

Information Theory

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Quantifying a Code

- ▶ How much information does a neural response carry about a stimulus?
- ▶ How efficient is a hypothetical code, given the statistical behaviour of the components?
- ▶ How much better could another code do, given the same components?
- ▶ Is the information carried by different neurons complementary, synergistic (whole is greater than sum of parts), or redundant?
- ▶ Can further processing extract more information about a stimulus?

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Information theory does not directly address:

- ▶ estimation (but there are some relevant bounds)
- ▶ computation (but “information bottleneck” might provide a motivating framework)
- ▶ representation (but redundancy reduction has obvious information theoretic connections)

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We need to start by considering the uncertainty in a probability distribution $\rightarrow$ called the **entropy**

Let $S \sim P(S)$. The entropy is the minimum number of bits needed, on average, to specify the value S takes, assuming $P(S)$ is known.

Equivalently, the minimum average number of yes/no questions needed to guess S .

Entropy

Definition

Properties

Examples

Applications

Conclusion

Entropy

- Suppose there are M equiprobable stimuli: $P(s_m) = 1/M$.

To specify which stimulus appears on a given trial, we would need assign each a (binary) number. This would take,

$$\begin{aligned} B_s &\leq \log_2 M + 1 & [2^B \geq M] \\ &= -\log_2 \frac{1}{M} + 1 \text{ bits} \end{aligned}$$

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- Now suppose we code N such stimuli, drawn iid, at once.

$$\begin{aligned} B_N &\leq \log_2 M^N + 1 \\ &\rightarrow -N \log_2 \frac{1}{M} \quad \text{as } N \rightarrow \infty \\ \Rightarrow B_s &\rightarrow -\log_2 p \text{ bits} \end{aligned}$$

This is called block coding. It is useful for extracting theoretical limits. The nervous system is unlikely to use block codes in time, but may in space.

Entropy

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$$P(S_1, S_2, \dots, S_N) = \prod_m p_m^{n_m} \quad [\text{where } n_m = (\# \text{ of } S_i = s_m)].$$

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Rather than appealing to typicality, we could instead have used the law of large numbers directly:

$$\frac{1}{N} \log_2 P(S_1, S_2, \dots, S_N) = \frac{1}{N} \log_2 \prod_i P(S_i) = \frac{1}{N} \sum_i \log_2 P(S_i) \xrightarrow{N \rightarrow \infty} \mathbb{E}[\log_2 P(S_i)]$$

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$$P(S|r) = \frac{P(r|S)P(S)}{\sum_s P(r|s)P(s)}$$

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$$\mathbf{H}[S|R] = \sum_r P(r) \left[- \sum_s P(s|r) \log_2 P(s|r) \right] = - \sum_{s,r} P(s, r) \log_2 P(s|r)$$

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It is easy to show that:

1. $H[S|R] \leq H[S]$
2. $H[S|R] = H[S, R] - H[R]$
3. $H[S|R] = H[S]$ iff $S \perp\!\!\!\perp R$

$$- \sum_{s,r} P(s,r) \log_2 P(s,r)$$

Average Mutual Information

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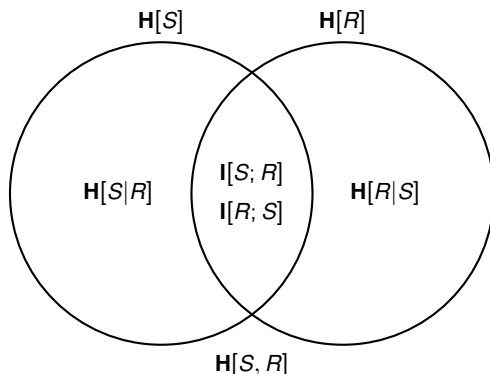
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It follows from the definition that

$$\begin{aligned}\mathbf{I}[S; R] &= \sum_s P(s) \log \frac{1}{P(s)} - \sum_{s,r} P(s, r) \log \frac{1}{P(s|r)} \\&= \sum_{s,r} P(s, r) \log \frac{1}{P(s)} + \sum_{s,r} P(s, r) \log P(s|r) \\&= \sum_{s,r} P(s, r) \log \frac{P(s|r)}{P(s)} \\&= \sum_{s,r} P(s, r) \log \frac{P(s, r)}{P(s)P(r)} \\&= \mathbf{I}[R; S]\end{aligned}$$

Average Mutual Information

The symmetry suggests a Venn-like diagram.



