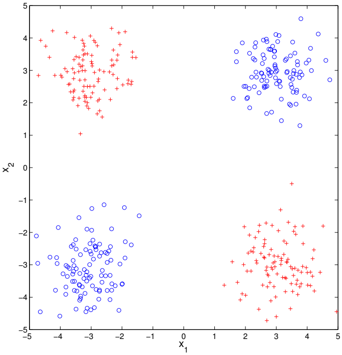
Lecture 1: Introduction to RKHS

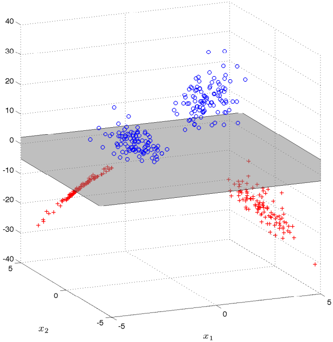
MLSS Tübingen, 2015

Gatsby Unit, CSML, UCL

July 22, 2015

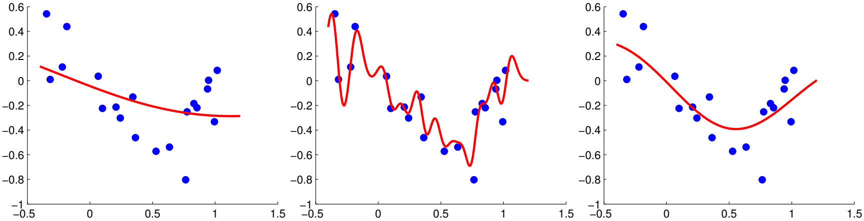
Kernels and feature space (1): XOR example





- No linear classifier separates red from blue
- Map points to **higher dimensional feature space**:
  $\phi(x) = \begin{bmatrix} x_1 & x_2 & x_1 x_2 \end{bmatrix} \in \mathbb{R}^3$

Kernels and feature space (2): smoothing



Kernel methods can control **smoothness** and **avoid overfitting/underfitting**.

Outline: reproducing kernel Hilbert space

We will describe in order:

1. Hilbert space
2. Kernel (lots of examples: e.g. you can build kernels from simpler kernels)
3. Reproducing property

Hilbert space

Definition (Inner product)

Let $\mathcal{H}$ be a vector space over $\mathbb{R}$. A function $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is an inner product on $\mathcal{H}$ if

1. Linear: $\langle \alpha_1 f_1 + \alpha_2 f_2, g \rangle_{\mathcal{H}} = \alpha_1 \langle f_1, g \rangle_{\mathcal{H}} + \alpha_2 \langle f_2, g \rangle_{\mathcal{H}}$
2. Symmetric: $\langle f, g \rangle_{\mathcal{H}} = \langle g, f \rangle_{\mathcal{H}}$
3. $\langle f, f \rangle_{\mathcal{H}} \geq 0$ and $\langle f, f \rangle_{\mathcal{H}} = 0$ if and only if $f = 0$.

Hilbert space

Definition (Inner product)

Let $\mathcal{H}$ be a vector space over $\mathbb{R}$. A function $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is an inner product on $\mathcal{H}$ if

1. Linear: $\langle \alpha_1 f_1 + \alpha_2 f_2, g \rangle_{\mathcal{H}} = \alpha_1 \langle f_1, g \rangle_{\mathcal{H}} + \alpha_2 \langle f_2, g \rangle_{\mathcal{H}}$
2. Symmetric: $\langle f, g \rangle_{\mathcal{H}} = \langle g, f \rangle_{\mathcal{H}}$
3. $\langle f, f \rangle_{\mathcal{H}} \geq 0$ and $\langle f, f \rangle_{\mathcal{H}} = 0$ if and only if $f = 0$.

Norm induced by the inner product: $\|f\|_{\mathcal{H}} := \sqrt{\langle f, f \rangle_{\mathcal{H}}}$

Hilbert space

Definition (Inner product)

Let $\mathcal{H}$ be a vector space over $\mathbb{R}$. A function $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is an inner product on $\mathcal{H}$ if

1. Linear: $\langle \alpha_1 f_1 + \alpha_2 f_2, g \rangle_{\mathcal{H}} = \alpha_1 \langle f_1, g \rangle_{\mathcal{H}} + \alpha_2 \langle f_2, g \rangle_{\mathcal{H}}$
2. Symmetric: $\langle f, g \rangle_{\mathcal{H}} = \langle g, f \rangle_{\mathcal{H}}$
3. $\langle f, f \rangle_{\mathcal{H}} \geq 0$ and $\langle f, f \rangle_{\mathcal{H}} = 0$ if and only if $f = 0$.

Norm induced by the inner product: $\|f\|_{\mathcal{H}} := \sqrt{\langle f, f \rangle_{\mathcal{H}}}$

Definition (Hilbert space)

Inner product space containing Cauchy sequence limits.

Kernel

Definition

Let $\mathcal{X}$ be a non-empty set. A function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a **kernel** if there exists an $\mathbb{R}$-Hilbert space and a map $\phi : \mathcal{X} \to \mathcal{H}$ such that $\forall x, x' \in \mathcal{X}$,

$$k(x, x') := \langle \phi(x), \phi(x') \rangle_{\mathcal{H}} .$$

- Almost no conditions on $\mathcal{X}$ (eg, $\mathcal{X}$ itself doesn't need an inner product, eg. documents).
- A single kernel can correspond to several possible features. A trivial example for $\mathcal{X} := \mathbb{R}$:

$$\phi_1(x) = x \qquad \text{and} \qquad \phi_2(x) = \begin{bmatrix} x/\sqrt{2} \\ x/\sqrt{2} \end{bmatrix}$$

New kernels from old: sums, transformations

Theorem (Sums of kernels are kernels)

Given $\alpha > 0$ and k, k_1 and k_2 all kernels on $\mathcal{X}$, then αk and $k_1 + k_2$ are kernels on $\mathcal{X}$.

(Proof via positive definiteness: later!) A difference of kernels may not be a kernel (**why?**)

New kernels from old: sums, transformations

Theorem (Sums of kernels are kernels)

Given $\alpha > 0$ and k, k_1 and k_2 all kernels on $\mathcal{X}$, then αk and $k_1 + k_2$ are kernels on $\mathcal{X}$.

(Proof via positive definiteness: later!) A difference of kernels may not be a kernel (**why?**)

Theorem (Mappings between spaces)

Let $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ be sets, and define a map $A : \mathcal{X} \to \widetilde{\mathcal{X}}$. Define the kernel k on $\widetilde{\mathcal{X}}$. Then the kernel $k(A(x), A(x'))$ is a kernel on $\mathcal{X}$.

Example: $k(x, x') = x^2 \, (x')^2$.

New kernels from old: products

Theorem (Products of kernels are kernels)

Given k_1 on $\mathcal{X}_1$ and k_2 on $\mathcal{X}_2$, then $k_1 \times k_2$ is a kernel on $\mathcal{X}_1 \times \mathcal{X}_2$. If $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$, then $k := k_1 \times k_2$ is a kernel on $\mathcal{X}$.

Proof: Main idea only!

