Coding (and computing with) Uncertainty

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  - activity varies systematically with small set of externally defined quantities (selectivity and invariance)
  - perhaps partitioning computation in terms of essential (causal) physical variables aides accuracy, efficiency and flexibility; and provides the substrate for unsupervised learning to accelerate learning of behaviour.

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- Accurate computation requires beliefs consistent with the calculus of probabilities.
- Thus, ultimately belief-like representation is arguably *more* natural than univalued representation.

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- Causal structure will also induce conditional independence is likely to extend amongst the z_i and x_j, so this computation may itself be achievable by local message passing.

Deterministic computation

- The core computations needed are:
  - Conditional marginalisation (prediction, message passing):

$$q(z_2) = \int dz_1\, p(z_2|z_1)q(z_1) = \mathbb{E}_{q(z_1)}\left[p(z_2|z_1)\right]$$

  - Action evaluation (Bayesian decision theory)

$$Q(a, b) = \mathbb{E}_{q(z)}\left[Q(a, z)\right]$$

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These require:

- multiplication of densities (in message passing)
- computation of expectations

Generally, there is *no* reason for neural circuits to decode the density $q(z)$ from representation (although some targets of expectation may include indicator functions).

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Stochastic vs. deterministic algorithm is somewhat orthogonal to belief representation. Stochastic approaches only obviate the need for distributional codes in a special case.

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  Thus, despite some attempts, the definitive experiment remains open.

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  - In particular, $p(X|z_i)$ is *not* experimentally defined distribution (unless the entire neural computation has adapted to the experimental environment perfectly).
- Cannot distinguish information content with encoding: does retinal activity "encode" everything about a visual scene?

Plausible coding schemes

Simple:

- Firing rate encoding of binary probabilities (Rao, Deneve)
- Explicit mean/variance encoding

Distributed:

- Linear density codes (NEF, Anderson Eliassmith)
- (Noisy) convolved density functions (DPC, Zemel Dayan Pouget)
- Expected value codes (DDPC/DDC, Sahani Dayan, current work)
  - exponential family mean parameters
  - current variant uses difference in expectation from prior
- Log-linear codes (Rao; also most common form of PPC, Ma Beck Latham Pouget)
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- In distributed forms, both operations depend on functions analagous to tuning curves.
- Actual form driven by learning useful $\mathbf{r}$, with implicit correspondence to q.

Linear density codes

$$q(z; \mathbf{r}) \propto \left[\sum_a \psi_a(z) r_a \right]_+$$

- Discussed by Anderson (90s); recent work by EliasSmith and others.
- Useful for “neural engineering framework” where operations defined on density can be mapped to basis function computations by hand.
- Computations linear in probability / density become easy.
- Encoding may be difficult.
- Basis functions ψ_a set a bound on possible precision.
- Noise in $\mathbf{r}$ enters decoder directly – suppressed if uncorrelated.

Convolved density functions

$$r_a = \left[\int dx \, \psi_a(z) q(z) \right]_+ = [\langle \psi_a(z) \rangle]_+$$

- "Distributional Population Code" (DPC) – Zemel, Dayan, Pouget.
- Decoding to histogram from noisy rates by maximum likelihood.
- Historically confused uncertainty and multiplicity. challenging – MaxEnt or EM-like algorithm if rates are noisy.
- Encoding can be learnt (delta rule) with access to z.
- Computations not discussed (but see DDC).

Log-linear codes

$$q(z;\mathbf{r}) \propto \exp\Big(\sum_a \psi_a(z) r_a\Big)$$

- Natural parameters of an exponential family.
- Message multiplication (e.g. cue combination) easy.
- Encoding may be difficult to learn.
- Uncorrelated noise in activities may average away.
- Basis functions set maximum log-precision.

Probabilistic Population Codes

Defined by Bayesian decoding:

$$q(z; \mathbf{r}) = p(z|\mathbf{r}) \propto p(\mathbf{r}|z)p(z)$$

but see previous discussion.

