Lecture 8: Support Vector Machines

Advanced Topics in Machine Learning: COMPGI13

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Overview

- The representer theorem
- Review of convex optimization
- Support vector classification, the C-SV and ν-SV machines

Representer theorem

Learning problem: setting

Given a set of paired observations $(x_1, y_1), \ldots (x_n, y_n)$ (regression or classification).
Find the function f^* in the RKHS $\mathcal{H}$ which satisfies

$$J(f^*) = \min_{f \in \mathcal{H}} J(f), \tag{1}$$

where

$$J(f) = L_y(f(x_1), \ldots, f(x_n)) + \Omega\left(\|f\|^2_{\mathcal{H}}\right),$$

Ω is non-decreasing, and y is the vector of y_i.

- Classification: $L_y(f(x_1), \ldots, f(x_n)) = \sum_{i=1}^n \mathbb{I}_{y_i f(x_i) \leq 0}$
- Regression: $L_y(f(x_1), \ldots, f(x_n)) = \sum_{i=1}^n (y_i - f(x_i))^2$

Representer theorem

The representer theorem: solution to

$$\min_{f\in\mathcal{H}}\left[L_y(f(x_1),\ldots,f(x_n))+\Omega\left(\|f\|^2_{\mathcal{H}}\right)\right]$$

takes the form

$$f^*=\sum_{i=1}^{n}\alpha_i k(x_i,\cdot).$$

If Ω is strictly increasing, all solutions have this form.

Proof: Denote f_s projection of f onto the subspace

$$\mathrm{span}\left\{k(x_i,\cdot):\ 1 \leq i \leq n\right\}, \tag{2}$$

such that

$$f = f_s + f_\perp,$$

where $f_s = \sum_{i=1}^n \alpha_i k(x_i,\cdot)$.
Regularizer:

$$\|f\|_{\mathcal{H}}^2 = \|f_s\|_{\mathcal{H}}^2 + \|f_\perp\|_{\mathcal{H}}^2 \geq \|f_s\|_{\mathcal{H}}^2,$$

then

$$\Omega\left(\|f\|_{\mathcal{H}}^2\right) \geq \Omega\left(\|f_s\|_{\mathcal{H}}^2\right),$$

so this term is minimized for $f = f_s$.

Representer theorem: proof

Proof (cont.): Individual terms $f(x_i)$ in the loss:

$$f(x_i) = \langle f, k(x_i,\cdot)\rangle_{\mathcal{H}} = \langle f_s + f_\perp, k(x_i,\cdot)\rangle_{\mathcal{H}} = \langle f_s, k(x_i,\cdot)\rangle_{\mathcal{H}},$$

so

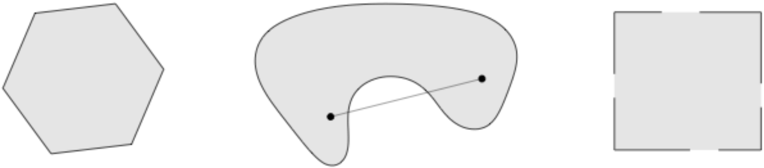
$$L_y(f(x_1),\ldots,f(x_n)) = L_y(f_s(x_1),\ldots,f_s(x_n)).$$

Hence

- Loss $L(\ldots)$ only depends on the component of f in the data subspace,
- Regularizer $\Omega(\ldots)$ minimized when $f = f_s$.
- If Ω is strictly non-decreasing, then $\|f_\perp\|_{\mathcal{H}} = 0$ is required at the minimum.

Short overview of convex optimization

Convex set



(Figure from Boyd and Vandenberghe)

Leftmost set is convex, remaining two are not.

Every point in the set can be seen from any other point in the set, along a straight line that never leaves the set.

Definition

C is convex if for all $x_1, x_2 \in C$ and any $0 \leq \theta \leq 1$ we have $\theta x_1 + (1-\theta)x_2 \in C$, i.e. every point on the line between x_1 and x_2 lies in C.

Convex function: no local optima

(Figure from Boyd and Vandenberghe)

Definition (Convex function)

A function f is **convex** if its domain $\mathrm{dom} f$ is a convex set and if $\forall x, y \in \mathrm{dom} f$, and any $0 \leq \theta \leq 1$,

$$f\left(\theta x + (1-\theta) y\right) \leq \theta f(x) + (1-\theta) f(y).$$

The function is **strictly convex** if the inequality is strict for $x \neq y$.