$\mathcal{H}_1$ space of kernels between **shapes**,

$$\phi_1(x) = \begin{bmatrix} \mathbb{I}_\square \\ \mathbb{I}_\triangle \end{bmatrix} \qquad \phi_1(\square) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad k_1(\square, \triangle) = 0.$$

$\mathcal{H}_2$ space of kernels between **colors**,

$$\phi_2(x) = \begin{bmatrix} \mathbb{I}_{\color{red}\bullet} \\ \mathbb{I}_{\color{blue}\bullet} \end{bmatrix} \qquad \phi_2({\color{blue}\bullet}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad k_2({\color{red}\bullet}, {\color{red}\bullet}) = 1.$$

New kernels from old: products

"Natural" feature space for **colored shapes**:

$$\Phi(x) = \begin{bmatrix} \mathbb{I}_{\color{red}\square} & \mathbb{I}_{\color{red}\triangle} \\ \mathbb{I}_{\color{blue}\square} & \mathbb{I}_{\color{blue}\triangle} \end{bmatrix} = \begin{bmatrix} \mathbb{I}_{\color{red}\bullet} \\ \mathbb{I}_{\color{blue}\bullet} \end{bmatrix} \begin{bmatrix} \mathbb{I}_{\square} & \mathbb{I}_{\triangle} \end{bmatrix} = \phi_2(x)\phi_1^\top(x)$$

New kernels from old: products

"Natural" feature space for **colored shapes**:

$$\Phi(x) = \begin{bmatrix} \mathbb{I}_{\square} & \mathbb{I}_{\triangle} \\ \mathbb{I}_{\square} & \mathbb{I}_{\triangle} \end{bmatrix} = \begin{bmatrix} \mathbb{I}_{\bullet} \\ \mathbb{I}_{\bullet} \end{bmatrix} \begin{bmatrix} \mathbb{I}_{\square} & \mathbb{I}_{\triangle} \end{bmatrix} = \phi_2(x)\phi_1^\top(x)$$

Kernel is:

$$k(x,x') = \sum_{i\in\{\bullet,\bullet\}} \sum_{j\in\{\square,\triangle\}} \Phi_{ij}(x)\Phi_{ij}(x') = \mathrm{tr}\left(\phi_1(x)\underbrace{\phi_2^\top(x)\phi_2(x')}_{k_2(x,x')}\phi_1^\top(x')\right)$$

$$= \mathrm{tr}\left(\underbrace{\phi_1^\top(x')\phi_1(x)}_{k_1(x,x')}\right) k_2(x,x') = k_1(x,x')k_2(x,x')$$

Sums and products $\implies$ polynomials

Theorem (Polynomial kernels)

Let $x, x' \in \mathbb{R}^d$ for $d \geq 1$, and let $m \geq 1$ be an integer and $c \geq 0$ be a positive real. Then

$$k(x, x') := \left(\langle x, x' \rangle + c\right)^m$$

is a valid kernel.

To prove: expand into a sum (with non-negative scalars) of kernels $\langle x, x' \rangle$ raised to integer powers. These individual terms are valid kernels by the product rule.

Infinite sequences

The kernels we've seen so far are dot products between finitely many features. E.g.

$$k(x, y) = \left[\begin{array}{ccc} \sin(x) & x^3 & \log x \end{array} \right]^\top \left[\begin{array}{ccc} \sin(y) & y^3 & \log y \end{array} \right]$$

where $\phi(x) = \left[\begin{array}{ccc} \sin(x) & x^3 & \log x \end{array} \right]$

Can a kernel be a dot product between infinitely many features?

Infinite sequences

Definition

The space ℓ_2 (**square** summable sequences) comprises all sequences $a := (a_i)_{i\geq 1}$ for which

$$\|a\|_{\ell_2}^2 = \sum_{i=1}^{\infty} a_i^2 < \infty.$$

Infinite sequences

Definition

The space ℓ_2 (**square** summable sequences) comprises all sequences $a := (a_i)_{i \geq 1}$ for which

$$\|a\|_{\ell_2}^2 = \sum_{i=1}^{\infty} a_i^2 < \infty.$$

Definition

Given sequence of functions $(\phi_i(x))_{i \geq 1}$ in ℓ_2 where $\phi_i : \mathcal{X} \to \mathbb{R}$ is the ith coordinate of $\phi(x)$. Then

$$k(x, x') := \sum_{i=1}^{\infty} \phi_i(x) \phi_i(x') \tag{1}$$

Infinite sequences (proof)

Why square summable? By Cauchy-Schwarz,

$$\left|\sum_{i=1}^{\infty} \phi_i(x)\phi_i(x')\right| \leq \|\phi(x)\|_{\ell_2} \left\|\phi(x')\right\|_{\ell_2},$$

so the sequence defining the inner product converges for all $x, x' \in \mathcal{X}$

Taylor series kernels

Definition (Taylor series kernel)

For $r \in (0, \infty]$, with $a_n \geq 0$ for all $n \geq 0$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad |z| < r, \; z \in \mathbb{R},$$

Define $\mathcal{X}$ to be the $\sqrt{r}$-ball in $\mathbb{R}^d$, so$\|x\| < \sqrt{r}$,

$$k(x, x') = f\left(\langle x, x' \rangle\right) = \sum_{n=0}^{\infty} a_n \langle x, x' \rangle^n .$$

Example (Exponential kernel)

$$k(x, x') := \exp\left(\langle x, x' \rangle\right) .$$

Taylor series kernel (proof)

Proof: Non-negative weighted sums of kernels are kernels, and products of kernels are kernels, so the following is a kernel **if it converges**:

$$k(x, x') = \sum_{n=0}^{\infty} a_n \left(\langle x, x' \rangle\right)^n$$

By Cauchy-Schwarz,

$$\left|\langle x, x' \rangle\right| \le \|x\| \|x'\| < r,$$

so the sum converges.

Gaussian kernel

Example (Gaussian kernel)

The Gaussian kernel on $\mathbb{R}^d$ is defined as

$$k(x, x') := \exp\left(-\gamma^{-2} \left\|x - x'\right\|^2\right).$$

Proof: an exercise! Use product rule, mapping rule, exponential kernel.

Positive definite functions

If we are given a function of two arguments, $k(x, x')$, how can we determine if it is a valid kernel?

1. Find a feature map?
   1. Sometimes this is not obvious (eg if the feature vector is infinite dimensional, e.g. the Gaussian kernel in the last slide)
   2. The feature map is not unique.
2. A direct property of the function: positive definiteness.

Positive definite functions

Definition (Positive definite functions)

A symmetric function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is positive definite if $\forall n \geq 1,\ \forall (a_1, \ldots a_n) \in \mathbb{R}^n,\ \forall (x_1, \ldots, x_n) \in \mathcal{X}^n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j k(x_i, x_j) \geq 0.$$

The function $k(\cdot, \cdot)$ is strictly positive definite if for mutually distinct x_i, the equality holds only when all the a_i are zero.

Kernels are positive definite

Theorem

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{X}$ a non-empty set and $\phi : \mathcal{X} \to \mathcal{H}$. Then $\langle \phi(x), \phi(y) \rangle_{\mathcal{H}} =: k(x, y)$ is positive definite.

