

# Bayesian inference with probabilistic population codes

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Recent psychophysical experiments indicate that humans perform near-optimal Bayesian inference in a wide variety of tasks, ranging from cue integration to decision making to motor control. This implies that neurons both represent probability distributions and combine those distributions according to a close approximation to Bayes' rule. At first sight, it would seem that the high variability in the responses of cortical neurons would make it difficult to implement such optimal statistical inference in cortical circuits. We argue that, in fact, this variability implies that populations of neurons automatically represent probability distributions over the stimulus, a type of code we call probabilistic population codes. Moreover, we demonstrate that the Poisson-like variability observed in cortex reduces a broad class of Bayesian inference to simple linear combinations of populations of neural activity. These results hold for arbitrary probability distributions over the stimulus, for tuning curves of arbitrary shape and for realistic neuronal variability.

Virtually all computations performed by the nervous system are subject to uncertainty and taking this into account is critical for making inferences about the outside world. For instance, imagine hiking in a forest and having to jump over a stream. To decide whether or not to jump, you could compute the width of the stream and compare it to your internal estimate of your jumping ability. If, for example, you can jump 2 m and the stream is 1.9 m wide, then you might choose to jump. The problem with this approach, of course, is that you ignored the uncertainty in the sensory and motor estimates. If you can jump  $2 \pm 0.4$  m and the stream is  $1.9 \pm 0.5$  m wide, jumping over it is very risky—and even life-threatening if it is filled with, say, piranhas.

Behavioral studies have confirmed that human observers not only take uncertainty into account in a wide variety of tasks, but do so in a way that is nearly optimal<sup>1–5</sup> (where 'optimal' is used in a Bayesian sense, as defined below). This has two important implications. First, neural circuits must represent probability distributions. For instance, in our example, the width of the stream could be represented in the brain by a Gaussian distribution with mean 1.9 m and s.d. 0.5 m. Second, neural circuits must be able to combine probability distributions nearly optimally, a process known as Bayesian inference.

Although it is clear experimentally that human behavior is nearly Bayes-optimal in a wide variety of tasks, very little is known about the neural basis of this optimality. In particular, we do not know how probability distributions are represented in neuronal responses, nor how neural circuits implement Bayesian inference. At first sight, it would seem that cortical neurons are not well suited to this task, as their responses are highly variable: the spike count of cortical neurons in response to the same sensory variable (such as the direction of motion of a visual stimulus) or motor command varies greatly from trial to trial, typically with Poisson-like statistics<sup>6</sup>. It is critical to realize, however, that variability and uncertainty go hand in hand: if neuronal

variability did not exist, that is, if neurons were to fire in exactly the same way every time you saw the same object, then you would always know with certainty what object was presented. Thus, uncertainty about the width of the river in the above example is intimately related to the fact that neurons in the visual cortex do not fire in exactly the same way every time you see a river that is 2 m wide. This variability is partly due to internal noise (like stochastic neurotransmitter release<sup>7</sup>), but the potentially more important component arises from the fact that rivers of the same width can look different, and thus give rise to different neuronal responses, when viewed from different distances or vantage points.

Neural variability, then, is not incompatible with the notion that humans can be Bayes-optimal; on the contrary, as we have just seen, neural variability is expected when subjects experience uncertainty. What is not clear, however, is exactly how optimal inference is achieved given the particular type of noise—Poisson-like variability—observed in the cortex. Here we show that Poisson-like variability makes a broad class of Bayesian inferences particularly easy. Specifically, this variability has a unique property: it allows neurons to represent probability distributions in a format that reduces optimal Bayesian inference to simple linear combinations of neural activities.

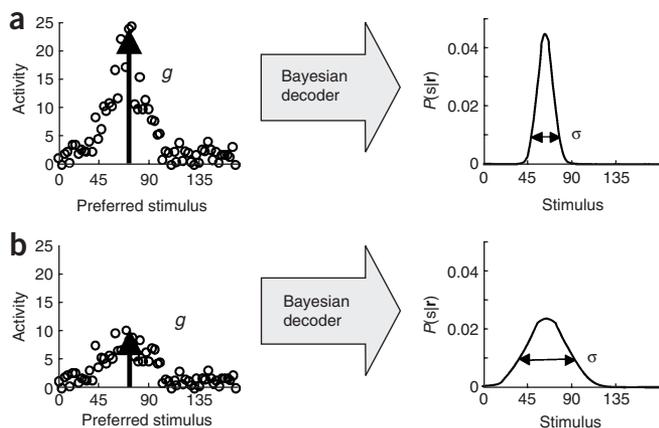
## RESULTS

### Probabilistic population codes (PPC)

Thinking of neurons as encoders of probability distributions is a departure from the more standard view, which is to think of them as encoding the values of variables (like the width of a stream, as in our previous example). However, as several authors have pointed out<sup>8–12</sup>, population activity automatically encodes probability distributions. This is because of the variability in neuronal responses, which implies that the population response,  $\mathbf{r} \equiv \{r_1, \dots, r_N\}$ , to a stimulus,  $s$ , is