Optimization and the Lagrangian

Optimization problem on $x \in \mathbb{R}^n$,

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots p. \end{array} \tag{3}$$

- p^* the optimal value of (3)
- $\mathcal{D} := \bigcap_{i=0}^{m} \mathrm{dom} f_i \cap \bigcap_{i=1}^{p} \mathrm{dom} h_i$ (nonempty).

The **Lagrangian** $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ associated with problem (3) is written

$$L(x, \lambda, \nu) := f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x),$$

and has domain $\mathrm{dom} L := \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$. The vectors λ and ν are called **Lagrange multipliers** or **dual variables**.

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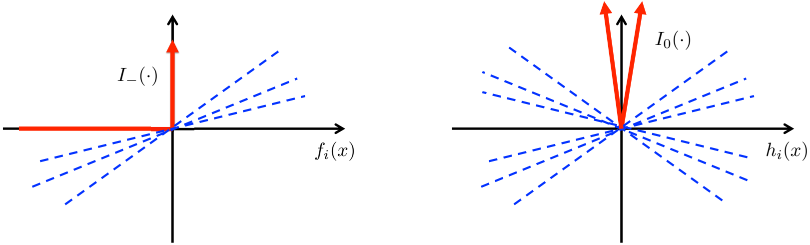
Lower bound interpretation of Lagrangian

Idealy we would want an unconstrained problem

$$\text{minimize } f_0(x) + \sum_{i=1}^{m} I_-\left(f_i(x)\right) + \sum_{i=1}^{p} I_0\left(h_i(x)\right),$$

where $I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0 \end{cases}$ and $I_0(u)$ is the indicator of 0.

To get a lower bound, we require $\lambda \succeq 0$.

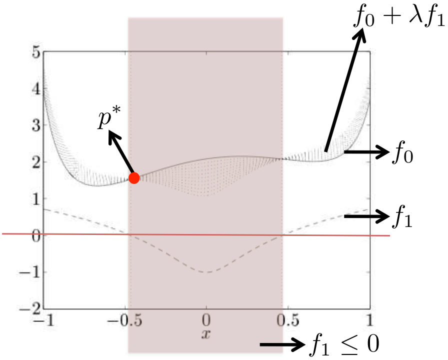


Lagrange dual: lower bound on optimum p^*

Lagrange dual: minimize Lagrangian (lower bound) over all $x \in \mathcal{D}$
When $\lambda \succeq 0$ and $f_i(x) \leq 0$, Lagrange dual function is

$$g(\lambda, \nu) := \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \leq p^*. \qquad (4)$$

A **dual feasible** pair (λ, ν) is a pair for which $\lambda \succeq 0$ and $(\lambda, \nu) \in \mathrm{dom}(g)$. (Figure from Boyd and Vandenberghe)



Reminders:

- f_0 is function to be minimized.
- $f_1 \leq 0$ is inequality constraint
- $\lambda \geq 0$ is Lagrange multiplier
- p^* is minimum f_0 *in constraint set*

Lagrange dual: lower bound on optimum p^*

When $\lambda \succeq 0$, then for all ν we have

$$g(\lambda, \nu) \leq p^* \qquad (5)$$

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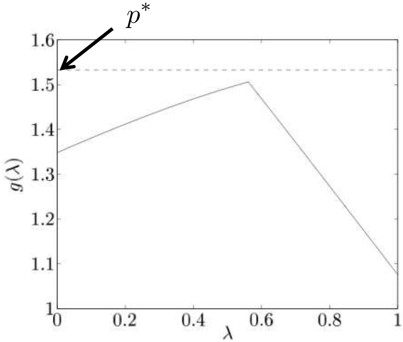

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- $g(\lambda, \nu) := \inf_{x \in \mathcal{D}} L(x, \lambda)$
- $\lambda \geq 0$ is Lagrange multiplier
- p^* is minimum f_0

  in constraint set

Lagrange dual is lower bound on p^* (proof)

We now give a formal proof that **Lagrange dual function** $g(\lambda, \nu)$ lower bounds p^*.