All of the additive and equality relationships implied by this picture hold for two variables. Unfortunately, we will see that this does not generalise to any more than two.

Kullback-Leibler Divergence

Another useful information theoretic quantity measures the difference between two distributions.

$$\begin{aligned}\text{KL}[P(S) \parallel Q(S)] &= \sum_s P(s) \log \frac{P(s)}{Q(s)} \\ &= \sum_s P(s) \log \frac{1}{Q(s)} - \mathbf{H}[P]\end{aligned}$$

Handwritten annotations:

- A blue bracket under the sum $\sum_s$ in the second equation is labeled $\mathbb{E}$.
- A blue bracket under the term $\log \frac{1}{Q(s)}$ in the second equation is labeled "cross entropy".
- A blue arrow points from the "cross entropy" label to the text "codeword length under Q".

$$H[P] = \underbrace{\sum_s P(s)}_{\mathbb{E}} \underbrace{\log \frac{1}{P(s)}}_{\text{codeword length}}$$

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Excess cost in bits paid by encoding according to Q instead of P .

$$\begin{aligned}-\mathbf{KL}[P \| Q] &= \sum_s P(s) \log \frac{Q(s)}{P(s)} \\ &\leq \log \sum_s P(s) \frac{Q(s)}{P(s)} \quad \text{by Jensen} \\ &= \log \sum_s Q(s) = \log 1 = 0\end{aligned}$$

So $\mathbf{KL}[P \| Q] \geq 0$. Equality iff $P = Q$

Mutual Information and KL

$$I[S; R] = \sum_{s,r} P(s, r) \log \frac{P(s, r)}{P(s)P(r)} = \mathbf{KL}[P(S, R) \| P(S)P(R)]$$

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Thus:

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Thus:

1. Mutual information is always non-negative

$$I[S; R] \geq 0$$

2. Conditioning never increases entropy

$$H[S|R] \leq H[S]$$

Multiple Responses

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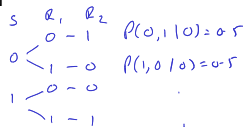
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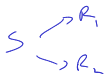
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Thus, the Venn-like diagram with three variables is misleading.

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Then,

$$\begin{aligned} P(R_2, S|R_1) &= P(R_2|R_1)P(S|R_1) \\ \Rightarrow P(S|R_1, R_2) &= P(S|R_1) \end{aligned}$$

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Equality holds iff $S \rightarrow R_2 \rightarrow R_1$ as well. In this case R_2 is called a **sufficient statistic** for S .

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$$\begin{aligned}\mathbf{H}[S_1, S_2, \dots, S_n] &= \mathbf{H}[S_n | S_1, S_2, \dots, S_{n-1}] + \mathbf{H}[S_1, S_2, \dots, S_{n-1}] \\ &= \mathbf{H}[S_n | S_1, S_2, \dots, S_{n-1}] + \mathbf{H}[S_{n-1} | S_1, S_2, \dots, S_{n-2}] + \dots + \mathbf{H}[S_1]\end{aligned}$$

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The **entropy rate** of $\mathcal{S}$ is defined as

$$\mathbf{H}[\mathcal{S}] = \lim_{n \rightarrow \infty} \frac{\mathbf{H}[S_1, S_2, \dots, S_n]}{N}$$

or alternatively as

$$\mathbf{H}[\mathcal{S}] = \lim_{n \rightarrow \infty} \mathbf{H}[S_n | S_1, S_2, \dots, S_{n-1}]$$

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$$\mathbf{H}[\mathcal{S}] = \lim_{n \rightarrow \infty} \frac{\mathbf{H}[S_1, S_2, \dots, S_n]}{N}$$

or alternatively as

$$\mathbf{H}[\mathcal{S}] = \lim_{n \rightarrow \infty} \mathbf{H}[S_n | S_1, S_2, \dots, S_{n-1}]$$

If $S_i \stackrel{\text{iid}}{\sim} P(S)$ then $\mathbf{H}[\mathcal{S}] = \mathbf{H}[S]$.

