Assignment 8
Theoretical Neuroscience

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All figure and equation numbers refer to *Theoretical Neuroscience*.

1. **Oja's rule convergence**

   Show that the averaged form of the single-trial Oja rule in equation 8.16 is given by

   $$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{Q} \cdot \mathbf{w} - \alpha(\mathbf{w} \cdot \mathbf{Q} \cdot \mathbf{w})\mathbf{w} \,.$$

   Prove that if it converges, the averaged learning rule produces a set of weights proportional to an eigenvector of the correlation matrix $\mathbf{Q}$, normalized so that $|\mathbf{w}|^2 = 1/\alpha$.

2. **Ocular dominance model**

   Simulate the ocular dominance model of figure 8.7 using a subtractively normalized version of equation 8.31 (i.e. equation 8.14) with saturation limits at 0 and 1, and cortical interactions generated as in figure 8.8 from

   $$\mathbf{K}_{aa'} = \exp\left(-\frac{(a-a')^2}{2\sigma^2}\right) - \frac{1}{9}\exp\left(-\frac{(a-a')^2}{18\sigma^2}\right) \,,$$

   where $\sigma = 0.066$ mm. Use 512 cortical cells with locations a spread evenly over a nominal 10 mm of cortex, **and periodic boundary conditions** (this means that you can use Fourier transforms to calculate the effect of the cortical interactions).

   Also use the discrete form of equation 8.31:

   $$\mathbf{W} \rightarrow \mathbf{W} + \epsilon \mathbf{K} \cdot \mathbf{W} \cdot \mathbf{Q}$$

   with a learning rate of $\epsilon = 0.01$.

   Plot $\mathbf{w}_-$ as it evolves from near $\mathbf{0}$ to the final form of ocular dominance. Calculate the magnitude of the discrete Fourier transform of $\mathbf{w}_-$. Repeat this around 100 times, work out the average of the magnitudes of the Fourier transforms. Why might one expect this average to resemble the Fourier transform of $\mathbf{K}$? Does it, in fact, do so?

3. **Learning rate effect**

   Consider minimizing the function $E(w) = (w-2)^2$ using the gradient descent rule for w,

   $$w \rightarrow w - \epsilon \frac{dE}{dw} \,.$$

   Plot $E(w)$ together with the trajectories of w starting from $w = 5$ for $\epsilon = 0.01, 0.1, 1, 2, 3$. Why does learning diverge as ϵ gets large?

4. **Extended delta rule**

Consider $E(\mathbf{w}) \propto \left\langle (h(s) - \mathbf{w} \cdot \mathbf{f}(s))^2 \right\rangle$, as in equation 8.52, in the case that matrix $\langle \mathbf{f}(s)\mathbf{f}(s) \rangle$ is invertible. An extended delta rule can be written as

$$\mathbf{w} \to \mathbf{w} + \langle (h(s) - \mathbf{w} \cdot \mathbf{f}(s))\mathbf{H} \cdot \mathbf{f}(s) \rangle \,,$$

where $\mathbf{H}$ is a matrix that generalizes the learning rate ϵ of the standard delta rule. For what matrix $\mathbf{H}$ does this rule go from any initial value $\mathbf{w}$ to the optimal weights in one single step. This amounts to a form of the Newton-Raphson method.

5. **Vectorised value function**

Consider a finite Markov chain with transition matrix P_{xy}. If a subject gets reward r_x at state x, and we define the long-run discounted value function V_x as

$$V_x = \mathcal{E}\left[\sum_{t=0}^{\infty} \gamma^t r_{x(t)} | x(0) = x\right]$$

following the stochastic dynamics of the chain, write down a matrix equation that the vector of all values, $\mathbf{V}$, satisfies.

6. **Equilibrium of partial reinforcement**

Consider the case of partial reinforcement (studied in figure 9.1) in which reward $r = 1$ is provided randomly with probability p on any given trial. Assume that there is a single stimulus with $u = 1$, so that $\epsilon\delta u$, with $\delta = r - v = r - wu$, is equal to $\epsilon(r - w)$. Calculate the self-consistent equilibrium values of the mean and variance of the weight w. What happens to your expression for the variance if $\epsilon = 2$ or $\epsilon > 2$? To what features of the learning rule do these effects correspond?