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**Figure 1** Certainty and gain. **(a)** The population activity,  $\mathbf{r}$ , on the left is the single trial response to a stimulus whose value was 70. All neurons were assumed to have a translated copy of the same generic Gaussian tuning curve to  $s$ . Neurons are ranked by their preferred stimulus (that is, the stimulus corresponding to the peak of their tuning curve). The plot on the right shows the posterior probability distribution over  $s$  given  $\mathbf{r}$ , as recovered using Bayes' theorem (equation (1)). When the neural variability follows an independent Poisson distribution (which is the case here), it is easy to show that the gain,  $g$ , of the population code (its overall amplitude) is inversely proportional to the variance of the posterior distribution,  $\sigma^2$ . **(b)** Decreasing the gain increases the width of the encoded distribution. Note that the population activity in **a** and **b** have the same widths; only their amplitudes are different.

given in terms of a probability distribution,  $p(\mathbf{r}|s)$ . This response distribution then very naturally encodes the posterior distribution over  $s$ ,  $p(s|\mathbf{r})$ , through Bayes' theorem<sup>8,9</sup>,

$$p(s|\mathbf{r}) \propto p(\mathbf{r}|s)p(s) \quad (1)$$

To take a specific example, for independent Poisson neural variability, equation (1) becomes,

$$p(s|\mathbf{r}) \propto \prod_i \frac{e^{-f_i(s)} f_i(s)^{r_i}}{r_i!} p(s),$$

where  $f_i(s)$  is the tuning curve of neuron  $i$ . In this case, the posterior distribution,  $p(s|\mathbf{r})$ , converges to a Gaussian as the number of neurons increases (assuming a flat prior over  $s$ , an assumption we make now only for convenience, but drop later). The mean of this distribution is close to the stimulus at which the population activity peaks (**Fig. 1**). The variance,  $\sigma^2$ , is also encoded in the population activity—it is inversely proportional to the amplitude of the hill of activity<sup>13–15</sup>. Using  $g$  (for gain; see **Fig. 1**) to denote the amplitude of the hill of activity, we have  $g \propto 1/\sigma^2$ . Thus, for independent Poisson neural variability (and, in fact, for many other noise models, as we discuss below), it is possible to encode any Gaussian probability distribution with population activity. This type of parameterization is sometimes known as a product of experts<sup>16</sup>.

### A simple case study: multisensory integration

Although it is clear that population activity can represent probability distributions, can they carry out any optimal computations—or inference—in ways consistent with human behavior? Before asking how neurons can do this, however, we need to define precisely what we mean by 'optimal'.

In a cue combination task, the goal is to integrate two cues,  $c_1$  and  $c_2$ , both of which provide information about the same stimulus,  $s$ . For

instance,  $s$  could be the spatial location of a stimulus,  $c_1$  could be a visual cue for the location, and  $c_2$  could be an auditory cue. Given observations of  $c_1$  and  $c_2$ , and under the assumption that these quantities are independent given  $s$ , the posterior over  $s$  is obtained via Bayes' rule,  $p(s|c_1, c_2) \propto p(c_1|s)p(c_2|s)p(s)$ .

When the prior is flat and the likelihood functions,  $p(c_1|s)$  and  $p(c_2|s)$ , are Gaussian with respect to  $s$  with means  $\mu_1$  and  $\mu_2$  and variances  $\sigma_1^2$  and  $\sigma_2^2$ , respectively, the mean and variance of the posterior,  $\mu_3$  and  $\sigma_3^2$ , are given by the following equations (from ref. 17):

$$\mu_3 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \mu_2 \quad (2)$$

$$\frac{1}{\sigma_3^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \quad (3)$$

Experiments show that humans perform a close approximation to this Bayesian inference—meaning their mean and variance, averaged over many trials, follow equations (2) and (3)—when tested on cue combination<sup>2,3,18,19</sup>.