If $\mathcal{S}$ is Markov (and stationary) then $\mathbf{H}[\mathcal{S}] = \mathbf{H}[S_n | S_{n-1}]$.

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We define the **differential entropy**:

$$h(S) = - \int ds p(s) \log p(s).$$

Note that $h(S)$ can be < 0 , and can be $\pm\infty$.

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The mutual information, however, is well-defined

$$\begin{aligned} I_{\Delta}[S; R] &= H_{\Delta}[S] - H_{\Delta}[S|R] \\ &= - \sum_i \Delta s p(s_i) \log p(s_i) - \log \Delta s \\ &\quad - \int dr p(r) \left(- \sum_i \Delta s p(s_i|r) \log p(s_i|r) - \log \Delta s \right) \\ &\rightarrow h(S) - h(S|R) \end{aligned}$$

as are other KL divergences.

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$$f(s) = s \quad \Rightarrow \quad p(s) = \frac{1}{Z} e^{\lambda_1 s}. \quad \text{Exponential (need } p(s) = 0 \text{ for } s < T).$$

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Both results together $\Rightarrow$ maximum entropy point process (for fixed mean arrival rate) is homogeneous Poisson – independent, exponentially distributed ISIs.

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We now direct our focus to the conditional $P(R|S)$ which defines the **channel** linking S to R .

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The mutual information

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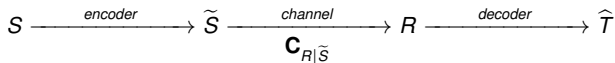
Instead, we characterise the channel by its **capacity**

$$C_{R|S} = \sup_{P(s)} I[S; R]$$

Thus the capacity gives the theoretical limit on the amount of information that can be transmitted over a channel. Clearly, this is limited by the properties of the noise.

Joint source-channel coding theorem

The remarkable central result of information theory.



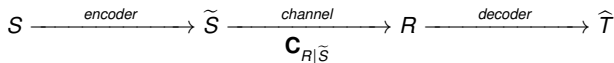
Any source ensemble S with entropy $H[S] < C_{R|\tilde{S}}$ can be transmitted (in sufficiently long blocks) with $P_{error} \rightarrow 0$.

The proof is beyond our scope.

Some of the key ideas that appear in the proof are:

- ▶ block coding
- ▶ error correction
- ▶ joint typicality
- ▶ random codes

The channel coding problem



Given channel $P(R|\tilde{S})$ and source $P(S)$, find **encoding** $P(\tilde{S}|S)$ (may be deterministic) to maximise $\mathbf{I}[S; R]$.

By data processing inequality, and defn of capacity:

$$\mathbf{I}[S; R] \leq \mathbf{I}[\tilde{S}; R] \leq \mathbf{C}_{R|\tilde{S}}$$

By JSCT, equality can be achieved (in the limit of increasing block size).

Thus $\mathbf{I}[\tilde{S}; R]$ should saturate $\mathbf{C}_{R|\tilde{S}}$.

See homework for an algorithm (Blahut-Arimoto) to find $P(\tilde{S})$ that saturates $\mathbf{C}_{R|\tilde{S}}$ for a general discrete channel.

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i.e.

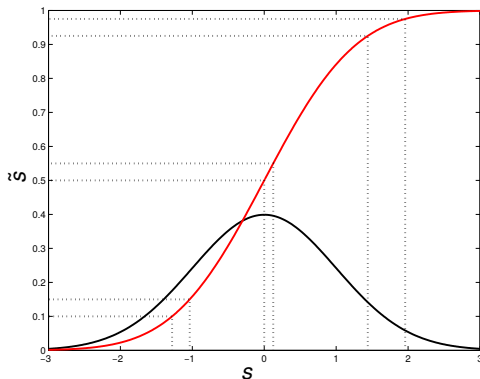
$$p(r) = \begin{cases} \frac{1}{r_{\max}} & r \in [0, r_{\max}] \\ 0 & \text{otherwise} \end{cases}$$

