Dynamic Mean Field Analysis

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March 26, 2019

We're interesting in understanding the behavior of the set of equations

$$\frac{dx_i}{dt} = -x_i + \eta_i \tag{1a}$$

$$\eta_i = \sum_{j=1}^{N} J_{ij}\phi(x_j). \tag{1b}$$

We're going to use the method outlined in "Intrinsically-generated fluctuating activity in excitatory-inhibitory networks," by Francesca Mastrogiuseppea and Srdjan Ostojic, *PLoS Computational Biology* 13(4):e1005498 (2017). But our version is a bit simpler.

One of the main things we want to compute is the population averaged firing rate, denoted μ,

$$\mu \equiv \frac{1}{N}\sum_i \langle x_i \rangle \tag{2}$$

where angle brackets indicate an average with respect to time. Using Eq. (1), we have

$$\mu = \frac{1}{N}\sum_{ij} J_{ij}\langle \phi(x_j)\rangle. \tag{3}$$

Interchanging indices, this becomes

$$\mu = \sum_j \langle \phi(x_j)\rangle \frac{1}{N}\sum_i J_{ij}. \tag{4}$$

Let

$$\bar{J} = \frac{1}{N}\sum_i J_{ij}. \tag{5}$$

Note that $\bar{J}$ should depend on j. However, in the large N limit that dependence should be weak, so we ignore it. We thus should think of $\bar{J}$ as an average over the distribution of the weight matrix. Strictly speaking this is valid only in the $N \to \infty$ limit, and in principle one should be careful about fluctuations, but it's not hard to show that they aren't important. Once we make this approximation, we have

$$\mu = \bar{J}\sum_j \langle \phi(x_j)\rangle. \tag{6}$$

As usual, to perform that average, we assume that x_j is a Gaussian random variable, and compute its mean and variance with respect to index, j. The variance, which we'll call $\Delta(0)$, is given by

$$\Delta(0) \equiv \frac{1}{N} \sum_i \langle x_i(t) x_i(t) \rangle - \mu^2. \tag{7}$$

If we knew $\Delta(0)$, we could solve for the mean, μ, self-consistently via

$$\mu = N \bar{J} \int d\xi \phi\big(\mu + \sqrt{\Delta(0)}\xi\big) \,. \tag{8}$$

If there's a stable equilibrium, it's pretty easy to compute $\Delta(0)$ – we can use standard methods (going back at least to Danny Amit, and probably further). However, for large enough coupling, $x_i(t)$ is time dependent, which makes things a lot harder. In that case, it turns out that we need the whole covariance of $x_i(t)$. The rest of this writeup is devoted to finding that.

The covariance is defined in the usual way,

$$\Delta(\tau) \equiv \frac{1}{N} \sum_i \langle x_i(t) x_i(t+\tau) \rangle - \mu^2. \tag{9}$$

To compute this, we write down a differential equation for it. Using Eq. (1a), we have

$$\begin{aligned} \frac{d\Delta(\tau)}{d\tau} &= -\frac{1}{N} \sum_i \langle x_i(t) x_i(t+\tau) \rangle + \frac{1}{N} \sum_i \langle x_i(t) \eta_i(t+\tau) \rangle \\ &= -\frac{1}{N} \sum_i \langle x_i(t) x_i(t+\tau) \rangle + \frac{1}{N} \sum_i \langle x_i(t-\tau) \eta_i(t) \rangle. \end{aligned} \tag{10}$$

Taking another derivative (using the second line in the above expression), again using Eq. (1a), and performing a very small amount of algebra, arrive at

$$\begin{aligned} \frac{d^2\Delta(\tau)}{d\tau^2} &= \frac{1}{N} \sum_i \langle x_i(t) x_i(t+\tau) \rangle - \frac{1}{N} \sum_i \langle x_i(t) \eta_i(t+\tau) \rangle \\ &\quad + \frac{1}{N} \sum_i \langle x_i(t) x_i(t+\tau) \rangle - \frac{1}{N} \sum_i \langle \eta_i(t) \eta_i(t+\tau) \rangle \\ &= \frac{1}{N} \sum_i \langle x_i(t) x_i(t+\tau) \rangle - \frac{1}{N} \sum_i \langle \eta_i(t) \eta_i(t+\tau) \rangle \\ &= \Delta(\tau) - \left[\frac{1}{N} \sum_i \langle \eta_i(t+\tau) \eta_i(t) \rangle - \mu^2 \right] . \end{aligned} \tag{11}$$