Proof.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j k(x_i, x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle a_i \phi(x_i), a_j \phi(x_j) \rangle_{\mathcal{H}} = \left\| \sum_{i=1}^{n} a_i \phi(x_i) \right\|_{\mathcal{H}}^{2} \geq 0.$$

Reverse also holds: positive definite $k(x, x')$ is inner product in a unique $\mathcal{H}$ (Moore-Aronsajn: coming later!). □

Sum of kernels is a kernel

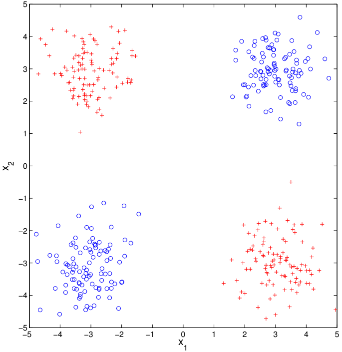
Consider two kernels $k_1(x, x')$ and $k_2(x, x')$. Then

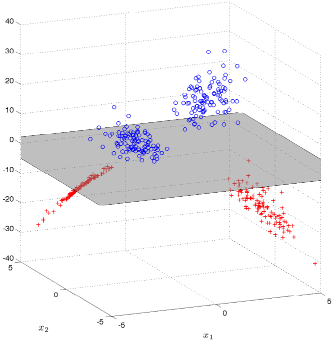
$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \left[k_1(x_i, x_j) + k_2(x_i, x_j)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j k_1(x_i, x_j) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j k_2(x_i, x_j) \\
&\geq 0
\end{aligned}
$$

The reproducing kernel Hilbert space

First example: finite space, polynomial features

Reminder: XOR example:





First example: finite space, polynomial features

Reminder: Feature space from XOR motivating example:

$$\begin{aligned} \phi : \mathbb{R}^2 &\to \mathbb{R}^3 \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\mapsto \phi(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{bmatrix}, \end{aligned}$$

with kernel

$$k(x, y) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \\ y_1 y_2 \end{bmatrix}$$

(the standard inner product in $\mathbb{R}^3$ between features). Denote this feature space by $\mathcal{H}$.

First example: finite space, polynomial features

Define a linear function of the inputs x_1, x_2, and their product $x_1 x_2$,

$$f(x) = f_1 x_1 + f_2 x_2 + f_3 x_1 x_2.$$

f in a space of functions mapping from $\mathcal{X} = \mathbb{R}^2$ to $\mathbb{R}$. Equivalent representation for f,

$$f(\cdot) = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}^\top .$$

$f(\cdot)$ refers to the function as an object (here as a vector in $\mathbb{R}^3$)
$f(x) \in \mathbb{R}$ is function evaluated at a point (a real number).

First example: finite space, polynomial features

Define a linear function of the inputs x_1, x_2, and their product $x_1 x_2$,

$$f(x) = f_1 x_1 + f_2 x_2 + f_3 x_1 x_2.$$

f in a space of functions mapping from $\mathcal{X} = \mathbb{R}^2$ to $\mathbb{R}$. Equivalent representation for f,

$$f(\cdot) = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}^\top.$$

$f(\cdot)$ refers to the function as an object (here as a vector in $\mathbb{R}^3$)
$f(x) \in \mathbb{R}$ is function evaluated at a point (a real number).

$$f(x) = f(\cdot)^\top \phi(x) = \langle f(\cdot), \phi(x) \rangle_{\mathcal{H}}$$

Evaluation of f at x is an **inner product in feature space** (here standard inner product in $\mathbb{R}^3$)
$\mathcal{H}$ is a space of functions mapping $\mathbb{R}^2$ to $\mathbb{R}$.

First example: finite space, polynomial features

I give you a vector:

$$g(\cdot) = \left[\begin{array}{ccc} 1 & -1 & -1 \end{array} \right]$$

Is this a function? Or is it a feature map $\phi(y) = \left[\begin{array}{ccc} y_1 & y_2 & y_1 y_2 \end{array} \right]$?

First example: finite space, polynomial features

I give you a vector:

$$g(\cdot) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$$

Is this a function? Or is it a feature map $\phi(y) = \begin{bmatrix} y_1 & y_2 & y_1 y_2 \end{bmatrix}$?
Both! All feature maps are also functions.

First example: finite space, polynomial features

I give you a vector:

$$g(\cdot) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$$

Is this a function? Or is it a feature map $\phi(y) = \begin{bmatrix} y_1 & y_2 & y_1 y_2 \end{bmatrix}$?
Both! All feature maps are also functions.
I give you a vector:

$$h(\cdot) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

Is this a function or a feature map?

First example: finite space, polynomial features

I give you a vector:

$$g(\cdot) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$$

Is this a function? Or is it a feature map $\phi(y) = \begin{bmatrix} y_1 & y_2 & y_1 y_2 \end{bmatrix}$?
Both! All feature maps are also functions.
I give you a vector:

$$h(\cdot) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

Is this a function or a feature map?
It is a function but not a feature map.

First example: finite space, polynomial features

I give you a vector:

$$g(\cdot) = \left[\begin{array}{ccc} 1 & -1 & -1 \end{array}\right]$$

Is this a function? Or is it a feature map $\phi(y) = \left[\begin{array}{ccc} y_1 & y_2 & y_1 y_2 \end{array}\right]$?
Both! All feature maps are also functions.
I give you a vector:

$$h(\cdot) = \left[\begin{array}{ccc} 1 & -1 & 2 \end{array}\right]$$

Is this a function or a feature map?
It is a function but not a feature map.
All feature maps are also functions. But the space of functions is larger: some functions are not feature maps.

First example: finite space, polynomial features

$\phi(y)$ is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$...
...which also parametrizes a function mapping $\mathbb{R}^2$ to $\mathbb{R}$.

$$k(\cdot, y) := \begin{bmatrix} y_1 & y_2 & y_1 y_2 \end{bmatrix}^\top = \phi(y),$$

We can *evaluate* this function at x

$$\langle k(\cdot, y), \phi(x) \rangle_{\mathcal{H}} = a x_1 + b x_2 + c x_1 x_2,$$

where $a = y_1$, $b = y_2$, and $c = y_1 y_2$

First example: finite space, polynomial features

$\phi(y)$ is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$...
...which also parametrizes a function mapping $\mathbb{R}^2$ to $\mathbb{R}$.

$$k(\cdot, y) := \begin{bmatrix} y_1 & y_2 & y_1 y_2 \end{bmatrix}^\top = \phi(y),$$

We can *evaluate* this function at x

$$\langle k(\cdot, y), \phi(x) \rangle_{\mathcal{H}} = a x_1 + b x_2 + c x_1 x_2,$$

where $a = y_1$, $b = y_2$, and $c = y_1 y_2$
...but due to symmetry,

$$\begin{aligned} \langle k(\cdot, x), \phi(y) \rangle &= u y_1 + v y_2 + w y_1 y_2 \\ &= k(x, y). \end{aligned}$$

We can write $\phi(x) = k(\cdot, x)$ and $\phi(y) = k(\cdot, y)$ without ambiguity: canonical feature map

The kernel trick

The kernel trick

This example illustrates the two defining features of an RKHS:

- **The reproducing property:** (kernel trick)
  $\forall x \in \mathcal{X}, \forall f(\cdot) \in \mathcal{H}, \ \ \langle f(\cdot), k(\cdot, x)\rangle_{\mathcal{H}} = f(x)$
  ...or use shorter notation $\langle f, \phi(x)\rangle_{\mathcal{H}}$.
- In particular, for any $x, y \in \mathcal{X}$,

$$k(x, y) = \langle k(\cdot, x), k(\cdot, y)\rangle_{\mathcal{H}}.$$

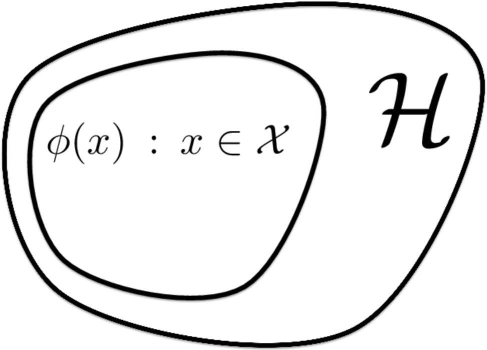
Note: the feature map of every point is in the feature space:
$\forall x \in \mathcal{X}, \ \ k(\cdot, x) = \phi(x) \in \mathcal{H}$,

First example: finite space, polynomial features

Another, more subtle point: $\mathcal{H}$ can be larger than all $\phi(x)$.

