Assignment 10
Theoretical Neuroscience

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1. **Linear analysis**

   Consider an equation of the form

$$\frac{d\mathbf{z}}{dt} = \mathbf{A} \cdot \mathbf{z} \tag{1}$$

   which in component form looks like $dz_i/dt = \sum_j A_{ij} z_j$. (The "." notation, favored by physicists worldwide, can be used for multiplying both matrices with vectors and vectors with vectors. For the former, the i^{th} component of $\mathbf{A} \cdot \mathbf{z}$ is $\sum_j A_{ij} z_j$, and for the latter $\mathbf{x} \cdot \mathbf{y} = \sum_i x_i y_i$.)

   Define $\mathbf{v}_k$, $\mathbf{v}_k^\dagger$, and λ_k via the equations

$$\begin{aligned} \mathbf{A} \cdot \mathbf{v}_k &= \lambda_k \mathbf{v}_k && \text{(2a)} \\ \mathbf{v}_k^\dagger \cdot \mathbf{A} &= \lambda_k \mathbf{v}_k^\dagger\,. && \text{(2b)} \end{aligned}$$

   The $\mathbf{v}_k$ and $\mathbf{v}_k^\dagger$ are eigenvectors and adjoint eigenvectors, respectively (the latter sometimes called left eigenvectors), and the λ_k are the associated eigenvalues. If $\mathbf{A}$ is an $n \times n$ matrix (which would mean that $\mathbf{z}$ has n components), there are n eigenvectors. Assume a normalization such that $\mathbf{v}_k \cdot \mathbf{v}_l^\dagger = \delta_{kl}$.

   Show that if $\mathbf{z}$ evolves according to Eq. (1) and $\mathbf{z}(t=0) = \mathbf{z}_0$, then

$$\mathbf{z}(t) = \sum_k \mathbf{v}_k \mathbf{v}_k^\dagger \cdot \mathbf{z}_0\, e^{\lambda_k t}\,. \tag{3}$$

   Remember this! If you stay in computational neuroscience, you will use it over and over and over.

2. **A memory network**

   Consider firing rate equations of the form

$$\tau \frac{d\nu_i}{dt} = \phi\left(\gamma\overline{\nu} + \frac{\beta}{Nf(1-f)} \sum_{j=1}^{N} \eta_i(\eta_j - f)\nu_j\right) - \nu_i \tag{4}$$

where N is the number of neurons, γ and β are constants, γ is negative, $\overline{\nu}$ is, as usual, the firing rate averaged over neurons,

$$\overline{\nu} = \frac{1}{N} \sum_i \nu_i \,, \tag{5}$$

η is a random binary vector,

$$\eta_i = \begin{cases} 1 & \text{probability } f \\ 0 & \text{probability } (1-f) \,, \end{cases} \tag{6}$$

and ϕ is sigmoidal (and thus monotonically increasing).

Let

$$m = \frac{1}{Nf(1-f)} \sum_i (\eta_i - f)\nu_i \,. \tag{7}$$

Note that m is the firing rate of the "memory" neurons relative to the mean firing rate, with an extra factor of $1/(1-f)$ thrown in to simplify the equations that you will derive.

2a. Derive *dynamical* mean field equations for $\overline{\nu}$ and m in the large N limit. By "dynamical," I mean derive equations for $d\overline{\nu}/dt$ and dm/dt.

2b. Sketch the nullclines for $\overline{\nu}$ and m assuming ϕ is sigmoidal. Work in a regime in which there are three equilibria, and indicate their stability. The bistability (two stable equilibria) is the reason we call this a memory network.

Take the $N \to \infty$ limit wherever applicable.

Assume the following:

- $\phi(0) > 0$.
- $\beta\phi'(\gamma\nu_0) < 1$ where ν_0 be the equilibrium mean firing rate when $m = 0$.
- When $\overline{\nu} = \nu_0$, m has three equilibria.

This is a hard, but important, problem.

2c. Suppose we have multiple memories; that is, in Eq. (4), we make the replacement

$$\sum_{j=1}^{N} \eta_i(\eta_j - f)\nu_j \to \sum_{\mu=1}^{p} \sum_{j=1}^{N} \eta_i^\mu(\eta_j^\mu - f)\nu_j \tag{8}$$

How would this affect your analysis? Can you get similar nullclines?

3. **Hopfield networks reduce energy**

Consider a Hopfield network that evolves *asynchronously* according to

$$S_i(t+1) = \text{sign}\left[\sum_j J_{ij} S_j(t)\right] \tag{9}$$

where J_{ij} is symmetric and has no diagonal elements,

$$J_{ij} = J_{ji} \tag{10a}$$

$$J_{ii} = 0\,. \tag{10b}$$

Define the energy,

$$H(t) = -\frac{1}{2}\sum_{ij} S_i(t) J_{ij} S_j(t)\,. \tag{11}$$

3a. Show that if the S_i obey the dynamics given in Eq. (10), then the energy never increases; i.e., $H(t+1) \le H(t)$.

3b. Show that if $J_{ii} \neq 0$, it is possible for the energy to increase. It is sufficient to find an example, with as few neurons as you want.