Proof: Assume $\tilde{x}$ is feasible, i.e. $f_i(\tilde{x}) \leq 0$, $h_i(\tilde{x}) = 0$, $\tilde{x} \in \mathcal{D}$, $\lambda \succeq 0$. Then

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq 0$$

Thus

$$\begin{aligned} g(\lambda, \nu) &:= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \\ &\leq f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \\ &\leq f_0(\tilde{x}). \end{aligned}$$

This holds for every feasible $\tilde{x}$, hence lower bound holds.

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Best lower bound $g(\lambda, \nu)$ on the optimal solution p^* of (3): **Lagrange dual problem**

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0. \end{array} \tag{6}$$

Dual feasible: (λ, ν) with $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$.
Dual optimal: solutions (λ^*, ν^*) to the dual problem, d^* is optimal value (**dual always easy to maximize**: next slide).
Weak duality always holds:

$$d^* \leq p^*.$$

...but what is the point of finding a **best lower bound** on a **minimization problem**?

Best lower bound: maximize the dual

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Strong duality: (does **not** always hold, conditions given later):

$$d^* = p^*.$$

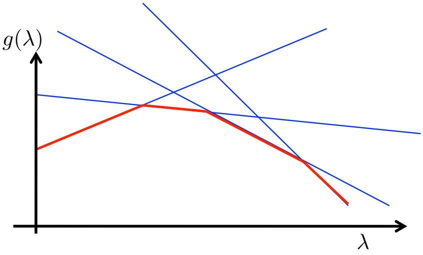
If S.D. holds: solve the **easy (concave) dual problem** to find p^*.

Maximizing the dual is always easy

The **Lagrange dual function:** minimize Lagrangian (lower bound)

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu).$$

Dual function is a pointwise infimum of affine functions of (λ, ν), hence **concave** in (λ, ν) with convex constraint set $\lambda \succeq 0$.



Example:

One inequality constraint,

$$L(x, \lambda) = f_0(x) + \lambda f_1(x),$$

and assume there are only four possible values for x. Each line represents a different x.

How do we know if strong duality holds?

Conditions under which strong duality holds are called **constraint qualifications** (they are sufficient, but not necessary)

(Probably) best known sufficient condition: Strong duality holds if

- Primal problem is convex, i.e. of the form

$$\begin{aligned} \text{minimize} \quad & f_0(x) & \\ \text{subject to} \quad & f_i(x) \leq 0 & i = 1, \ldots, n \\ & Ax = b & \end{aligned} \tag{8}$$

  for convex $f_0, \ldots, f_m$, *and*

- Slater's condition holds: there exists some *strictly* feasible point[1] $\tilde{x} \in \mathrm{relint}(\mathcal{D})$ such that

$$f_i(\tilde{x}) < 0 \quad i = 1, \ldots, m \quad A\tilde{x} = b.$$

[1] We denote by $\mathrm{relint}(\mathcal{D})$ the relative interior of the set $\mathcal{D}$. This looks like the interior of the set, but is non-empty even when the set is a subspace of a larger space.

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  for convex $f_0, \ldots, f_m$, *and*
- **Slater's condition** for the case of **affine** f_i is trivial:

$$f_i(\tilde{x}) \leq 0 \quad i = 1, \ldots, m \quad A\tilde{x} = b.$$

A consequence of strong duality...

Assume primal is equal to the dual. What are the consequences?

- x^* solution of original problem (minimum of f_0 under constraints),
- (λ^*, ν^*) solutions to dual

$$
\begin{aligned}
f_0(x^*) &\underset{\text{(assumed)}}{=} g(\lambda^*, \nu^*) \\
&\underset{\text{(g definition)}}{=} \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right) \\
&\underset{\text{(inf definition)}}{\leq} f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) \\
&\underset{(4)}{\leq} f_0(x^*),
\end{aligned}
$$

(4): (x^*, λ^*, ν^*) satisfies $\lambda^* \succeq 0$, $f_i(x^*) \leq 0$, and $h_i(x^*) = 0$.

From previous slide,

$$\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0, \tag{9}$$

which is the condition of **complementary slackness**. This means

$$\begin{aligned} \lambda_i^* > 0 \quad &\implies \quad f_i(x^*) = 0, \\ f_i(x^*) < 0 \quad &\implies \quad \lambda_i^* = 0. \end{aligned}$$

From λ_i, read off which inequality constraints are strict.