First example: finite space, polynomial features

Another, more subtle point: $\mathcal{H}$ can be larger than all $\phi(x)$.

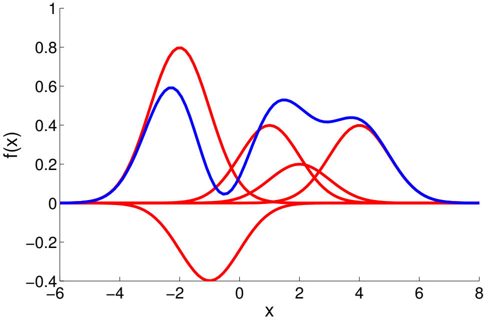


E.g. $f = [1\,1\,-1] \in \mathcal{H}$ cannot be obtained by $\phi(x) = [x_1\, x_2\, (x_1 x_2)]$.

Second example: infinite feature space

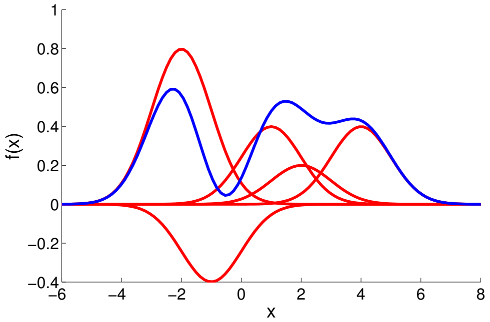
Second example: infinite feature space

Reproducing property for function with Gaussian kernel:
$f(x) := \sum_{i=1}^{m} \alpha_i k(x_i, x) = \left\langle \sum_{i=1}^{m} \alpha_i \phi(x_i), \phi(x) \right\rangle_{\mathcal{H}}$.



Second example: infinite feature space

Reproducing property for function with Gaussian kernel:
$f(x) := \sum_{i=1}^{m} \alpha_i k(x_i, x) = \left\langle \sum_{i=1}^{m} \alpha_i \phi(x_i), \phi(x) \right\rangle_{\mathcal{H}}$.



- What do the features $\phi(x)$ look like (warning: there are infinitely many of them!)
- What do these features have to do with smoothness?

Second example: infinite feature space

Under certain conditions (Mercer's theorem and extensions), we can write

$$k(x,x') = \sum_{i=1}^{\infty} \lambda_i e_i(x) e_i(x'), \qquad \int_{\mathcal{X}} e_i(x) e_j(x) d\mu(x) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

where this sum is guaranteed to converge whatever the x and x'.

Infinite dimensional feature map: $$\phi(x) = \begin{bmatrix} \vdots \\ \sqrt{\lambda_i} e_i(x) \\ \vdots \end{bmatrix} \in \ell_2.$$

Second example: infinite feature space

Under certain conditions (Mercer's theorem and extensions), we can write

$$k(x,x') = \sum_{i=1}^{\infty} \lambda_i e_i(x) e_i(x'), \qquad \int_{\mathcal{X}} e_i(x) e_j(x) d\mu(x) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

where this sum is guaranteed to converge whatever the x and x'.

Infinite dimensional feature map: $$\phi(x) = \begin{bmatrix} \vdots \\ \sqrt{\lambda_i} e_i(x) \\ \vdots \end{bmatrix} \in \ell_2.$$

Define $\mathcal{H}$ to be the space of functions: for $\{f_i\}_{i=1}^{\infty} \in \ell_2$,

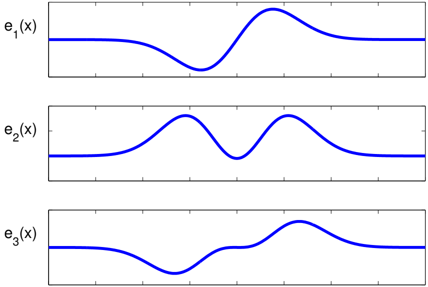
$$f(x) = \langle f, \phi(x) \rangle_{\mathcal{H}} = \sum_{i=1}^{\infty} f_i \sqrt{\lambda_i} e_i(x).$$

Second example: infinite feature space

Gaussian kernel, $k(x,y) = \exp\left(-\frac{\|x-y\|^2}{2\sigma^2}\right)$,

$$\begin{aligned} \lambda_k &\propto b^k \qquad b < 1 \\ e_k(x) &\propto \exp(-(c-a)x^2) H_k(x\sqrt{2c}), \end{aligned}$$

a, b, c are functions of σ, and H_k is kth order Hermite polynomial.



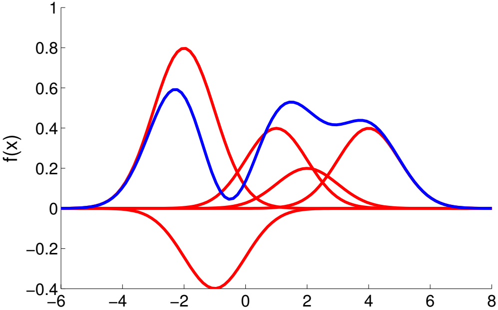
$$k(x,x') = \sum_{i=1}^{\infty} \lambda_i e_i(x) e_i(x')$$

Second example: infinite feature space

Example RKHS function, Gaussian kernel:

$$f(x) := \sum_{i=1}^{m} \alpha_i k(x_i, x) = \sum_{i=1}^{m} \alpha_i \left[\sum_{j=1}^{\infty} \lambda_j e_j(x_i) e_j(x) \right] = \sum_{j=1}^{\infty} f_j \underbrace{\left[\sqrt{\lambda_j} e_j(x) \right]}_{\phi_j(x)}$$

where $f_j = \sum_{i=1}^{m} \alpha_i \sqrt{\lambda_j} e_j(x_i)$.



NOTE that this enforces smoothing: λ_j decay as e_j become rougher, f_j decay since $\sum_j f_j^2 < \infty$.

Third (infinite) example: fourier series

Third (infinite) example: fourier series

Function on the torus $\mathbb{T} := [-\pi, \pi]$ with periodic boundary. Fourier series:

$$f(x) = \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell \exp(\imath \ell x) = \sum_{l=-\infty}^{\infty} \hat{f}_\ell \left(\cos(\ell x) + \imath \sin(\ell x)\right).$$