In practice, commonly assumes "Poisson-like" $p(\mathbf{r}|z)$ (expfam with linear sufficient statistic):

$$p(\mathbf{r}|z) = e^{\psi(z)^{\mathsf{T}}\mathbf{r} - A(z)}\nu(\mathbf{r})$$
$$\Rightarrow q(z; \mathbf{r}) \propto e^{\mathbf{r}^{\mathsf{T}}\psi(z)}\nu(z)$$

so gives log-linear/natural parameter encoding

- Poisson-like intuition derives from measure neural variability, but this is conditioned on S not z, and so neglects realistic $P(S|z)$.
- Neural variability conditioned on stimulus cannot sensibly be part of deterministic coding (though could reflect stochastic computation).
- Gain modulation of tuned population appears to encode changes in confidence without change in width of activity.
  - If true, consistent with observations of contrast-invariant orientation tuning in V1.
  - However, for uncertainty to be non-negligible, noise must be strongly correlated
    $\Rightarrow$ stochastic shifts in bump of activity
    $\Rightarrow$ predicted widening at greater uncertainty.

Distributed distributional codes

$$\mathbf{r} = \langle \psi(z) \rangle_{q(z)} = \int dz\, \psi(z) q(z)$$

- Encoding essentially the same as DPC, but intepretation differs.
- Doubly-DPC (Sahani, Dayan) proposed expectation form, based on (thresholded) DPC encoding of *multiplicities*. Let $z = z(x)$ be a feature map (e.g. motion strength as function of angle).

$$\mathbf{r} = \Big\langle \underbrace{\Big[\int dx\, z(x) \phi(x) \Big]^{+}}_{\psi(z)} \Big\rangle_{q[z(x)]}$$

- Maxent intepretation: q maximally uncertain given constraints $\Rightarrow$

$$q(z; \mathbf{r}) \propto e^{\boldsymbol{\eta}^{\mathsf{T}} \psi(z)}$$

  exponential family, with **r** representing the *mean* parameters.
  (DDPC paper also discussed decoding to a mixture by ML)
- Maxent interpetation may be important for unsupervised learning; but supervised learning and computation can be formulated without it.
- [Related to belief states, predictive state representations and RKHS mean embeddings]

DDC computation

- Many computations depend on finding expectations wrt q.
- If the $\psi_a(z)$ form an adequate basis for the required functions of z, then these expectations can be computed as linear combinations of r_a:

$$f(z) = \sum_a \alpha_a \psi_a(z)$$
$$\Rightarrow \mathbb{E}\left[f(z)\right] = \sum_a \alpha_a \mathbb{E}\left[\psi_a(z)\right] = \sum_a \alpha_a r_a$$

- Marginalisation, value computation and some forms of learning reduce to linear operations.
- Message combination may require mapping to natural parameters.

Supervised learning

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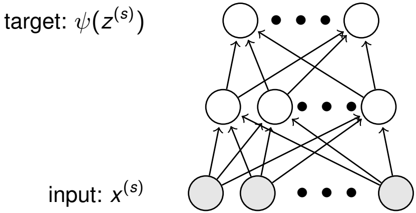
- Consider a network that learns a parameter(ρ)-dependent function $R(x; \rho)$



Supervised learning

- Expectations are easily learned from samples:

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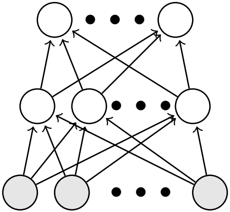
target: $\psi(z^{(s)})$



input: $x^{(s)}$

$$\Delta\rho \propto \left(\psi(z^{(s)}) - R(x^{(s)}; \rho)\right) \cdot \nabla_\rho R(x^{(s)}; \rho)$$