KKT conditions for global optimum

Assume functions f_i, h_i are differentiable and **strong duality**. Since x^* minimizes $L(x, \lambda^*, \nu^*)$, derivative at x^* is zero,

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0.$$

KKT conditions we are at **global optimum**, $(x, \lambda, \nu) = (x^*, \lambda^*, \nu^*)$ when **strong duality** holds, and

$$\begin{aligned} f_i(x) &\leq 0, \; i = 1, \ldots, m \\ h_i(x) &= 0, \; i = 1, \ldots, p \\ \lambda_i &\geq 0, \; i = 1, \ldots, m \\ \lambda_i f_i(x) &= 0, \; i = 1, \ldots, m \\ \nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) &= 0 \end{aligned}$$

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\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) &= 0
\end{aligned}$$

KKT conditions for global optimum

In summary: if

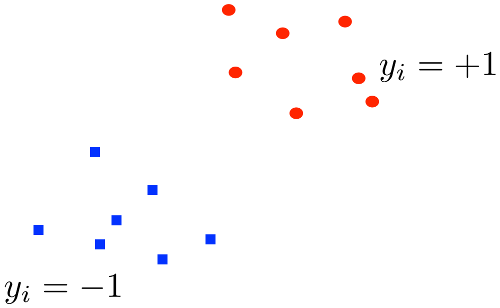
- primal problem convex and
- constraint functions satisfy Slater's conditions and
- functions f_i, h_i differentiable

then KKT conditions *necessary and sufficient* for optimality.

Support vector classification

Linearly separable points

Classify two clouds of points, where there exists a hyperplane which linearly separates one cloud from the other without error.



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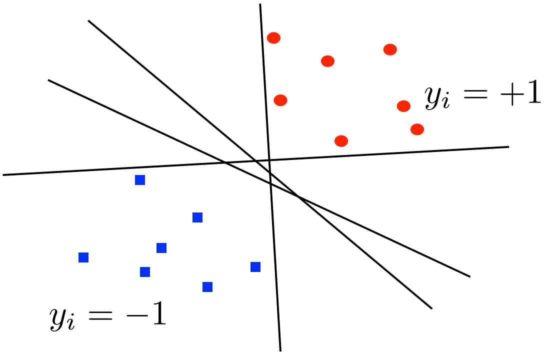

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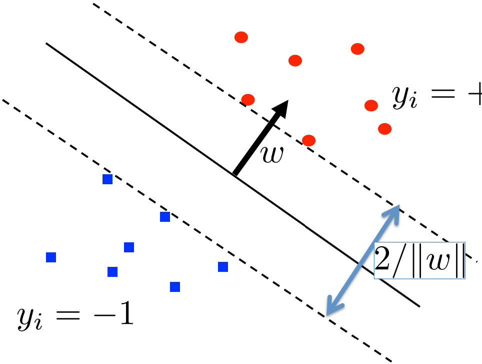

Smallest distance from each class to the separating hyperplane (w, b) is called the **margin**.

Maximum margin classifier, linearly separable case

This problem can be expressed as follows:

$$\max_{w,b}(\text{margin}) = \max_{w,b}\left(\frac{2}{\|w\|}\right) \qquad (10)$$

subject to

$$\begin{cases} \min\left(w^\top x_i + b\right) = 1 & i : y_i = +1, \\ \max\left(w^\top x_i + b\right) = -1 & i : y_i = -1. \end{cases} \qquad (11)$$

The resulting classifier is

$$y = \text{sign}(w^\top x + b),$$

We can rewrite to obtain

$$\max_{w,b} \frac{1}{\|w\|} \quad \text{or} \quad \min_{w,b} \|w\|^2$$

subject to

$$y_i(w^\top x_i + b) \geq 1. \qquad (12)$$

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Maximum margin classifier: with errors allowed

Allow "errors": points within the margin, or even on the wrong side of the decision boudary. Ideally:

$$\min_{w,b} \left(\frac{1}{2}\|w\|^2 + C \sum_{i=1}^{n} \mathbb{I}[y_i \left(w^\top x_i + b \right) < 0] \right),$$

where C controls the tradeoff between maximum margin and loss.
Replace with **convex upper bound**:

$$\min_{w,b} \left(\frac{1}{2}\|w\|^2 + C \sum_{i=1}^{n} \theta \left(y_i \left(w^\top x_i + b \right) \right) \right).$$

with hinge loss,

$$\theta(\alpha) = (1-\alpha)_+ = \begin{cases} 1-\alpha & 1-\alpha > 0 \\ 0 & \text{otherwise.} \end{cases}$$