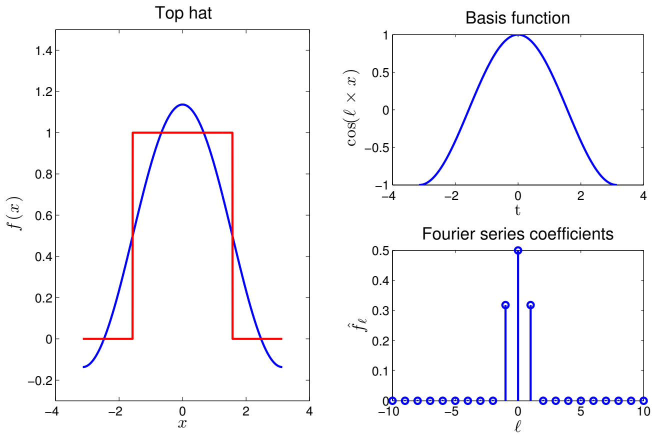
Example: "top hat" function,

$$f(x) = \begin{cases} 1 & |x| < T, \\ 0 & T \leq |x| < \pi. \end{cases}$$

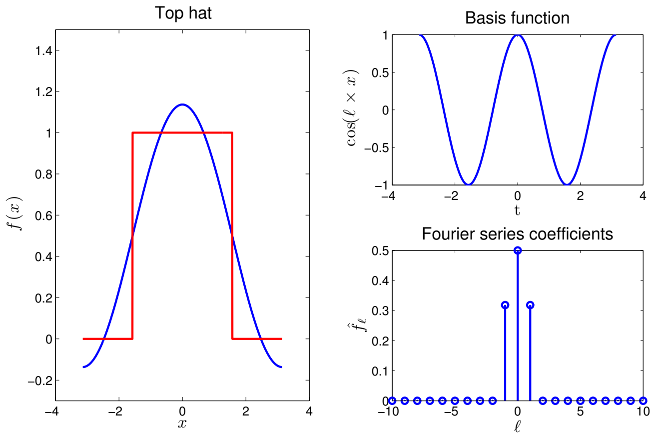
Fourier series:

$$\hat{f}_\ell := \frac{\sin(\ell T)}{\ell \pi} \qquad f(x) = \sum_{\ell=0}^{\infty} 2\hat{f}_\ell \cos(\ell x).$$

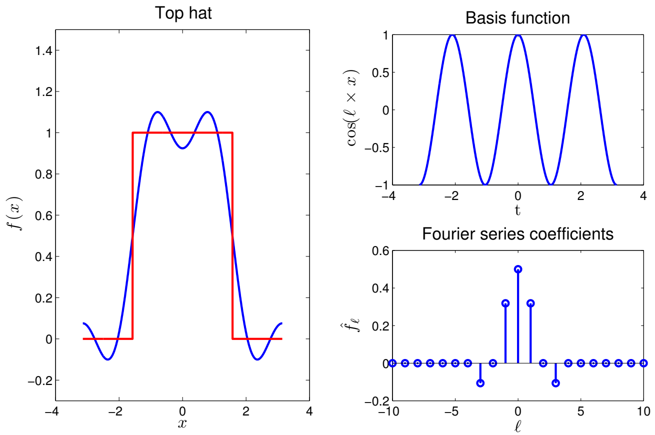
Fourier series for top hat function



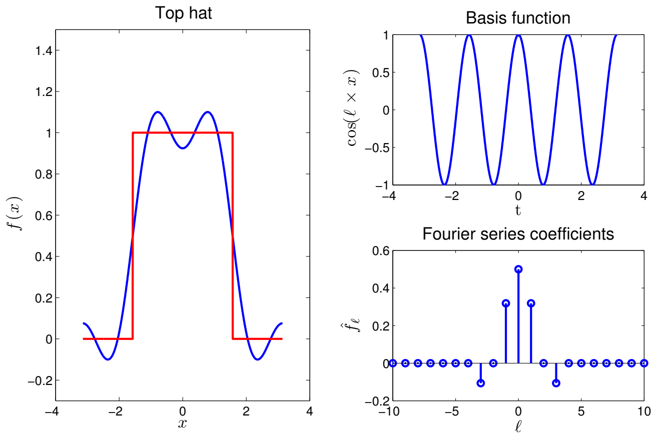
Fourier series for top hat function



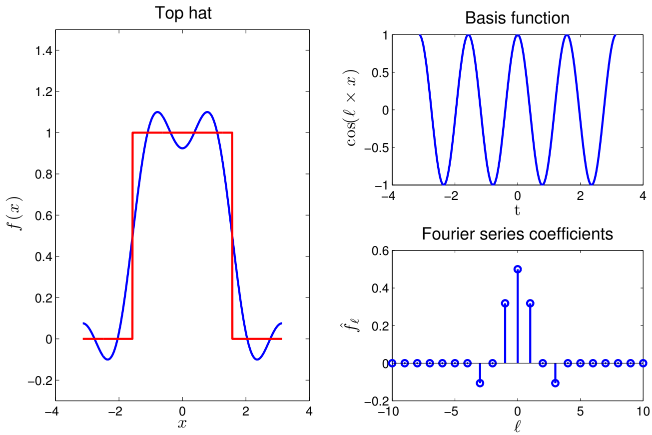
Fourier series for top hat function



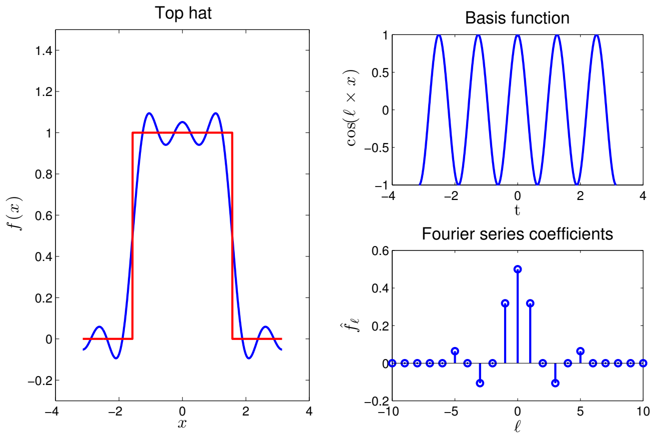
Fourier series for top hat function



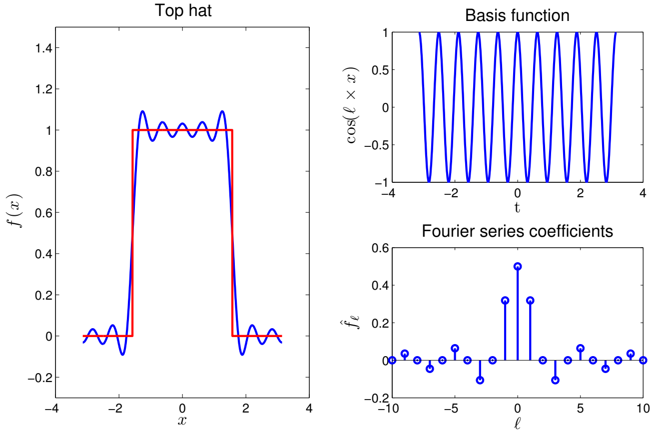
Fourier series for top hat function



Fourier series for top hat function



Fourier series for top hat function



Fourier series for kernel function

Kernel takes a single argument,

$$k(x, y) = k(x - y),$$

Define the Fourier series representation of k

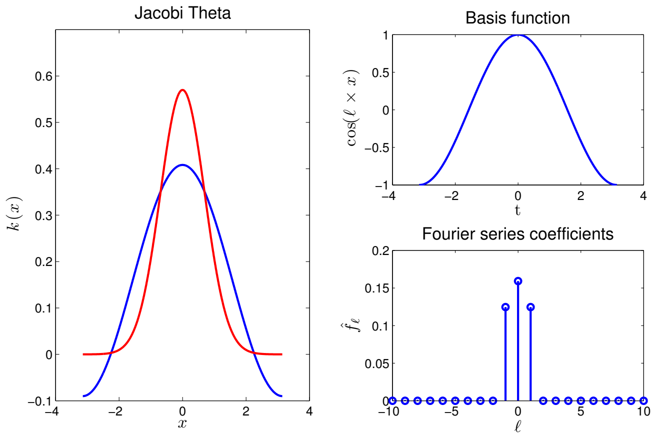
$$k(x) = \sum_{\ell=-\infty}^{\infty} \hat{k}_\ell \exp(\imath \ell x),$$