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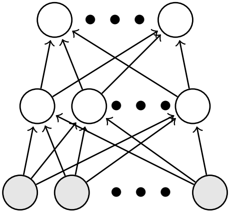
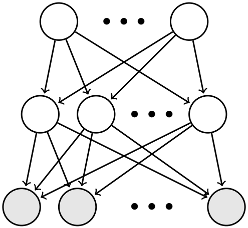


$$\Delta\rho \propto \big(\psi(z^{(s)}) - R(x^{(s)};\rho)\big) \cdot \nabla_\rho R(x^{(s)};\rho)$$

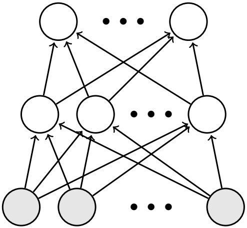
$\Rightarrow$ $R(x;\rho) \rightsquigarrow \langle\psi(z)\rangle_{p(z|x)}$.

Unsupervised learning: the Helmholtz machines

The Helmholtz Machine (Dayan et al. 1995). Approximate inference by recognition network.



Generative or causal network
A model of the data

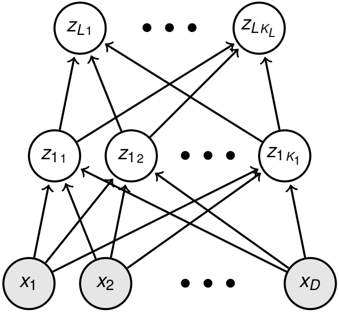


Recognition or inference network
Reasons about causes of a datum

Learning:

- Wake phase: estimate mean-field representation $\hat{z} = q(z) = R(x; \rho)$. Update generative parameters θ to increase a likelihood-related function F.
- Sleep phase: sample from generative model. Update recognition parameters ρ.

Distributed Distributional Recognition for a Helmholtz Machine



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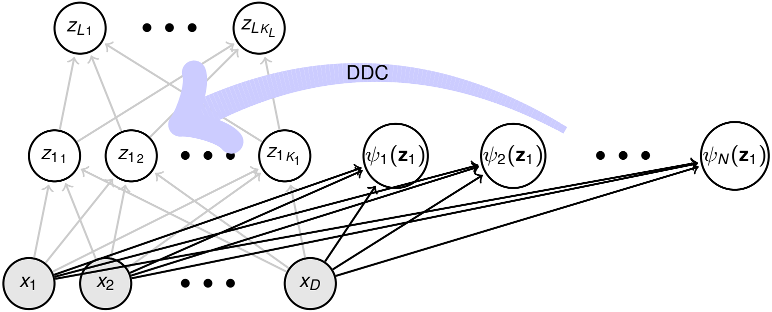

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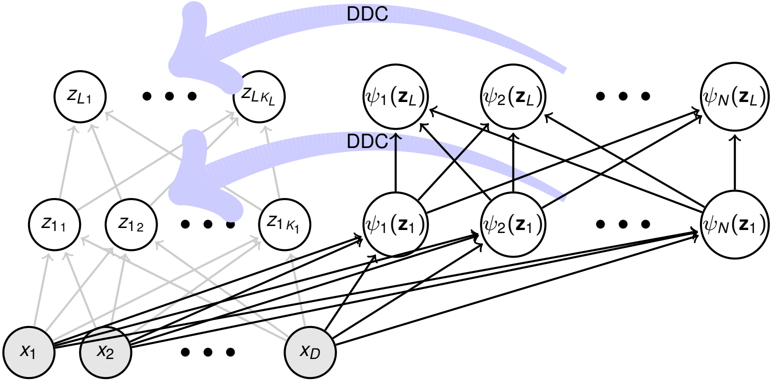
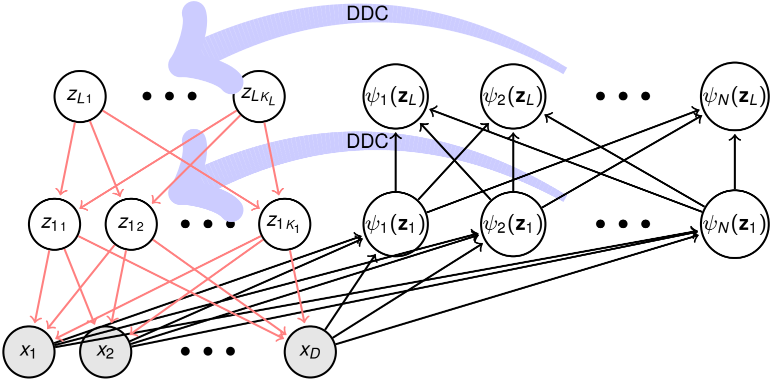


Wake phase – learning the model

Learning requires expected gradients of joint likelihood.