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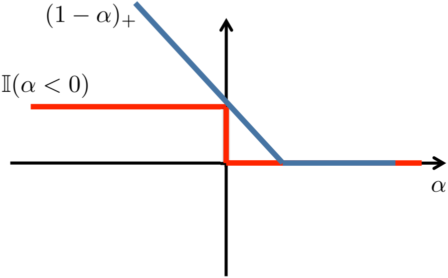

Substituting in the hinge loss, we get

$$\min_{w,b}\left(\frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\theta\left(y_i\left(w^\top x_i + b\right)\right)\right).$$

How do you implement hinge loss with simple **inequality constraints** (for optimization)?

$$\min_{w,b,\xi}\left(\frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i\right) \tag{13}$$

subject to[2]

$$\xi_i \geq 0 \qquad y_i\left(w^\top x_i + b\right) \geq 1 - \xi_i$$

[2]Either $y_i\left(w^\top x_i + b\right) \geq 1$ and $\xi_i = 0$ as before, or $y_i\left(w^\top x_i + b\right) < 1$, and then $\xi_i > 0$ takes the value satisfying $y_i\left(w^\top x_i + b\right) = 1 - \xi_i$.

Substituting in the hinge loss, we get

$$\min_{w,b}\left(\frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\theta\left(y_i\left(w^\top x_i + b\right)\right)\right).$$

How do you implement hinge loss with simple **inequality constraints** (for optimization)?

$$\min_{w,b,\xi}\left(\frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i\right) \tag{13}$$

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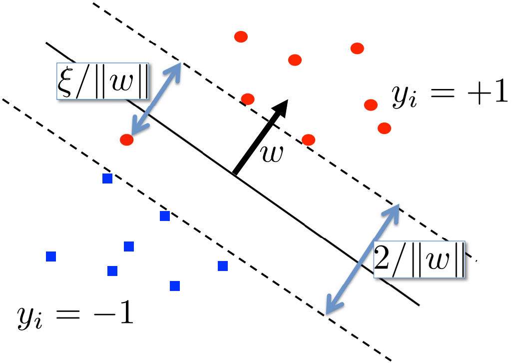
Support vector classification



Does strong duality hold?

1. Is the optimization problem convex wrt the variables w, b, ξ?

$$\begin{aligned} \text{minimize} \quad & f_0(w, b, \xi) := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i \\ \text{subject to} \quad & f_i(w, b, \xi) := 1 - \xi_i - y_i\left(w^\top x_i + b\right) \le 0 \quad i = 1, \ldots, n \\ & Ax = b \quad \text{(absent)} \end{aligned}$$

Each of $f_0, f_1, \ldots, f_n$ are convex.

2. Does Slater's condition hold? **Trivial** since inequality constraints **affine**, and there always exists some

$$\begin{aligned} \xi_i &\geq 0 \\ y_i\left(w^\top x_i + b\right) &\geq 1 - \xi_i \end{aligned}$$

Thus **strong duality** holds, the problem is differentiable, hence the KKT conditions hold at the global optimum.

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The Lagrangian: $L(w, b, \xi, \alpha, \lambda)$

$$= \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - y_i\left(w^\top x_i + b\right) - \xi_i\right) + \sum_{i=1}^{n}\lambda_i(-\xi_i)$$

with dual variable constraints

$$\alpha_i \geq 0, \qquad \lambda_i \geq 0.$$

Minimize wrt the primal variables w, b, and ξ.

Support vector classification: Lagrangian

Derivative wrt w:

$$\frac{\partial L}{\partial w} = w - \sum_{i=1}^{n} \alpha_i y_i x_i = 0 \qquad w = \sum_{i=1}^{n} \alpha_i y_i x_i. \tag{14}$$

Derivative wrt b:

$$\frac{\partial L}{\partial b} = \sum_{i} y_i \alpha_i = 0. \tag{15}$$

Derivative wrt ξ_i:

$$\frac{\partial L}{\partial \xi_i} = C - \alpha_i - \lambda_i = 0 \qquad \alpha_i = C - \lambda_i. \tag{16}$$

Noting that $\lambda_i \geq 0$,

$$\alpha_i \leq C.$$

Now use complementary slackness:

Non-margin SVs: $\alpha_i = C \neq 0$:

1. We immediately have $1 - \xi_i = y_i \left(w^\top x_i + b \right)$.
2. Also, from condition $\alpha_i = C - \lambda_i$, we have $\lambda_i = 0$, hence $\xi_i > 0$.