k and its Fourier transform are real and symmetric. E.g. ,

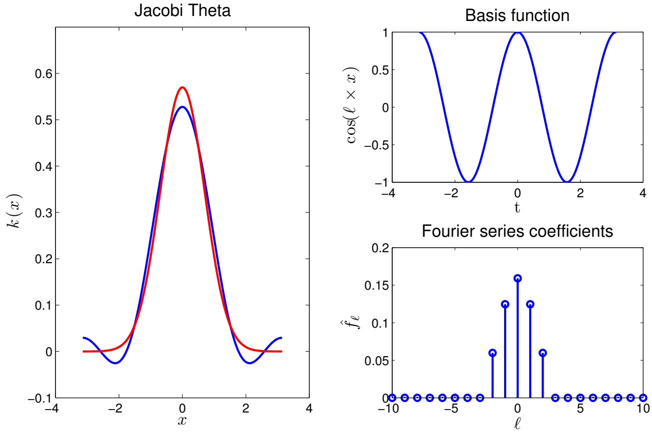
$$k(x) = \frac{1}{2\pi}\vartheta\left(\frac{x}{2\pi}, \frac{\imath\sigma^2}{2\pi}\right), \qquad \hat{k}_\ell = \frac{1}{2\pi}\exp\left(\frac{-\sigma^2\ell^2}{2}\right).$$

ϑ is the Jacobi theta function, close to Gaussian when σ^2 sufficiently narrower than $[-\pi, \pi]$.

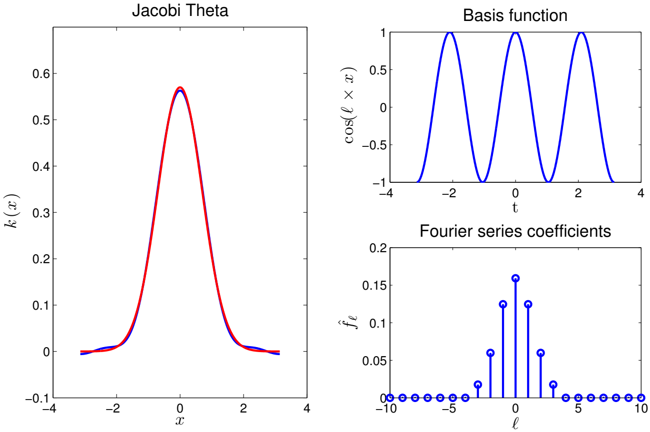
Fourier series for Gaussian-spectrum kernel



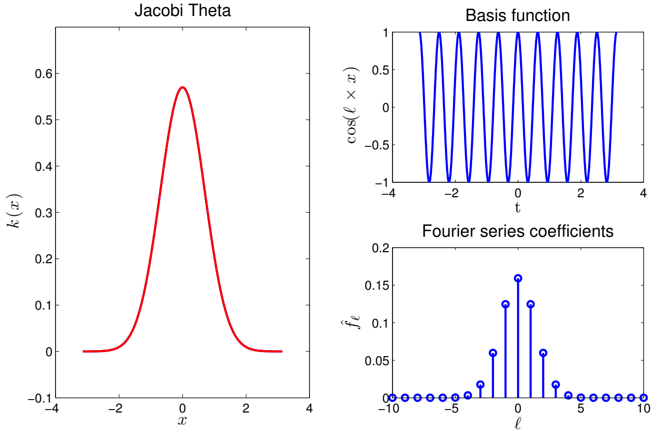
Fourier series for Gaussian-spectrum kernel



Fourier series for Gaussian-spectrum kernel



Fourier series for Gaussian-spectrum kernel



Feature space via fourier series

Define $\mathcal{H}$ to be the space of functions with (infinite) feature space representation

$$f(\cdot) = \begin{bmatrix} \ldots & \hat{f}_\ell/\sqrt{\hat{k}_\ell} & \ldots \end{bmatrix}^\top.$$

Feature space via fourier series

Define $\mathcal{H}$ to be the space of functions with (infinite) feature space representation

$$f(\cdot) = \begin{bmatrix} \ldots & \hat{f}_\ell / \sqrt{\hat{k}_\ell} & \ldots \end{bmatrix}^\top.$$

Define the feature map

$$k(\cdot, x) = \phi(x) = \begin{bmatrix} \ldots & \sqrt{\hat{k}_\ell} \exp(-\imath \ell x) & \ldots \end{bmatrix}^\top$$

Feature space via fourier series

The reproducing theorem holds,

$$
\begin{aligned}
\langle f(\cdot), k(\cdot, x)\rangle_{\mathcal{H}} &= \sum_{\ell=-\infty}^{\infty} \left(\frac{\hat{f}_\ell}{\sqrt{\hat{k}_\ell}}\right) \overline{\sqrt{\hat{k}_\ell} \exp(-\imath \ell x)} \\
&= \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell \exp(\imath \ell x) = f(x),
\end{aligned}
$$

Feature space via fourier series

The reproducing theorem holds,

$$\begin{aligned}\langle f(\cdot), k(\cdot, x)\rangle_{\mathcal{H}} &= \sum_{\ell=-\infty}^{\infty} \left(\frac{\hat{f}_\ell}{\sqrt{\hat{k}_\ell}}\right) \overline{\sqrt{\hat{k}_\ell} \exp(-\imath\ell x)} \\ &= \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell \exp(\imath\ell x) = f(x),\end{aligned}$$

...including for the kernel itself,

$$\begin{aligned}\langle k(\cdot, x), k(\cdot, y)\rangle_{\mathcal{H}} &= \sum_{\ell=-\infty}^{\infty} \left(\sqrt{\hat{k}_\ell} \exp(-\imath\ell x)\right) \left(\overline{\sqrt{\hat{k}_\ell} \exp(-\imath\ell y)}\right) \\ &= \sum_{\ell=-\infty}^{\infty} \hat{k}_\ell \exp(\imath\ell(y - x)) = k(x - y).\end{aligned}$$

Fourier series and smoothness

The squared norm of a function f in $\mathcal{H}$ is:

$$\|f\|_{\mathcal{H}}^2 = \langle f, f \rangle_{\mathcal{H}} = \sum_{l=-\infty}^{\infty} \frac{\hat{f}_\ell \overline{\hat{f}_\ell}}{\hat{k}_\ell}.$$

If $\hat{k}_\ell$ decays fast, then so must $\hat{f}_\ell$ if we want $\|f\|_{\mathcal{H}}^2 < \infty$.

Fourier series and smoothness

The squared norm of a function f in $\mathcal{H}$ is:

$$\|f\|^2_{\mathcal{H}} = \langle f, f \rangle_{\mathcal{H}} = \sum_{l=-\infty}^{\infty} \frac{\hat{f}_\ell \overline{\hat{f}_\ell}}{\hat{k}_\ell}.$$

If $\hat{k}_\ell$ decays fast, then so must $\hat{f}_\ell$ if we want $\|f\|^2_{\mathcal{H}} < \infty$.
Recall

$$f(x) = \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell \left(\cos(\ell x) + \imath \sin(\ell x)\right).$$

Enforces smoothness.