Now that we have a target for optimality—equations (2) and (3)—we can ask how neurons can achieve it. Again we consider two cues,  $c_1$  and  $c_2$ , but here we encode them in population activities,  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , respectively, with gains  $g_1$  and  $g_2$  (**Fig. 2**). These probabilistic population codes (PPCs) represent two likelihood functions,  $p(\mathbf{r}_1|s)$  and  $p(\mathbf{r}_2|s)$ . We also assume (for now) that (i)  $\mathbf{r}_1$  and  $\mathbf{r}_2$  have the same number of neurons, and (ii) two neurons with the same index  $i$  share the same tuning curve profile; that is, the mean value of both  $r_{1i}$  and  $r_{2i}$  are proportional to  $f_i(s)$ . What we now show is that when the prior is flat ( $p(s) = \text{constant}$ ), taking the sum of the two population codes,  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , is equivalent to optimal Bayesian inference. By taking the sum, we mean that we construct a third population,  $\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2$ , which is the sum of  $\mathbf{r}_1$  and  $\mathbf{r}_2$  on a neuron-by-neuron basis:  $r_{3i} = r_{1i} + r_{2i}$ . If  $\mathbf{r}_1$  and  $\mathbf{r}_2$  follow Poisson distributions, so will  $\mathbf{r}_3$ . Therefore,  $\mathbf{r}_3$  encodes a likelihood function with variance  $\sigma_3^2$ , where  $\sigma_3^2$  is inversely proportional to the gain of  $\mathbf{r}_3$ . Notably, the gain of the third population, denoted  $g_3$ , is simply the sum of the gains of the first two:  $g_3 = g_1 + g_2$  (**Fig. 2**). Because  $g_k$  is proportional to  $1/\sigma_k^2$  ( $k = 1, 2, 3$ ), with a constant of proportionality that is independent of  $k$ , this relationship between the gains implies that  $1/\sigma_3^2 = 1/\sigma_1^2 + 1/\sigma_2^2$ . This is exactly equation (3). Consequently, the variance of the distribution encoded by  $\mathbf{r}_3$  is precisely the variance of the posterior distribution,  $p(s|c_1, c_2)$ .

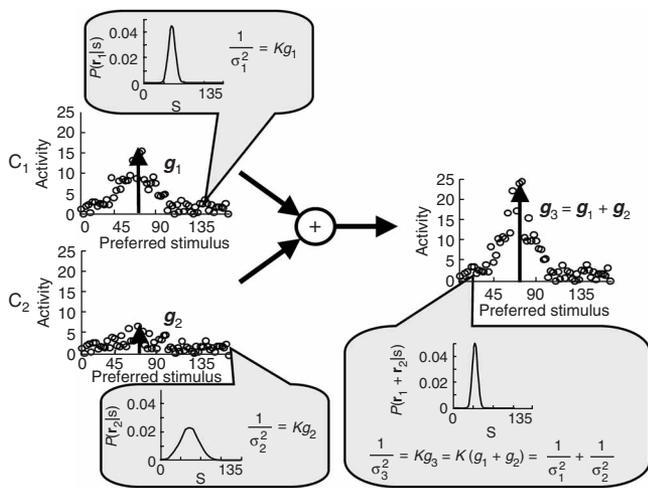
### General theory and the exponential family of distributions

Does the strategy of adding population codes lead to optimal inference under more general conditions, such as non-Gaussian distributions over the stimulus and non-Poisson neural variability? In general, the sum,  $\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2$ , is Bayes-optimal if  $p(s|\mathbf{r}_3)$  is equal to  $p(s|\mathbf{r}_1)p(s|\mathbf{r}_2)$  or, equivalently, if  $p(\mathbf{r}_1 + \mathbf{r}_2|s) \propto p(\mathbf{r}_1|s)p(\mathbf{r}_2|s)$ . This is not the case for most probability distributions (such as additive Gaussian noise with fixed variance; see **Supplementary Note** online) but, as shown in **Supplementary Note**, the sum is Bayes-optimal if all distributions are what we call Poisson-like; that is, distributions of the form

$$p(\mathbf{r}_k|s, g_k) = \phi_k(\mathbf{r}_k, g_k) \exp(\mathbf{h}^T(s) \mathbf{r}_k) \quad (4)$$

where the index  $k$  can take the value, 1, 2 or 3, and the kernel  $\mathbf{h}(s)$  obeys

$$\mathbf{h}'(s) = \sum_k^{-1} (s, g_k) \mathbf{f}'_k(s, g_k) \quad (5)$$



**Figure 2** Inference with probabilistic population codes for Gaussian probability distributions and Poisson variability. The left plots correspond to population codes for two cues,  $c_1$  and  $c_2$ , related to the same variable  $s$ . Each of these encodes a probability distribution with a variance inversely proportional to the gains,  $g_1$  and  $g_2$ , of the population codes ( $K$  is a constant depending on the width of the tuning curve and the number of neurons). Adding these two population codes leads to the output population activity shown on the right. This output also encodes a probability distribution with a variance inversely proportional to the gain. Because the gain of this code is  $g_1 + g_2$ , and  $g_1$  and  $g_2$  are inversely proportional to  $\sigma_1^2$  and  $\sigma_2^2$ , respectively, the inverse variance of the output population code is the sum of the inverse variances associated with  $c_1$  and  $c_2$ . This is precisely the variance expected from an optimal Bayesian inference (equation (3)). In other words, taking the sum of two population codes is equivalent to taking the product of their encoded distributions.