Margin SVs: $0 < \alpha_i < C$:

1. We again have $1 - \xi_i = y_i \left(w^\top x_i + b \right)$
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Non-SVs: $\alpha_i = 0$

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The support vectors

We observe:

1. The solution is sparse: points which are not on the margin, or "margin errors", have $\alpha_i = 0$
2. The support vectors: only those points on the decision boundary, or which are margin errors, contribute.
3. Influence of the non-margin SVs is bounded, since their weight cannot exceed C.

Support vector classification: dual function

Thus, our goal is to maximize the dual,

$$
\begin{aligned}
g(\alpha,\lambda) &= \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - y_i\left(w^\top x_i + b\right) - \xi_i\right) \\
&\qquad + \sum_{i=1}^{n}\lambda_i(-\xi_i) \\
&= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j + C\sum_{i=1}^{m}\xi_i - \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j \\
&\qquad \underbrace{-b\sum_{i=1}^{m}\alpha_i y_i}_{0} + \sum_{i=1}^{m}\alpha_i - \sum_{i=1}^{m}\alpha_i\xi_i - \sum_{i=1}^{m}(C-\alpha_i)\xi_i \\
&= \sum_{i=1}^{m}\alpha_i - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j.
\end{aligned}
$$

Support vector classification: dual function

Maximize the dual,

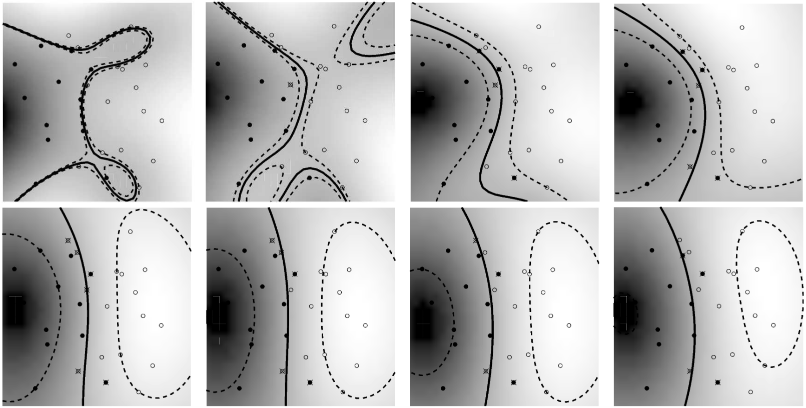
$$g(\alpha) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j x_i^\top x_j,$$

subject to the constraints

$$0 \leq \alpha_i \leq C, \quad \sum_{i=1}^{n} y_i \alpha_i = 0$$

This is a quadratic program.
Offset b: for the margin SVs, we have $1 = y_i \left(w^\top x_i + b \right)$. Obtain b from any of these, or take an average.

Support vector classification: kernel version



Taken from Schoelkopf and Smola (2002)

Maximum margin classifier in RKHS: write the hinge loss formulation

$$\min_w \left(\frac{1}{2}\|w\|^2_{\mathcal{H}} + C \sum_{i=1}^n \theta\left(y_i \left\langle w, k(x_i,\cdot)\right\rangle_{\mathcal{H}}\right) \right)$$

for the RKHS $\mathcal{H}$ with kernel $k(x,\cdot)$. Use the result of the representer theorem,

$$w(\cdot) = \sum_{i=1}^n \beta_i k(x_i,\cdot).$$

Maximizing the margin equivalent to minimizing $\|w\|^2_{\mathcal{H}}$: for many RKHSs a smoothness constraint (e.g. Gaussian kernel).

Support vector classification: kernel version

Substituting and introducing the ξ_i variables, get

$$\min_{w,b} \left(\frac{1}{2} \beta^\top K \beta + C \sum_{i=1}^{n} \xi_i \right) \tag{17}$$

where the matrix K has i, jth entry $K_{ij} = k(x_i, x_j)$, subject to

$$\xi_i \geq 0 \qquad y_i \sum_{j=1}^{n} \beta_j k(x_i, x_j) \geq 1 - \xi_i$$

Convex in β since K is positive definite: hence strong duality holds.