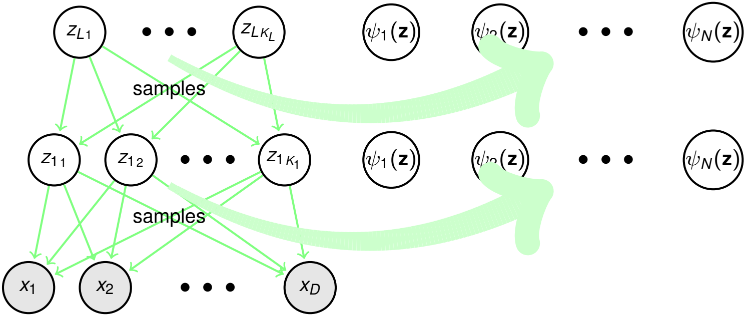


$$\nabla_\theta F(\mathbf{z}_l, \theta) \approx \sum_i \gamma_l^i \psi_i(\mathbf{z}_l) \;\Rightarrow\; \langle \nabla_\theta F(\mathbf{z}_l, \theta) \rangle_q \approx \sum_i \gamma_l^i \langle \psi_i(\mathbf{z}_l) \rangle$$

Vértes & Sahani, NeurIPS 2018

Sleep phase – learning to recognise and to learn

Sleep phase simulation are used to learn the recognition model and the gradients needed for learning.



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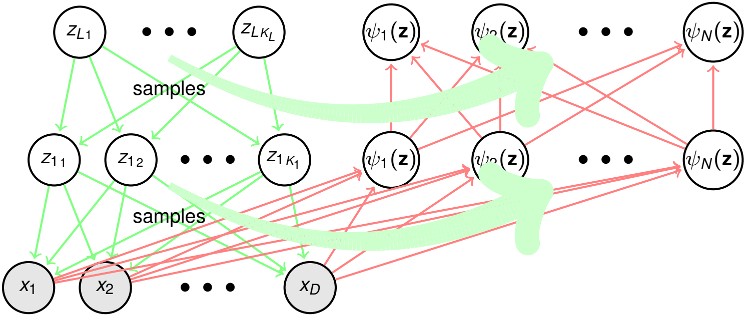

- Train ρ to map x to $\langle\psi(\mathbf{z})\rangle_{|\mathbf{x}}$

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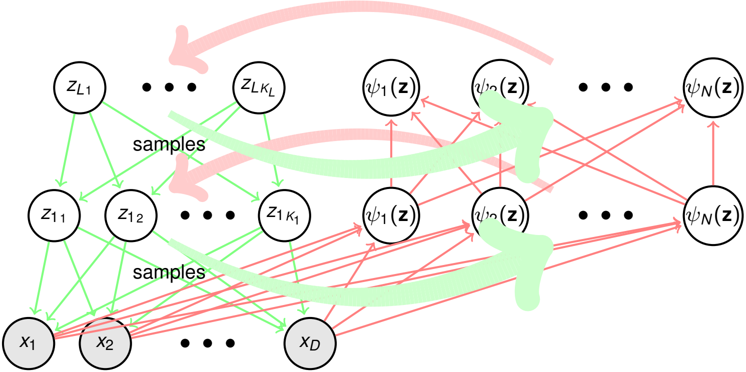

- Train ρ to map x to $\langle\psi(\mathbf{z})\rangle_{|\mathbf{x}}$
- Train weights γ to map $\psi(\mathbf{z})$ to $\nabla_\theta F$