So now all we need to do is compute the term in brackets. Using Eq. (1a), we have

$$\frac{1}{N}\sum_i \langle \eta_i(t)\eta_i(t+\tau)\rangle = \frac{1}{N}\sum_{ijk} J_{ij}J_{ik}\langle \phi(x_j(t))\phi(x_k(t+\tau))\rangle\,. \tag{12}$$

Rearranging terms gives us

$$\frac{1}{N}\sum_i \langle \eta_i(t)\eta_i(t+\tau)\rangle = \sum_{jk}\langle \phi(x_j(t))\phi(x_k(t+\tau))\rangle \frac{1}{N}\sum_i J_{ij}J_{ik}. \tag{13}$$

As above, we take the large N limit, which means we can compute the last term by averaging over the distribution of the weight matrix. Let us define (for lack of better notation)

$$\frac{1}{N}\sum_i J_{ij}J_{ik} \equiv \begin{cases} \overline{J_1J_2} & j\neq k \\ \overline{J^2} & j=k. \end{cases} \tag{14}$$

The second term, $\overline{J^2}$, is the second moment of the weight matrix. The first term, $\overline{J_1J_2}$, takes into account correlations in the elements of the weight matrix (if they exist). In the absence of correlations, $\overline{J_1J_2}$ is simply $\bar{J}^2$. Inserting this definition into Eq. (11), and using Eqs. (6) and (19), we have

$$\begin{aligned}\Delta(\tau) - \frac{d^2\Delta(\tau)}{d\tau^2} &= \sum_{j\neq k}\left[\overline{J_1J_2}\langle \phi(x_j(t))\phi(x_k(t+\tau))\rangle - \bar{J}^2\langle\phi(x_j(t))\rangle\langle\phi(x_k(t))\rangle\right] \\ &\quad + \sum_j \left[\overline{J^2}\langle \phi(x_j(t))\phi(x_j(t+\tau))\rangle - \bar{J}^2\langle\phi(x_j(t))\rangle^2\right]\,. \end{aligned} \tag{15}$$

We'll assume x_j and x_k are uncorrelated when $j\neq k$, something that needs to be shown. We don't do that here; see Francesca and Srdjan's paper. But with this assumption, the above expression simplifies,

$$\begin{aligned}\Delta(\tau) - \frac{d^2\Delta(\tau)}{d\tau^2} &= \mathrm{Cov}[J]\sum_{j\neq k}\langle\phi(x_j(t))\rangle\langle\phi(x_k(t))\rangle \\ &\quad + \sum_j \left[\overline{J^2}\langle \phi(x_j(t))\phi(x_j(t+\tau))\rangle - \bar{J}^2\langle\phi(x_j(t))\rangle^2\right] \end{aligned} \tag{16}$$

where

$$\mathrm{Cov}[J] \equiv \overline{J_1J_2} - \bar{J}^2. \tag{17}$$

It is convenient to make another simplification to Eq. (16). Including the $j = k$ term in the first sum and then subtracting it from the second gives us

$$\Delta(\tau) - \frac{d^2\Delta(\tau)}{d\tau^2} = \mathrm{Cov}[J] \sum_{jk} \langle \phi(x_j(t)) \rangle \langle \phi(x_k(t)) \rangle + \sum_j \left[\overline{J^2} \langle \phi(x_j(t)) \phi(x_j(t+\tau)) \rangle - \overline{J_1 J_2} \langle \phi(x_j(t)) \rangle^2 \right] \tag{18}$$

To write this in a more compact form, we make the definition

$$C(\tau) \equiv \frac{1}{N} \sum_j \langle \phi(x_j(t)) \phi(x_j(t+\tau)) \rangle \,. \tag{19}$$