$\Sigma_k$  is the covariance matrix of  $\mathbf{r}_k$ , and  $\mathbf{f}'_k$  is the derivative of the tuning curves. In the case of independent Poisson noise, identically shaped tuning curves,  $\mathbf{f}(s)$ , in the two populations, and different gains, it turns out that  $\mathbf{h}(s) = \log \mathbf{f}(s)$ , and  $\phi_k(\mathbf{r}_k, g_k) = \exp(-cg_k) \prod_i \exp(r_{ki} \log g_k) / r_{ki}!$  with  $c$  a constant.

As indicated by equation (5), for addition of population codes to be optimal, the right-hand side of this equation must be independent of both  $g_k$  and  $k$ . As  $\mathbf{f}'$  is clearly proportional to the gain, for the first condition to be satisfied  $\Sigma_k(s, g_k)$  must also be proportional to the gain. This is exactly what is observed in cortex, where it is found that the covariance matrix is proportional to the mean spike count<sup>6,20</sup>, which in turn is proportional to the gain. This applies in particular to independent Poisson noise, for which the variance is equal to the mean, but is not limited to that distribution. For instance, we do not require that the neurons be independent (that is, that  $\Sigma_k(s, g_k)$  be diagonal). Also, although we need the covariance to be proportional to the mean, the constant of proportionality does not have to be 1. This is important because how the diagonal elements of the covariance matrix scale with  $g$  determines the Fano factor, and values reported in cortex for these scaling are not always 1 (as would be the case for purely Poisson neurons) but instead range from 0.3 to 1.8 (refs. 6,20).

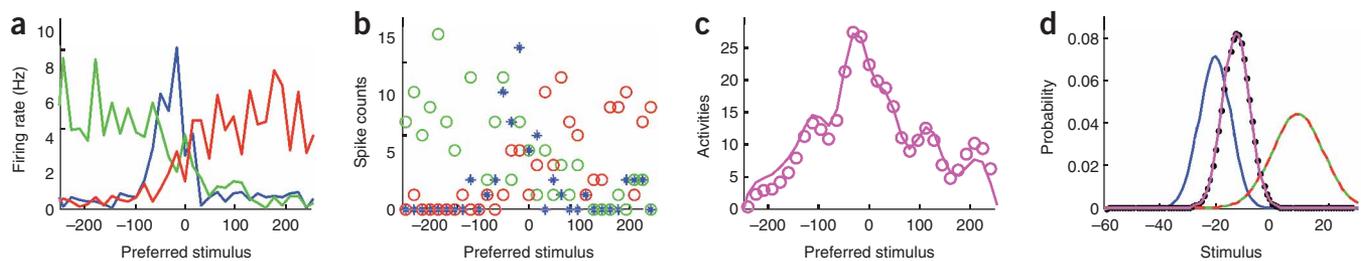
The second condition, that  $\mathbf{h}'(s)$  must be independent of  $k$ , requires that  $\mathbf{h}(s)$  be identical, up to an additive constant, in all input layers. This

occurs, for instance, when the input tuning curves are identical and the noise is independent and Poisson. When the  $\mathbf{h}(s)$ 's are not the same, so that  $\mathbf{h}(s) \rightarrow \mathbf{h}_k(s)$ , addition is no longer optimal, but optimality can still be achieved with linear combinations of activity, that is, a dependence of the form  $\mathbf{r}_3 = \mathbf{A}_1 \mathbf{T}_1 + \mathbf{A}_2 \mathbf{T}_2$  (provided the functions of  $s$  that make up the components of the  $\mathbf{h}_k(s)$ 's are drawn from a common basis set; details in **Supplementary Note**). Therefore, even if the tuning curves and covariance structures are completely different in the two population codes—for instance, Gaussian tuning curves in one and sigmoidal curves in the other—optimal Bayesian inference can be achieved with linear combinations of population codes.

