Assignment 6
Theoretical Neuroscience

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1. **Fisher information for a Gaussian distribution**

   Consider a conditional response distribution of the form

   $$p(\mathbf{r}|x) = \frac{e^{-(\mathbf{r}-\mathbf{f}(x))\cdot\boldsymbol{\Sigma}^{-1}(x)\cdot(\mathbf{r}-\mathbf{f}(x))/2}}{\mathrm{Det}(2\pi\boldsymbol{\Sigma}(x))^{1/2}}\,.$$

   Compute the Fisher information,

   $$I(x) = \left\langle -\frac{\partial^2 \log p(\mathbf{r}|x)}{\partial x^2} \right\rangle,$$

   where the average is with respect to $p(\mathbf{r}|x)$. Note that the covariance matrix depends on x; this complicated the expression.

2. **Bias in a locally optimal linear estimator**

   Consider a population of neurons whose firing activity is given by the usual tuning curve plus noise model,

   $$\mathbf{r} = \mathbf{f}(x) + \boldsymbol{\xi}. \tag{1}$$

   The noise is zero mean and has covariance matrix $\boldsymbol{\Sigma}$,

   $$\langle \boldsymbol{\xi}\boldsymbol{\xi} \rangle = \boldsymbol{\Sigma}.$$

   Consider a linear estimator, $\mathbf{w}$,

   $$\hat{x} - x_0 = \mathbf{w}\cdot(\mathbf{r} - \mathbf{f}(x_0)).$$

   Show that if $\boldsymbol{\Sigma}$ is independent of x, $\mathbf{w}$ is chosen to be a optimal (in the sense of minimum variance) and unbiased at $x = x_0$, and x is close to x_0, then the bias, defined to be

   $$b(x) = \frac{\partial \langle \hat{x} \rangle}{\partial x} - 1,$$

   is given, to lowest order in $x - x_0$, by

   $$b(x_0) = \frac{1}{2}\frac{\partial \log I(x_0)}{\partial x_0}(x - x_0).$$

3. **Differential correlations**

Consider a covariance matrix, $\mathbf{\Sigma}_0$, perturbed by a rank one matrix,

$$\mathbf{\Sigma} = \mathbf{\Sigma}_0 + \epsilon \mathbf{u}(x)\mathbf{u}(x).$$

Assume a tuning curve plus noise model, as in Eq. (1). Show that

$$I(x) = I_0(x)\sin^2\theta + \frac{I_0(x)\cos^2(\theta)}{1+\epsilon I_u(x)}$$

where

$$\begin{aligned} I_0(x) &= \mathbf{f}'(x)\cdot\mathbf{\Sigma}_0^{-1}\cdot\mathbf{f}'(x) \\ I_u(x) &= \mathbf{u}(x)\cdot\mathbf{\Sigma}_0^{-1}\cdot\mathbf{u}(x) \\ \cos^2\theta &= \frac{[\mathbf{f}'(x)\cdot\mathbf{\Sigma}_0^{-1}\cdot\mathbf{u}(x)]^2}{\mathbf{f}'(x)\cdot\mathbf{\Sigma}_0^{-1}\cdot\mathbf{f}'(x)\ \mathbf{u}(x)\cdot\mathbf{\Sigma}_0^{-1}\cdot\mathbf{u}(x)}\,. \end{aligned}$$

If both $I_0(x)$ and $I_u(x)$ are $\mathcal{O}(n)$, where n is the number of neurons, then the only way to have $\mathcal{O}(1)$ information is to have $\theta = 0$, for which $\mathbf{u}(x) = \mathbf{f}'(x)$.