Histogram Equalisation

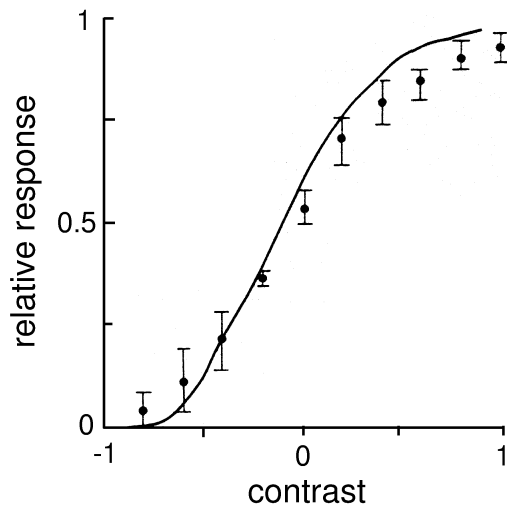
Suppose $r = \tilde{s} + \eta$ where η represents a (relatively small) source of noise. Consider deterministic encoding $\tilde{s} = f(s)$. How do we ensure that $p(r) = 1/r_{\max}$?

$$\frac{1}{r_{\max}} = p(r) \approx p(\tilde{s}) = \frac{p(s)}{f'(s)} \quad \Rightarrow f'(s) = r_{\max} p(s)$$

$$\Rightarrow f(s) = r_{\max} \int_{-\infty}^s ds' p(s')$$



Histogram Equalisation



Laughlin (1981)

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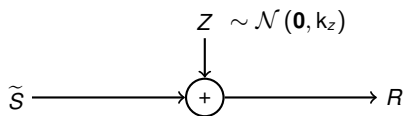
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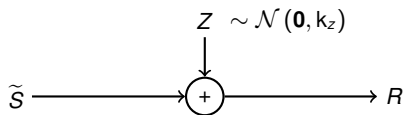
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Gaussian channel – white noise

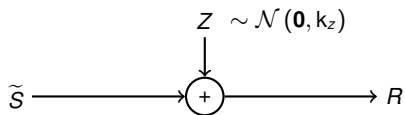


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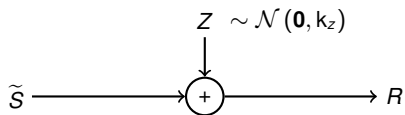
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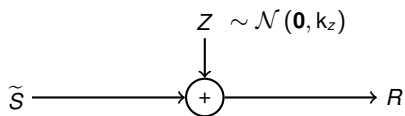
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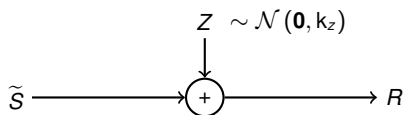
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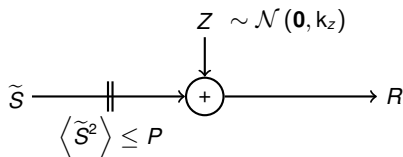


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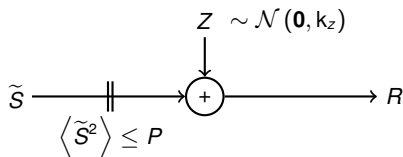
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Therefore, constrain $\frac{1}{n} \sum_{i=1}^n \tilde{s}_i^2 \leq P$.