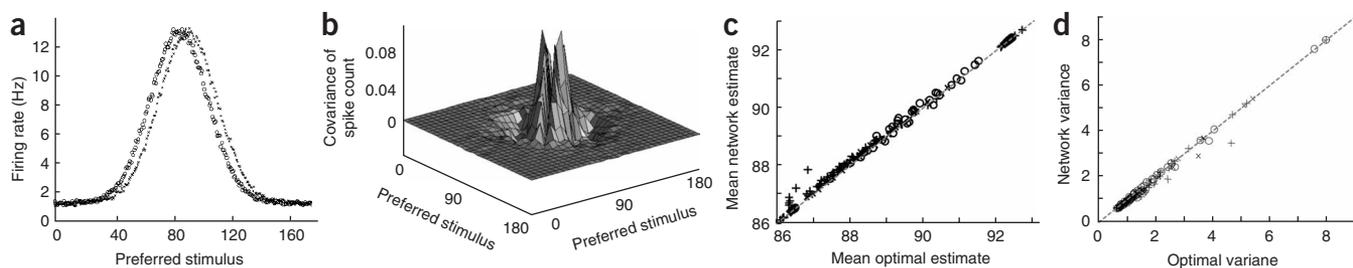
To illustrate this point, we show a simulation (**Fig. 3**) in which there are three input layers in which the tuning curves are Gaussian, sigmoidal increasing and sigmoidal decreasing, and the parameters of the tuning curves, such as the widths, slopes, amplitude and baseline activity, vary within each layer (that is, the tuning curves are not perfectly translation invariant). As predicted, with an appropriate choice of the matrices  $\mathbf{A}_1$ ,  $\mathbf{A}_2$  and  $\mathbf{A}_3$  (**Supplementary Note**), a linear combination of the input activities,  $\mathbf{r}_3 = \mathbf{A}_1 \mathbf{T}_1 + \mathbf{A}_2 \mathbf{T}_2 + \mathbf{A}_3 \mathbf{T}_3$ , is optimal.

Another important property of equation (4) worth emphasizing is that it imposes no constraint on the shape of the probability distribution with respect to  $s$ , so long as  $\mathbf{h}(s)$  forms a basis set. In other words, our scheme works for a large class of distributions over  $s$ , not just Gaussian distributions.

Finally, it is easy to incorporate prior distributions. We encode the desired prior in a population code (using equation (1)) and add that to



**Figure 3** Inference with non-translation invariant Gaussian and sigmoidal tuning curves. **(a)** Mean activity in the three input layers. Blue curves, input layer with Gaussian tuning curves. Red curves, input layers with sigmoidal tuning curves with positive slopes. Green curves, input layers with sigmoidal tuning curves with negative slopes. The noise in the curves is due to variability in the baseline, widths, slopes and amplitudes of the tuning curves and to the fact that the tuning curves are not equally spaced along the stimulus axis. **(b)** Activity in the three input layers on a given trial. These activities were sampled from Poisson distributions with means as in **a**. Color legend as in **a**. **(c)** Solid lines, mean activity in the output layer. Circles, output activity on a given trial, obtained by a linear combination of the input activities shown in **b**. **(d)** Blue curves, probability distribution encoded by the blue stars in **b** (input layer with Gaussian tuning curves). Red-green curve, probability distribution encoded by the red and green circles in **b** (the two input layers with sigmoidal tuning curves). Magenta curve, probability distribution encoded by the activity shown in **c** (magenta circles). Black dots, probability distribution obtained with Bayes rule (that is, the product of the blue and red-green curves appropriately normalized). The fact that the black dots are perfectly lined up with the magenta curve demonstrates that the output activity shown in **c** encodes the probability distribution expected from Bayes rule.



**Figure 4** Near-optimal inference with a two-layer network of integrate-and-fire neurons similar in spirit to the network shown in **Figure 2**. The network consisted of two input layers that sent feedforward connections to the output layer. The output layer contained both excitatory and inhibitory neurons and was recurrently connected; the input layers were purely excitatory and had no recurrent connections. **(a)** Average activity in the two input layers for identical gains. The positions of the two hills differ on average by 6 to simulate a cue conflict (the units are arbitrary). **(b)** Covariance matrix of the spike count in the output layer. The diagonal terms (the variances) were set to zero in this plot because they swamp the signal from the covariance (and are uninformative). Because of lateral connections and correlations in the input, output units with similar tuning are correlated. **(c)** Mean of the probability distribution encoded in the output layer when inputs 1 and 2 are presented together (mean network estimate) versus mean predicted by an optimal Bayesian estimator (mean optimal estimate, obtained from equation (2); see Methods). Each point corresponds to the means averaged over 1,008 trials for a particular combination of gains in the input layers. The symbols correspond to different types of input. Circles, same tuning curves and same covariance matrix for both inputs. Plus signs, same tuning curves and different covariance matrices. Crosses, different tuning curves and different covariance matrices (see Methods). **(d)** Same as in **c** but for the variance. The optimal variance is obtained from equation (3). In both **c** and **d**, the data lie near the line with slope = 1 (diagonal dashed line), indicating that the network performs a close approximation to Bayesian inference.

the population code representing the likelihood function. This predicts that in an area encoding a prior, neurons should fire before the start of the trial. Moreover, if the prior at a particular spatial location is increased, all neurons with receptive fields at that location should fire more strongly (their gain should increase). This is indeed what has been reported in area LIP (ref. 21) and in the superior colliculus<sup>22</sup>. One problem with this approach is that the encoded prior will vary from trial to trial due to the Poisson variability. Whether such a variability in the encoded prior is observed in human subjects is not presently known<sup>5</sup>.

