Assignment 9
Theoretical Neuroscience

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1. **Linear analysis**

   Consider an equation of the form

$$\frac{d\mathbf{z}}{dt} = \mathbf{A} \cdot \mathbf{z} \tag{1}$$

   which in component form looks like $dz_i/dt = \sum_j A_{ij} z_j$. (The "." notation, favored by physicists worldwide, can be used for multiplying both matrices with vectors and vectors with vectors. For the former, the i^{th} component of $\mathbf{A} \cdot \mathbf{z}$ is $\sum_j A_{ij} z_j$, and for the latter $\mathbf{x} \cdot \mathbf{y} = \sum_i x_i y_i$.)

   Define $\mathbf{v}_k$, $\mathbf{v}_k^\dagger$, and λ_k via the equations

$$\mathbf{A} \cdot \mathbf{v}_k = \lambda_k \mathbf{v}_k \tag{2a}$$
$$\mathbf{v}_k^\dagger \cdot \mathbf{A} = \lambda_k \mathbf{v}_k^\dagger \,. \tag{2b}$$

   The $\mathbf{v}_k$ and $\mathbf{v}_k^\dagger$ are eigenvectors and adjoint eigenvectors, respectively (the latter sometimes called left eigenvectors), and the λ_k are the associated eigenvalues. If $\mathbf{A}$ is an $n \times n$ matrix (which would mean that $\mathbf{z}$ has n components), there are n eigenvectors. Assume a normalization such that $\mathbf{v}_k \cdot \mathbf{v}_l^\dagger = \delta_{kl}$.

   Show that if $\mathbf{z}$ evolves according to Eq. (1) and $\mathbf{z}(t=0) = \mathbf{z}_0$, then

$$\mathbf{z}(t) = \sum_k \mathbf{v}_k \mathbf{v}_k^\dagger \cdot \mathbf{z}_0 \, e^{\lambda_k t} \,. \tag{3}$$

   Remember this! If you stay in computational neuroscience, you will use it over and over and over.

2. **Stability analysis**

   Consider two dynamical variables, x and y, that evolve according to

$$dx/dt = ax + by \tag{4a}$$
$$dy/dt = cx + dy \tag{4b}$$

As usual, the trace (T) and determinant (D) are given by $T = a + d$ and $D = ad - bc$.

2a. Consider the following four cases:

1) $T < 0, D > 0$,

2) $T < 0, D < 0$,

3) $T > 0, D > 0$,

4) $T > 0, D < 0$.

Choose values of a, b, c and d that are consistent with each case, and sketch the nullclines and trajectories in x-y space. To make life easier for the person grading this, draw the x-nullcline in blue and the y-nullcline in red. To make contact with what we did in class, you may want to make a and c positive and b and d negative, but this is not necessary.

2b. Show that when $T^2 < 4D$, $x(t)$ and $y(t)$ are given by

$$x(t) = x_0 e^{\lambda_r t} \cos(\lambda_i t + \phi_x) \tag{5a}$$
$$y(t) = y_0 e^{\lambda_r t} \cos(\lambda_i t + \phi_y) \tag{5b}$$

where λ_r and λ_i are the real and imaginary parts of the eigenvalues (see problem 1) and x_0, y_0, ϕ_x and ϕ_y are constants.

At some point in your lives you should compute x_0, y_0, ϕ_x and ϕ_y in terms of a, b, c, d and the initial conditions, but this is *not* part of the homework assignment. Merry Christmas.

3. **Mean field analysis**

Consider firing rate equations of the form

$$\tau \frac{d\nu_i}{dt} = \phi\left(\gamma\overline{\nu} + \frac{\beta}{Nf(1-f)} \sum_{j=1}^{N} \eta_i(\eta_j - f)\nu_j\right) - \nu_i \tag{6}$$

where N is the number of neurons, γ and β are constants, γ is negative, $\overline{\nu}$ is, as usual, the firing rate averaged over neurons,

$$\overline{\nu} = \frac{1}{N}\sum_i \nu_i\,, \tag{7}$$

η is a random binary vector,

$$\eta_i = \begin{cases} 1 & \text{probability } f \\ 0 & \text{probability } (1-f)\,, \end{cases} \tag{8}$$

and ϕ is monotonically increasing.

Let

$$m = \frac{1}{Nf(1-f)} \sum_i (\eta_i - f)\nu_i \,. \tag{9}$$

Note that m is the firing rate of the "memory" neurons relative to the mean firing rate, with an extra factor of $1/(1-f)$ thrown in to simplify the equations that you will derive.

3a. Derive *dynamical* mean field equations for $\overline{\nu}$ and m in the large N limit. By "dynamical," I mean derive equations for $d\overline{\nu}/dt$ and dm/dt.

3b. Sketch the nullclines for $\overline{\nu}$ and m assuming ϕ is approximately sigmoidal.

4. **Hopfield networks reduce energy**

Consider a Hopfield network that evolves *asynchronously* according to

$$S_i(t+1) = \text{sign}\left[\sum_i J_{ij} S_j(t)\right] \tag{10}$$

where J_{ij} is symmetric and has no diagonal elements,

$$\begin{aligned} J_{ij} &= J_{ji} && \text{(11a)} \\ J_{ii} &= 0\,. && \text{(11b)} \end{aligned}$$

Define the energy,

$$H(t) = -\frac{1}{2} \sum_i S_i(t) J_{ij} S_j(t)\,. \tag{12}$$

Show that if the S_i obey the dynamics given in Eq. (10), then the energy never increases; i.e., $H(t{+}1) \leq H(t)$.

This is an important result: since the energy is bounded from below, it implies that the dynamics eventually reaches a fixed point.