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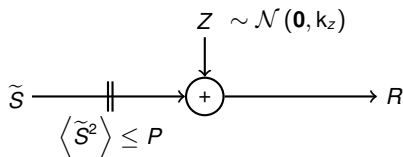
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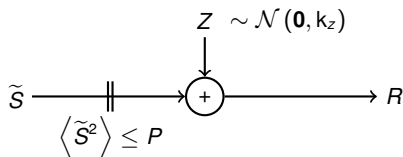
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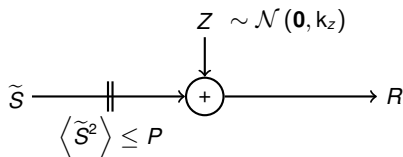
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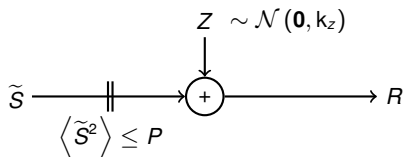
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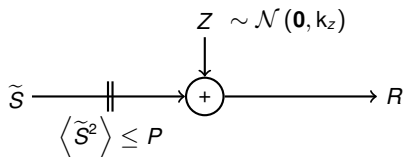
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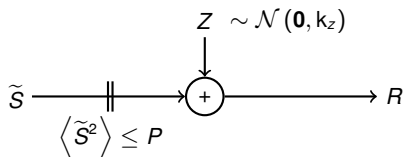
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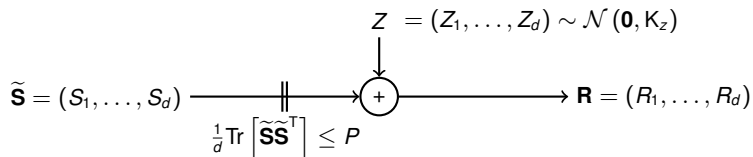
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The capacity is achieved iff $R \sim \mathcal{N}(0, P + k_z) \Rightarrow \tilde{S} \sim \mathcal{N}(0, P)$.

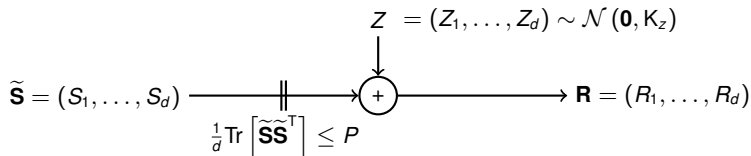
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Now consider a vector Gaussian channel:



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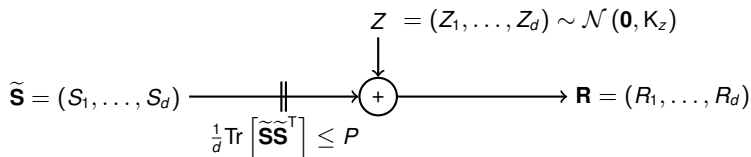


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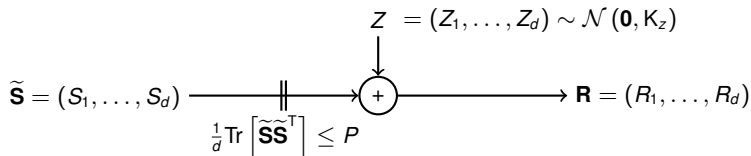
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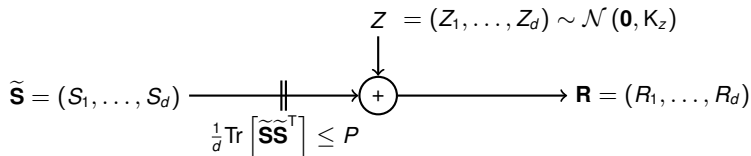
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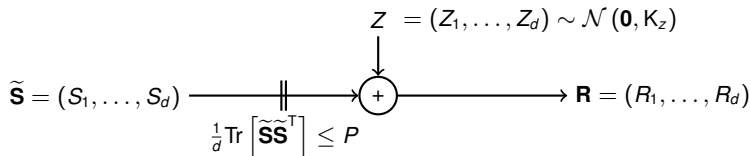
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$$k_{\tilde{\mathbf{S}}}^*(\omega) = \arg\max_{\omega} \prod (k_{\tilde{\mathbf{S}}}(\omega) + k_{\mathbf{Z}}(\omega)) \quad \text{such that} \quad \frac{1}{d} \sum k_{\tilde{\mathbf{S}}}(\omega) \leq P$$

Water filling

Assume that optimum is achieved for max. input power.

$$k_s^*(\omega) = \operatorname{argmax} \left[\sum_{\omega} \log (k_s(\omega) + k_z(\omega)) - \lambda \left(\frac{1}{d} \sum_{\omega} k_s(\omega) - P \right) \right]$$

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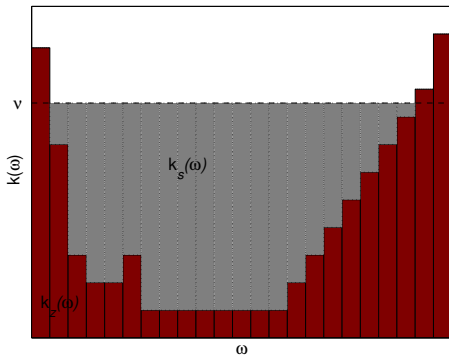
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Waterfilling: choose ν so

$$\sum_{\omega} k_s(\omega) = d \cdot P$$



R is white or decorrelated (within power budget) $\Rightarrow$ variance equalisation.