Fourier series and smoothness

The squared norm of a function f in $\mathcal{H}$ is:

$$\|f\|_{\mathcal{H}}^2 = \langle f, f\rangle_{\mathcal{H}} = \sum_{l=-\infty}^{\infty} \frac{\hat{f}_\ell \overline{\hat{f}_\ell}}{\hat{k}_\ell}.$$

If $\hat{k}_\ell$ decays fast, then so must $\hat{f}_\ell$ if we want $\|f\|_{\mathcal{H}}^2 < \infty$.
Recall

$$f(x) = \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell \left(\cos(\ell x) + \imath \sin(\ell x)\right).$$

Enforces smoothness.

Question: is the **top hat** function in the Gaussian-spectrum RKHS?

Some reproducing kernel Hilbert space theory

Reproducing kernel Hilbert space (1)

Definition

$\mathcal{H}$ a Hilbert space of $\mathbb{R}$-valued functions on non-empty set $\mathcal{X}$. A function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a reproducing kernel of $\mathcal{H}$, and $\mathcal{H}$ is a reproducing kernel Hilbert space, if

- $\forall x \in \mathcal{X}, \quad k(\cdot, x) \in \mathcal{H}$,
- $\forall x \in \mathcal{X}, \forall f \in \mathcal{H}, \quad \langle f(\cdot), k(\cdot, x) \rangle_{\mathcal{H}} = f(x)$ (the reproducing property).

In particular, for any $x, y \in \mathcal{X}$,

$$k(x, y) = \langle k(\cdot, x), k(\cdot, y) \rangle_{\mathcal{H}}. \tag{2}$$

Original definition: kernel an inner product between feature maps. Then $\phi(x) = k(\cdot, x)$ a valid feature map.

Reproducing kernel Hilbert space (2)

Another RKHS definition:

Define δ_x to be the operator of evaluation at x, i.e.

$$\delta_x f = f(x) \quad \forall f \in \mathcal{H},\ x \in \mathcal{X}.$$

Definition (Reproducing kernel Hilbert space)

$\mathcal{H}$ is an RKHS if the evaluation operator δ_x is bounded: $\forall x \in \mathcal{X}$ there exists $\lambda_x \geq 0$ such that for all $f \in \mathcal{H}$,

$$|f(x)| = |\delta_x f| \leq \lambda_x \|f\|_{\mathcal{H}}$$

$\Longrightarrow$ two functions identical in RHKS norm agree at every point:

$$|f(x) - g(x)| = |\delta_x (f - g)| \leq \lambda_x \|f - g\|_{\mathcal{H}} \quad \forall f, g \in \mathcal{H}.$$

RKHS definitions equivalent

Theorem (Reproducing kernel equivalent to bounded δ_x)

$\mathcal{H}$ is a reproducing kernel Hilbert space (i.e., its evaluation operators δ_x are bounded linear operators), if and only if $\mathcal{H}$ has a reproducing kernel.

Proof: If $\mathcal{H}$ has a reproducing kernel $\implies$ δ_x bounded

$$
\begin{aligned}
|\delta_x[f]| &= |f(x)| \\
&= |\langle f, k(\cdot,x)\rangle_{\mathcal{H}}| \\
&\le \|k(\cdot,x)\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \\
&= \langle k(\cdot,x), k(\cdot,x)\rangle_{\mathcal{H}}^{1/2} \|f\|_{\mathcal{H}} \\
&= k(x,x)^{1/2} \|f\|_{\mathcal{H}}
\end{aligned}
$$

Cauchy-Schwarz in 3rd line . Consequently, $\delta_x : \mathcal{F} \to \mathbb{R}$ bounded with $\lambda_x = k(x,x)^{1/2}$.

RKHS definitions equivalent

Proof: δ_x bounded $\implies \mathcal{H}$ has a reproducing kernel
We use...

Theorem

(Riesz representation) In a Hilbert space $\mathcal{H}$, all bounded linear functionals are of the form $\langle \cdot, g \rangle_{\mathcal{H}}$, for some $g \in \mathcal{H}$.

If $\delta_x : \mathcal{F} \to \mathbb{R}$ is a bounded linear functional, by Riesz $\exists f_{\delta_x} \in \mathcal{H}$ such that

$$\delta_x f = \langle f, f_{\delta_x} \rangle_{\mathcal{H}}, \ \forall f \in \mathcal{H}.$$

Define $k(x', x) = f_{\delta_x}(x'), \ \forall x, x' \in \mathcal{X}$. By its definition, both $k(\cdot, x) = f_{\delta_x} \in \mathcal{H}$ and $\langle f, k(\cdot, x) \rangle_{\mathcal{H}} = \delta_x f = f(x)$. Thus, k is the reproducing kernel.

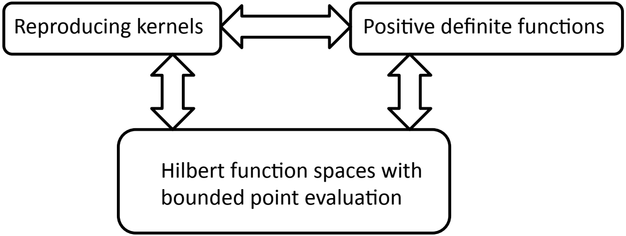
Moore-Aronszajn Theorem

Theorem (Moore-Aronszajn)

Let $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be positive definite. There is a ***unique RKHS*** *$\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$ with reproducing kernel k.*

Recall feature map is *not* unique (as we saw earlier): only kernel is.

Main message #1

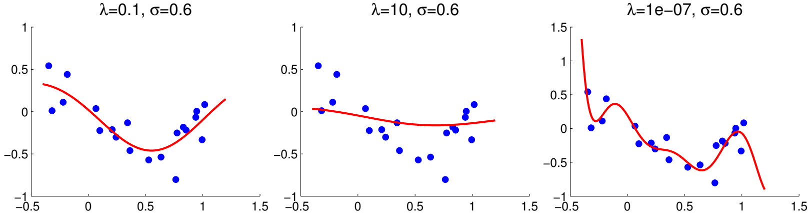


Main message #2

Small RKHS norm results in smooth functions.

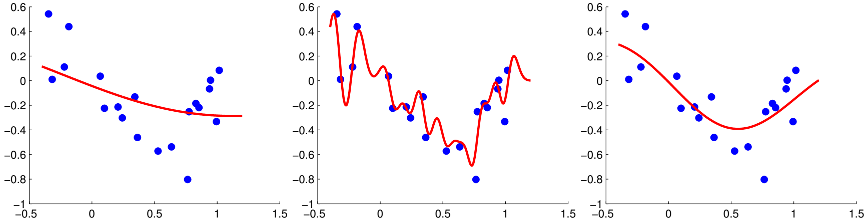
E.g. kernel ridge regression with Gaussian kernel:

$$f^* = \arg\min_{f \in \mathcal{H}} \left(\sum_{i=1}^{n} \left(y_i - \langle f, \phi(x_i) \rangle_{\mathcal{H}}\right)^2 + \lambda \|f\|_{\mathcal{H}}^2 \right).$$



Kernel Ridge Regression

Kernel ridge regression



Very simple to implement, works well when no outliers.