### Simulations with integrate-and-fire neurons

So far, our results rely on the assumption that neurons can compute linear combinations of spike counts, which is only an approximation of what actual neurons do. Neurons are nonlinear devices that integrate their inputs and fire spikes. To determine whether it is possible to perform near-optimal Bayesian inference with realistic neurons, we simulated a network like the one shown in **Figure 2** but with conductance-based integrate-and-fire neurons. The network consisted of two input layers, denoted 1 and 2, that sent feedforward connections to the output layer, denoted layer 3. The activity in the input layers formed noisy hills with the peak in layer 1 centered at  $s = 86.5$  and the peak in layer 2 at  $s = 92.5$  (**Fig. 4a** shows the mean input activities in both layers). We used different values of the positions of the input hills to simulate cue conflict, as is commonly done in psychophysics experiments. The amplitude of each input hill was determined by the reliability of the cue it encoded: the higher the reliability, the higher the hill, as expected for a PPC with Poisson-like variability (**Fig. 1**). The activity in the output layer also formed a hill, which was decoded using a locally optimal linear estimator<sup>23</sup>. Parameters were chosen such that the spike counts of the output neurons exhibit realistic Fano factors (Fano factors ranging from 0.76 to 1.0). As we have seen, Fano factors that are independent of the gain are one of the key properties required for optimality. Additionally, the conductances of the feedforward and lateral connections were adjusted to ensure that the average firing rates of the output neurons were approximately linear functions of the average firing rates of the input neurons. Because of the convergent feedforward connectivity and the cortical connections, output units

with similar tuning ended up being correlated (**Fig. 4b**; additional details of the model in Methods and **Supplementary Note**).

The goal of these simulations was to assess whether the mean and variance of the distributions encoded in the output layer are consistent with optimal Bayesian inference (equations (2) and (3)). To simulate psychophysical experiments, we first presented one cue at a time; that is, we activated either layer 1 or layer 2, but not both. We systematically varied the certainty of the cue by changing the value of the gain of the activated input layer. For each gain, we computed the mean and variance of the distribution encoded in the output layer when only one cue was presented. These were denoted  $\mu_1$  and  $\sigma_1^2$ , respectively, when only input 1 was active, and  $\mu_2$  and  $\sigma_2^2$  when only input 2 was active. We then presented both cues together, which gave us  $\mu_3$  and  $\sigma_3^2$ , the mean and variance of the distribution encoded in the output layer when both cues are presented simultaneously. To test whether the network was Bayes-optimal, we plotted (**Fig. 4c**)  $\mu_3$  against

$$\mu_1 \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} + \mu_2 \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

(equation (2)), and (**Fig. 4d**)  $\sigma_3^2$  against

$$\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

(equation (3)) over a wide range of values of certainty for the two cues (corresponding to gains of the two input hills). If the network is performing a close approximation to Bayesian inference, the data should lie close to a line with slope 1 and intercept 0.

It is clear (**Fig. 4c,d**) that the network is indeed nearly optimal on average for all combinations of gains tested, as has been found in human data<sup>1-4</sup>. This result holds even when the input layers use different sets of tuning curves and different patterns of correlations (**Fig. 4d**), thus confirming the applicability of our analytical findings. Therefore, linear combinations of probabilistic population codes are Bayes-optimal for Poisson-like noise.

### Experimental predictions

These ideas can be tested experimentally in different domains, as Bayesian inference seems to be involved in many sensory, motor and

cognitive tasks. We now consider three specific predictions that can be tested with single- or multiunit recordings:

First, we predict that if an animal exhibits Bayes-optimal behavior in a cue combination task, and the variability of multisensory neurons is Poisson-like (as defined by equation (4)), one should find that the responses of these neurons to multisensory inputs should be the sum of the responses to the unisensory inputs. This prediction seems at odds with the main result that has been emphasized in the literature, namely, superadditivity. Superadditivity refers to a multimodal response that is greater than the value predicted by the sum of the unimodal responses<sup>24</sup>. Recent studies<sup>25,26</sup>, however, have shown that the vast majority of multisensory neurons exhibit additive responses in anesthetized animals. What is needed now to test our hypothesis is similar data in awake animals performing optimal multisensory integration.