Inserting this definition into Eq. (18), using Eq. (6), and noting that $x_j(t)$ and $x_j(t+\tau)$ are temporally uncorrelated in the limit $\tau \to \infty$, we arrive at

$$\frac{d^2\Delta(\tau)}{d\tau^2} = \Delta(\tau) - \left[\mu^2 \mathrm{Cov}[J]/\bar{J}^2 + N\overline{J^2} C(\tau) - N\overline{J_1 J_2} C(\infty) \right] . \tag{20}$$

To solve this equation, we need to compute compute averages over weight matrices. We'll consider two cases, stochastic and fixed in-degree. We'll consider stochastic in-degree first, as it is easier. In this case,

$$J_{ij} = \begin{cases} J_0 & \text{prob } C/N \\ 0 & \text{prob } 1 - C/N, \end{cases} \tag{21}$$

where K is the number of connections per neuron. We'll assume K is fixed and let N go to infinity.

Because the weights are independent, $\overline{J_1 J_2} = \bar{J}^2$. Consequently, $\mathrm{Cov}[J]$ vanishes. We thus need only compute the first and second moments of J; these are given by

$$\bar{J} = \frac{K}{N} J_0 \tag{22a}$$

$$\overline{J^2} = \frac{K}{N} J_0^2 \,. \tag{22b}$$

Thus,

$$\frac{d^2\Delta(\tau)}{d\tau^2} = \Delta(\tau) - K J_0^2 C(\tau) + \frac{K^2}{N} J_0^2 C(\infty) \to \Delta(\tau) - K J_0^2 C(\tau) \tag{23}$$

where the "$\to$" is valid in the limit $N \to \infty$ with K fixed.

For fixed in-degree, for each i exactly K of the J_{ij} are nonzero, and all nonzero elements are equal to J_0. This doesn't change the first and second moments of J; they're still given by Eq. (22). However, there are correlations among the J_{ij}, so $\overline{J_1 J_2}$ is no longer equal to $\bar{J}^2$. Instead, we have

$$\overline{J_1 J_2} = \frac{1}{N} \sum_i J_{ij} J_{ik} = J_0^2 \frac{K}{N} \frac{K-1}{N} \tag{24}$$

where in the above expression $j \neq k$. Consequently,

$$\text{Cov}[J] = -\frac{K}{N^2} J_0^2, \tag{25}$$

reflecting the fact that the J_{ij} are very weakly anti-correlated. Inserting this into Eq. (20), using Eq. (22) for the first two moments of J_{ij}, and again taking the large N limit with K fixed, we arrive at

$$\frac{d^2 \Delta(\tau)}{d\tau^2} = \Delta(\tau) - K J_0^2 \left[C(\tau) - \mu^2 \right]. \tag{26}$$

To solve either of our two equations – Eq. (23) or (26) – we need to relate $C(\tau)$ to $\Delta(\tau)$. That's relatively straightforward: from Eq. (19) and the definition of $\Delta(\tau)$, Eq. (9), we have

$$C(\tau) = \big\langle \phi(\mu + \xi_1) \phi(\mu + \xi_2) \big\rangle_{\xi_1, \xi_2} \tag{27}$$

where ξ_1 and ξ_2 are zero mean Gaussian random variables with with $\langle \xi_1^2 \rangle = \langle \xi_2 \rangle = \Delta(0)$ and $\langle \xi_1 \xi_2 \rangle = \Delta(\tau)$. Alternatively, if $\Delta(\tau) \geq 0$, this can we written

$$C(\tau) = \big\langle \phi\big(\mu + \sqrt{\Delta(0) - \Delta(\tau)}\xi_1 + \sqrt{\Delta(0)}\xi\big) \phi\big(\mu + \sqrt{\Delta(0) - \Delta(\tau)}\xi_2 + \sqrt{\Delta(0)}\xi\big) \big\rangle_{\xi, \xi_1, \xi_2} \tag{28}$$

where now ξ, ξ_1 and ξ_2 are independent, zero mean, unit variance Gaussian random variables.

So now Eqs. (23) and (26) can be solved, at least in principle. However, that's not trivial. For details, see Francesca and Srdjan's paper.