Kernel ridge regression

Use features of $\phi(x_i)$ in the place of x_i:

$$f^* = \arg\min_{f \in \mathcal{H}} \left(\sum_{i=1}^{n} \left(y_i - \langle f, \phi(x_i) \rangle_{\mathcal{H}}\right)^2 + \lambda \|f\|_{\mathcal{H}}^2 \right).$$

E.g. for finite dimensional feature spaces,

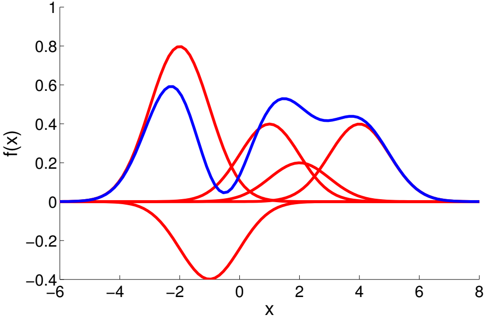
$$\phi_p(x) = \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^\ell \end{bmatrix} \qquad \phi_s(x) = \begin{bmatrix} \sin x \\ \cos x \\ \sin 2x \\ \vdots \\ \cos \ell x \end{bmatrix}$$

a is a vector of length ℓ giving weight to each of these features so as to find the mapping between x and y. Feature vectors can also have *infinite* length (more soon).

Kernel ridge regression

Solution easy if we already know f is a linear combination of feature space mappings of points: representer theorem.

$$f = \sum_{i=1}^{n} \alpha_i \phi(x_i) = \sum_{i=1}^{n} \alpha_i k(x_i, \cdot).$$



Representer theorem

Given a set of paired observations $(x_1, y_1), \ldots (x_n, y_n)$ (regression or classification).
Find the function f^* in the RKHS $\mathcal{H}$ which satisfies

$$J(f^*) = \min_{f \in \mathcal{H}} J(f), \tag{3}$$

where

$$J(f) = L_y(f(x_1), \ldots, f(x_n)) + \Omega\left(\|f\|^2_{\mathcal{H}}\right),$$

Ω is non-decreasing, and y is the vector of y_i.

- Classification: $L_y(f(x_1), \ldots, f(x_n)) = \sum_{i=1}^n \mathbb{I}_{y_i f(x_i) \leq 0}$
- Regression: $L_y(f(x_1), \ldots, f(x_n)) = \sum_{i=1}^n (y_i - f(x_i))^2$

Representer theorem

The representer theorem: (simple version) solution to

$$\min_{f \in \mathcal{H}} \left[L_y(f(x_1), \ldots, f(x_n)) + \Omega\left(\|f\|_{\mathcal{H}}^2 \right) \right]$$

takes the form

$$f^* = \sum_{i=1}^{n} \alpha_i k(x_i, \cdot).$$

If Ω is strictly increasing, all solutions have this form.

Representer theorem: proof

Proof: Denote f_s projection of f onto the subspace

$$\mathrm{span}\left\{k(x_i,\cdot):\ 1\leq i\leq n\right\}, \tag{4}$$

such that

$$f = f_s + f_\perp,$$

where $f_s = \sum_{i=1}^n \alpha_i k(x_i,\cdot)$.
Regularizer:

$$\|f\|_{\mathcal{H}}^2 = \|f_s\|_{\mathcal{H}}^2 + \|f_\perp\|_{\mathcal{H}}^2 \geq \|f_s\|_{\mathcal{H}}^2,$$

then

$$\Omega\left(\|f\|_{\mathcal{H}}^2\right) \geq \Omega\left(\|f_s\|_{\mathcal{H}}^2\right),$$

so this term is minimized for $f = f_s$.

Representer theorem: proof

Proof (cont.): Individual terms $f(x_i)$ in the loss:

$$f(x_i) = \langle f, k(x_i, \cdot)\rangle_{\mathcal{H}} = \langle f_s + f_\perp, k(x_i, \cdot)\rangle_{\mathcal{H}} = \langle f_s, k(x_i, \cdot)\rangle_{\mathcal{H}} ,$$

so

$$L_y(f(x_1), \ldots, f(x_n)) = L_y(f_s(x_1), \ldots, f_s(x_n)).$$

Hence

- Loss $L(\ldots)$ only depends on the component of f in the data subspace,
- Regularizer $\Omega(\ldots)$ minimized when $f = f_s$.
- If Ω is strictly non-decreasing, then $\|f_\perp\|_{\mathcal{H}} = 0$ is required at the minimum.

Kernel ridge regression: proof

We *begin* knowing f is a linear combination of feature space mappings of points (representer theorem)

$$f = \sum_{i=1}^{n} \alpha_i \phi(x_i).$$

Then

$$\sum_{i=1}^{n} \left(y_i - \langle f, \phi(x_i) \rangle_{\mathcal{H}}\right)^2 + \lambda \|f\|_{\mathcal{H}}^2 \quad = \quad \|y - K\alpha\|^2 + \lambda \alpha^\top K \alpha$$

Differentiating wrt α and setting this to zero, we get

$$\alpha^* = (K + \lambda I_n)^{-1} y.$$

Reminder: smoothness

What does $\|a\|_{\mathcal{H}}$ have to do with smoothing?

Example 1: The Fourier series representation on torus $\mathbb{T}$:

$$f(x) = \sum_{l=-\infty}^{\infty} \hat{f}_l \exp(\imath l x),$$

and

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{l=-\infty}^{\infty} \frac{\hat{f}_l \overline{\hat{g}_l}}{\hat{k}_l}.$$

Thus,

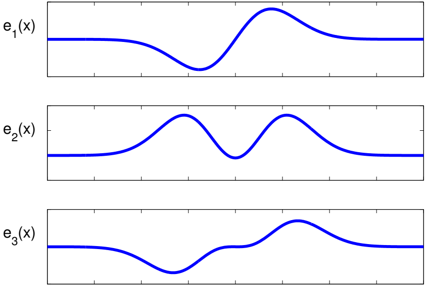
$$\|f\|_{\mathcal{H}}^2 = \langle f, f \rangle_{\mathcal{H}} = \sum_{l=-\infty}^{\infty} \frac{\left|\hat{f}_l\right|^2}{\hat{k}_l}.$$

Reminder: smoothness

What does $\|a\|_{\mathcal{H}}$ have to do with smoothing?

Example 2: The Gaussian kernel on $\mathbb{R}$. Recall

$$f(x) = \sum_{i=1}^{\infty} a_i \sqrt{\lambda_i} e_i(x), \qquad \|f\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} a_i^2.$$



Parameter selection for KRR

Given the objective

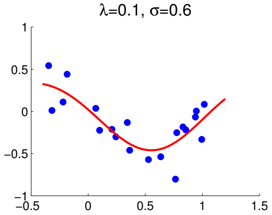
$$f^* = \arg\min_{f \in \mathcal{H}} \left(\sum_{i=1}^{n} (y_i - \langle f, \phi(x_i) \rangle_{\mathcal{H}})^2 + \lambda \|f\|_{\mathcal{H}}^2 \right).$$

How do we choose

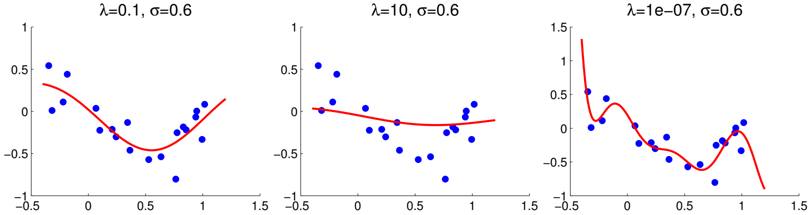
- The regularization parameter λ?
- The kernel parameter: for Gaussian kernel, σ in

$$k(x, y) = \exp\left(\frac{-\|x - y\|^2}{\sigma} \right).$$

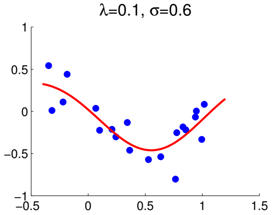
Choice of λ



Choice of λ



Choice of σ



Choice of σ