Dual:

$$g(\alpha) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j k(x_i, x_j),$$

subject to the constraints $0 \leq \alpha_i \leq C$, and

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$$w(\cdot) = \sum_{i=1}^{n} y_i \alpha_i k(x, \cdot).$$

Hard to interpret C. Modify the formulation to get a more intuitive parameter ν.
Again, we drop b for simplicity. Solve

$$\min_{w,\rho,\xi} \left(\frac{1}{2}\|w\|^2 - \nu\rho + \frac{1}{n}\sum_{i=1}^{n} \xi_i \right)$$

subject to

$$\begin{aligned} \rho &\geq 0 \\ \xi_i &\geq 0 \\ y_i w^\top x_i &\geq \rho - \xi_i, \end{aligned}$$

(now directly adjust margin width ρ).

The ν-SVM: Lagrangian

$$\frac{1}{2}\|w\|_{\mathcal{H}}^2+\frac{1}{n}\sum_{i=1}^n \xi_i-\nu\rho+\sum_{i=1}^n \alpha_i\left(\rho-y_i w^\top x_i-\xi_i\right)+\sum_{i=1}^n \beta_i(-\xi_i)+\gamma(-\rho)$$

for dual variables $\alpha_i \geq 0$, $\beta_i \geq 0$, and $\gamma \geq 0$.

Differentiating and setting to zero for each of the primal variables w, ξ, ρ,

$$\begin{aligned} w &= \sum_{i=1}^n \alpha_i y_i x_i \\ \alpha_i+\beta_i &= \frac{1}{n} \quad (18) \\ \nu &= \sum_{i=1}^n \alpha_i - \gamma \quad (19) \end{aligned}$$

From $\beta_i \geq 0$, equation (18) implies

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Complementary slackness (1)

Complementary slackness conditions:
Assume $\rho > 0$ at the global solution, hence $\gamma = 0$, and

$$\sum_{i=1}^{n} \alpha_i = \nu. \tag{20}$$

Case of $\xi_i > 0$: complementary slackness states $\beta_i = 0$, hence from (18) we have $\alpha_i = n^{-1}$. Denote this set as $N(\alpha)$. Then

$$\sum_{i \in N(\alpha)} \frac{1}{n} = \sum_{i \in N(\alpha)} \alpha_i \le \sum_{i=1}^{n} \alpha_i = \nu,$$

so

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and ν is an upper bound on the number of non-margin SVs.

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Complementary slackness (2)

Case of $\xi_i = 0$: then $\beta_i > 0$ and so $\alpha_i < n^{-1}$. Denote by $M(\alpha)$ the set of points $n^{-1} > \alpha_i > 0$. Then from (20),

$$\nu = \sum_{i=1}^{n} \alpha_i = \sum_{i \in N(\alpha)} \frac{1}{n} + \sum_{i \in M(\alpha)} \alpha_i \leq \sum_{i \in M(\alpha) \cup N(\alpha)} \frac{1}{n},$$

thus

$$\nu \leq \frac{|N(\alpha)| + |M(\alpha)|}{n},$$

and ν is a lower bound on the number of support vectors with non-zero weight (both on the margin, and "margin errors").

Substituting into the Lagrangian, we get

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j + \frac{1}{n}\sum_{i=1}^{n}\xi_i - \rho\nu - \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j$$

$$+\sum_{i=1}^{n}\alpha_i\rho - \sum_{i=1}^{n}\alpha_i\xi_i - \sum_{i=1}^{n}\left(\frac{1}{n}-\alpha_i\right)\xi_i - \rho\left(\sum_{i=1}^{n}\alpha_i - \nu\right)$$

$$= -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j$$

Maximize:

$$g(\alpha) = -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j,$$

subject to

$$\sum_{i=1}^{n}\alpha_i \geq \nu \qquad 0 \leq \alpha_i \leq \frac{1}{n}.$$

Dual for ν-SVM

Substituting into the Lagrangian, we get

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j + \frac{1}{n}\sum_{i=1}^{n}\xi_i - \rho\nu - \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j$$

$$+\sum_{i=1}^{n}\alpha_i\rho - \sum_{i=1}^{n}\alpha_i\xi_i - \sum_{i=1}^{n}\left(\frac{1}{n}-\alpha_i\right)\xi_i - \rho\left(\sum_{i=1}^{n}\alpha_i - \nu\right)$$

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Questions?