Decorrelation at the retina

Atick and Redlich (1992) argued that the retina decorrelates natural spatial statistics.

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RGCs exhibit roughly linear (centre-surround) processing:

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and thus output decorrelation requires

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Spatial correlations of natural images fall off with f^{-2} :

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and the optical filter of the eye introduces (crudely) a low-pass term $\propto e^{-\alpha|\mathbf{k}|}$.

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$$\mathbf{s} + \boldsymbol{\eta} \xrightarrow{D_\eta} \hat{\mathbf{s}} \xrightarrow{D_s} \mathbf{r}$$

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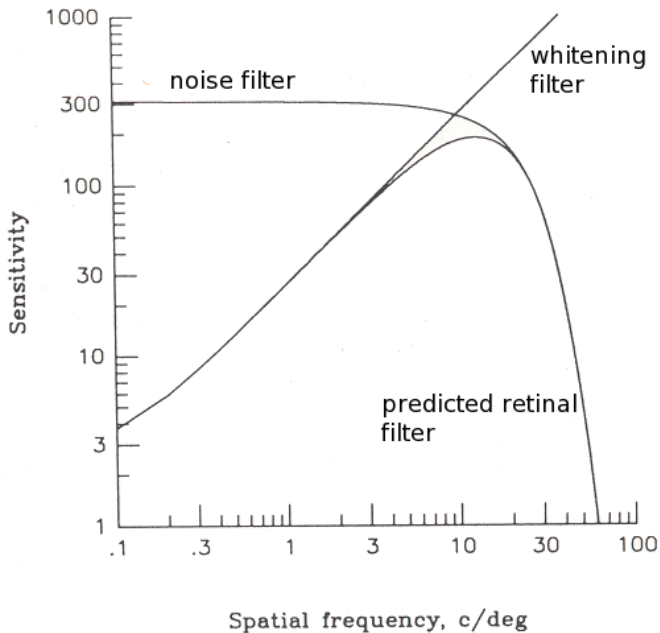
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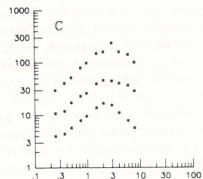
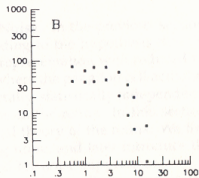
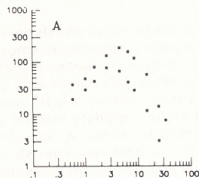
Thus the combined RGC filter is predicted to be:

$$|\tilde{D}_s(\mathbf{k})| \tilde{D}_\eta(\mathbf{k}) \propto \frac{\sqrt{\tilde{Q}_s(\mathbf{k})}}{\tilde{Q}_s(\mathbf{k}) + \tilde{Q}_\eta(\mathbf{k})}$$

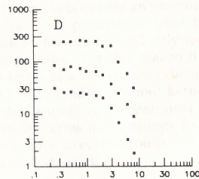
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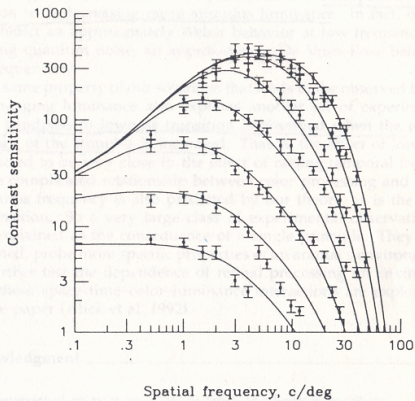
Decorrelation at the retina



Spatial frequency, c/deg



Spatial frequency, c/deg



Related ideas

- ▶ efficient channel utilisation
- ▶ output entropy maximisation
- ▶ variance equalisation
- ▶ redundancy reduction
- ▶ decorrelation
- ▶ discovery of independent projections or components