have the same number of elements.

3 Normed spaces

We say that a space V is a normed space if its a vector space endowed with a norm $\|.\|$. A norm captures the notion of length of a vector or the distance between a vector and the null O or the origin of the space V. For a norm to be valid it needs to satisfy the following properties:

- $\|.\|$ is defined from V to the set of non-negative real numbers R^+.
- if $\|x\| = 0$ then $x = 0$
- Triangular inequality: for all x ,y we have that $\|x + y\| \leq \|x\| + \|y\|$
- Scaling : For any scalar λ: $\|\lambda x\| = |\lambda|\|x\|$. Where $|\lambda|$ is the absolute value of λ

Example of normed spaces:

- $V = R^2$ with the application $\|x\| = \sqrt{x_1^2 + x_2^2}$ is a normed space.
- more generally, any euclidien space of the form $V = R^n$ is a normed space , if endowed with the following norm: $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. It is called the euclidien norm and is generally denoted by $\|x\|_2$ Another norm in the same space R^n is the 1-norm: $\|x\|_1 = |x_1| + |x_2| + ... + |x_n|$

  A third interesting norm is called the supremum norm or max-norm: $\|x\|_\infty = \max(|x_1|, |x_2|, ..., |x_n|)$
- The set of continuous functions f that that are defined on the interval $[0, 1]$ and that takes real values is a normed space, if we endow it with the following norm: $\|f\| = \max_{x \in [0,1]} |f(x)|$ It is also called the supremum norm.

3.1 Complete spaces, also called Banach spaces

An interesting kind of normed spaces are the ones where every Cauchy sequence is convergent . (More on this coming soon). Every normed space that has a finite dimension is complete. More specifically all Euclidian spaces are complete R^n.

The notion of completeness is more tricky for spaces of infinite dimensions (like spaces of functions and sequences).

4 Hilbert space

Coming soon

5 Notation for sets

A set is a collection of distinct objects, considered as an object in its own right. For example, the numbers 2, 4, and 6 are distinct objects when considered separately, but when they are considered collectively they form a single set of size three, written $\{2, 4, 6\}$.

Example notation	Meaning
$\{1, 2, 3\}$	The set containing the numbers 1, 2 and 3
$\{vec(2, 3), \text{'hello'}, 42\}$	The set containing the items listed
$\{x \mid x \text{ is a natural number}\}$	The set of all x such that x is a natural number
$\{x \in \mathbb{R} : x^2 = 1\}$	The set of all real numbers such that $x^2 = 1$
$x \in A$	x is a member of set A
$x \notin A$	x is not a member of set A
$\{\}$, or Ø, or $\varnothing$	The null/empty set
$\{1, 2\} \subseteq \{1, 2, 3\}$	$\{1, 2\}$ is a subset of $\{1, 2, 3\}$. The horizontal line implies 'proper subset of, *or* the same set'. A set is a subset of itself.
$\{1, 2\} \subset \{1, 2, 3\}$	$\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$. A set is *not* a proper subset of itself.
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$	The union of A and B is the set of all items which are in set A and/or set B. e.g. $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$	The intersection of A and B is the set of all items which are in A and in B. e.g. $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$
$A \cap B = \varnothing$	The intersection of A and B is the empty set. A and B are *disjoint*.
$A = B$	Sets A and B are equal iff every element of A is also an element of B, and every element of B is an element of A.
$\{f : [0, 1] \to \mathbb{R}\}$	The set of functions f that map from the range $[0, 1]$ to a real number