Our second prediction concerns decision making, more specifically, binary decision making (as in ref. 27). In these experiments, animals are trained to decide between two saccades (in opposite directions) given the direction of motion in a random-dot kinematogram. In a Bayesian framework, the first step in decision making is to compute the posterior distribution over the decision variable,  $s$ , given the available evidence. In this particular task, the evidence takes the form of a population pattern of activity from motion-sensitive neurons, probably from area MT. Denoting  $\mathbf{r}_t^{\text{MT}}$  to be the population pattern of activity in area MT at time  $t$ , the posterior distribution over  $s$  since the beginning of the trial can be computed recursively using Bayes' rule,

$$p(s|\mathbf{r}_t^{\text{MT}}, \dots, \mathbf{r}_1^{\text{MT}}) \propto p(\mathbf{r}_t^{\text{MT}}|s)p(s|\mathbf{r}_{t-1}^{\text{MT}}, \dots, \mathbf{r}_1^{\text{MT}}) \quad (6)$$

Note that this inference involves, as with cue combination, multiplying probability distributions. Thus, if we represent the posterior distribution at time  $t-1$ ,  $p(s|\mathbf{r}_{t-1}^{\text{MT}}, \dots, \mathbf{r}_1^{\text{MT}})$ , in a probabilistic population code (say in area LIP) then, upon observing a new pattern of activity from MT, we can simply add this pattern to LIP activity. In other words, LIP neurons will automatically implement equation (6) simply by accumulating activity coming from MT. This predicts that LIP neurons behave like neural integrators of MT activity, which is consistent with what a previous study has found<sup>28</sup>. In addition, this predicts that the profile of tuning curves of LIP neurons over time should remain identical; only the gain and the baseline should change. This prediction has yet to be tested.

Third, our theory makes a general prediction regarding population codes in the cortex and their relation to behavioral performance. If a stimulus parameter is varied in such a way that the subject is less certain about the stimulus, the probability distribution over stimuli recovered by equation (1) (as assumed by PPCs) should reflect that uncertainty (in the case of a Gaussian posterior, for example, the distribution should get wider). This prediction has been verified in two cases in which it has been tested experimentally: motion coherence<sup>29,30</sup> and contrast<sup>31,32</sup>.

This last prediction may not be valid in all areas of the brain. For instance, it is conceivable that motor neurons encode a single action, not a full distribution over possible actions (as would be the case for any network computing maximum-likelihood estimates; see for instance ref. 33). If that were the case, applying Bayes' rule to the activity of motor neurons would not return a posterior distribution that reflects the subject's certainty about this action being correct.

## DISCUSSION

We have argued that the nervous system may use probabilistic population codes (PPCs) to encode probability distributions over variables in the outside world (such as the orientation of a bar or the

speed of a moving object). This notion is not entirely new. Several groups<sup>8-10,34</sup> have pointed out that probability distributions can be recovered from neuronal responses through equation (1). However, we go beyond this observation in two ways. First, we show that Bayesian inference—a nontrivial and critically important computation in the brain—is particularly simple when using PPCs with Poisson-like variability. Second, we do not merely propose that population activity encodes distributions—this part is always true, in the sense that equation (1) can always be applied to a population code. The new aspect of our claim is that the probability distributions encoded in some areas of the cortex reflect the uncertainty about the stimulus, whereas in other areas they do not (in particular in motor areas, as discussed at the end of the previous section).

Other types of neural codes beside PPCs have been proposed for encoding probability distributions that reflect the observer's uncertainty<sup>3,11,12,28,35-43</sup>. In most of these, however, the Poisson-like variability is either ignored altogether or treated as a nuisance factor that corrupts the codes. In only one of them was Poisson-like variability taken into account and, in fact, used to compute explicitly the log likelihood of the stimulus<sup>43</sup>, presumably because log-likelihood representations have the advantage that they turn products of probability distributions into sums<sup>28,35,41-43</sup>. A crucial point of our work, however, is to show that, when the neural variability belongs to the exponential family with linear sufficient statistics (as is the case in ref. 43), products turn into sums without any need for an explicit computation of the log likelihood. This is important because there are a number of problems associated with the explicit computation of the log likelihood. For instance, the model described in ref. 43 is limited to independent Poisson noise, unimodal probability distributions and winner-take-all readout. This is problematic, as the noise in the cortex is correlated, probability distributions can have multiple peaks (for example, the Necker cube), and winner-take-all is a particularly inefficient read-out technique. More importantly, the log-likelihood approach runs into severe computational limitations when applied to many Bayesian inference problems such as ones involved in Kalman filters<sup>41</sup>. By contrast, the PPC approach works for correlated Poisson-like noise and a wide variety of tuning curves, the latter being crucial for optimal nonlinear computations<sup>34,44</sup>. Our framework can also be readily extended to Kalman filters (J. Beck, W.J. Ma, P.E. Latham & A. Pouget, *Cosyne Abstr.* 47, 2006). Finally, it has the advantage of being recursive: with PPCs, all cortical areas use the same scheme to represent probability distributions (as opposed to log-likelihood schemes, in which some areas use the standard tuning curve plus noise model while others explicitly compute log likelihood). Recursive schemes map very naturally onto the stereotyped nature of cortical microcircuitry<sup>45</sup>.

One limitation of our scheme, and of any scheme that reduces Bayesian inference to addition of activities, is that neural activities are likely to saturate when sequential inferences are required. To circumvent this problem, a nonlinearity is needed to keep neurons within their dynamical range. A nonlinearity like divisive normalization<sup>46,47</sup> would be ideal because it is near linear for low firing rates, where uncertainty is large and thus there is much to be gained from performing exact inference, and saturating at high firing rates, where uncertainty is small and there is little to be gained from exact inference (see Fig. 1).

In conclusion, our notion of probabilistic population codes offers a new perspective on the role of Poisson-like variability. The presence of such variability throughout the cortex suggests that the entire cortex represents probability distributions, not just estimates, which is precisely what would be expected from a Bayesian perspective (see also ref. 48 for related ideas). We propose that these distributions are

collapsed onto estimates only when decisions are needed, a process that may take place in motor cortex or in subcortical structures. Notably, our previous work shows that attractor dynamics in these decision networks could perform this step optimally by computing maximum *a posteriori* estimates<sup>33</sup>.

## METHODS

**Spiking neuron simulations.** A detailed description of the network is given in **Supplementary Note**; here we give a brief overview. The network we simulated is a variation of the model reported in ref. 23. It contains two input layers and one output layer. Each input layer consists of 1,008 excitatory neurons. These neurons exhibit bell-shaped tuning curves with preferred stimuli evenly distributed over the range [0,180] (stimulus units are arbitrary). The input spike trains are near-Poisson with mean rates determined by the tuning curves. The output layer contains 1,260 conductance-based integrate-and-fire neurons, of which 1,008 are excitatory and 252 inhibitory. Each of those neurons receives connections from the input neurons. The conductances associated with the input connections follow a Gaussian profile centered on the preferred stimulus of each input unit.

The connectivity in the output layer is chosen so that the output units exhibit Gaussian tuning curves whose widths are close to the widths of the convolved input (that is, the width after the input tuning curves have been convolved with the feedforward weights). The balance of excitation and inhibition in the output layer was adjusted to produce high Fano factors (0.7–1.0), within the range observed *in vivo*<sup>6,20</sup>. Finally, additional tuning of connection strengths was performed to ensure that the firing rates of the output neurons were approximately linear functions of the firing rates of the input neurons.

We simulated three different networks. In the first (blue dots in **Fig. 4c,d**), for both populations the widths of the input tuning curves were 20 and the widths of the feedforward weights were 15. In the second (red dots in **Fig. 4c,d**), the widths of the input tuning curves were 15 and 25, and the widths of the corresponding feedforward weights were 20 and 10. The effective inputs for the two populations had identical tuning curves (with a width of 35) but, unlike in the first network, different covariance matrices. Finally, in the third network (green dots in **Fig. 4c,d**), the widths of the input tuning curves were 15 and 25, and the width of the feedforward weights was 15. In this case both the tuning curves and the covariance matrices of the effective inputs were different.

**Estimating the mean and variance of the encoded distribution.** To determine whether this network is Bayes-optimal, we need to estimate the mean and variance of the probability distribution encoded in the output layer. In principle, all we need is  $p(r|s)$ , equation (1). The response, however, is 1,008-dimensional. Estimating a distribution in 1,008 dimensions requires an unreasonably large amount of data—more than we could collect in several billion years. We thus used a different approach. The variances can be estimated using a locally optimal linear estimator, as described in ref. 23. For the mean, we fit a Gaussian to the output spike count on every trial and used the position of the Gaussian as an estimate of the mean of the encoded distribution. The best fit was found by minimizing the Euclidean distance between the Gaussian and the spike counts. The points in **Figure 4c,d** are the means and variances averaged over 1,008 trials (details in **Supplementary Note**).

Note: Supplementary information is available on the Nature Neuroscience website.

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## COMPETING INTERESTS STATEMENT

The authors declare that they have no competing financial interests.

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