Gatsby Computational Neuroscience Unit
Theoretical Neuroscience

Final Examination
22 Jan 2014

Part II – long questions

There are four questions, one from each main section of the course. Please answer three out of the four, starting the answers for each new question on a new page. Don't forget to write your name at the top of the answer to each question.

You are allowed to use your own handwritten notes, but no other online or offline reference sources.

You have a maximum of 7 hours for this exam.

Good luck!

Note: red text was added after the exam, to correct typos.

1 Biophysics

Morris-Lecar neuron

Spikes are generally caused by rapid influx of sodium. But neurons (and, more often, dendrites) also exhibit calcium spikes. The equations describing these spikes have the form

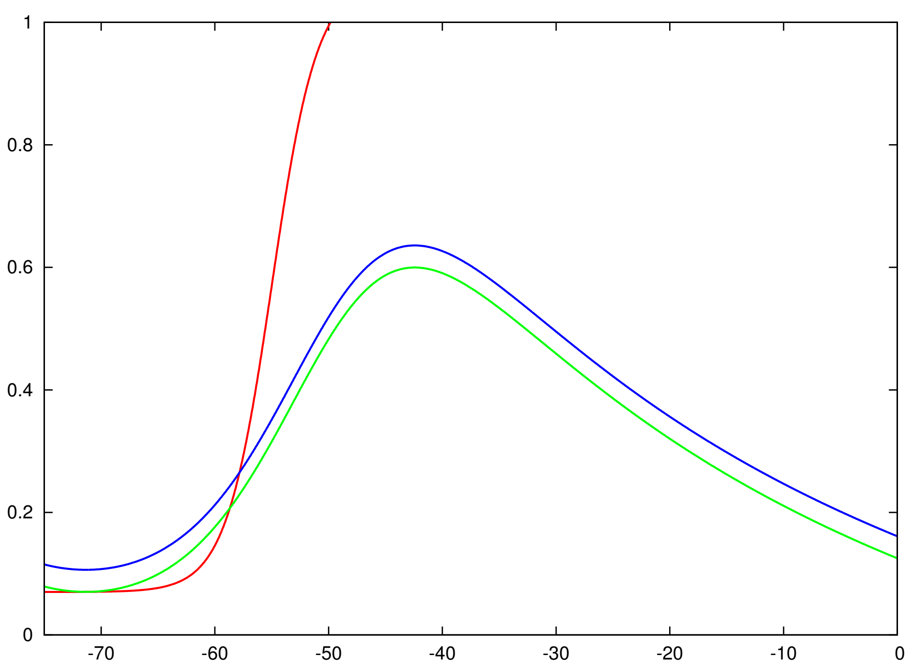
$$\tau_V \frac{dV}{dt} = -(V - \mathcal{E}_L) - g_{Ca} m_\infty(V)(V - \mathcal{E}_{Ca}) - g_K w (V - \mathcal{E}_K) + I$$
$$\tau_w \frac{dw}{dt} = w_\infty(V) - w\,.$$

where

$$\begin{aligned} \tau_V &= 10 \text{ ms} \\ \mathcal{E}_{Ca} &= 30 \text{ mV} \\ \mathcal{E}_K &= -90 \text{ mV} \\ \mathcal{E}_L &= -70 \text{ mV} \\ g_{Ca} &= 20 \\ g_K &= 40\,. \end{aligned}$$

The m-channel activation curve, $m_\infty(V)$, is monotonic increasing. The two remaining parameters, τ_w and I will vary.

The nullclines for these equations are shown below. The w-nullcline, which doesn't completely inactivate, is shown in red; V-nullclines are shown for two values of I (blue and green).



1. Sketch the approximate shape of $m_\infty(V)$ versus V that will give rise to the V-nullclines shown in the figure. Include actual voltages on the V-axis.

   (10 marks).

2. Show that where the blue and red nullclines intersect, the determinant of the linearized dynamics is positive. Recall that the determinant is computed as follows: if we write the equations as

$$\frac{dV}{dt} = f(V, w)$$
$$\frac{dw}{dt} = g(V, w)\,,$$

   then the determinant of the linearized dynamics, D, is

$$D = \partial_V f(V, w)\partial_w g(V, w) - \partial_w f(V, w)\partial_V g(V, w)\,,$$

   with the derivatives evaluated at the values of V and w for which $f(V, w) = g(V, w) = 0$.

   (5 marks)

3. Consider the fixed point where the blue and red nullclines intersect. Show that when $\tau_w \ll \tau_V$, the fixed point is stable; when $\tau_w \gg \tau_V$, the fixed point is unstable; and somewhere in between it goes unstable via a Hopf bifurcation (meaning a complex conjugate pair of eigenvalues develops a positive real part). Show that the frequency of the resulting oscillations at the point where the fixed point just becomes unstable is the square root of the determinant of the linearized dynamics. Note: you will need the fact that the determinant is positive.

   (10 marks)

4. Show that as the input current, I, decreases, the V-nullcline drops (e.g., it goes from the blue to green nullcline).

   (5 marks)

5. Consider the limit that $\tau_w \gg \tau_V$. Assume for simplicity that as the input current decreases, the V-nullcline drops uniformly. Sketch the firing rate as a function of (decreasing I), up to the point where the w and V-nullclines first develop a second intersection (like the green nullcline in the figure). Sketch the trajectory right where that first intersection occurs. Start the sketch of the trajectory just below the intersection.

   (10 marks)

2 Networks

Correlated connectivity matrix.

Consider a static rate model of the form

$$\begin{aligned} \nu_i &= \phi(h_i) \\ h_i &= \sum_{j=1}^{N} W_{ij}\nu_j \end{aligned} \tag{1}$$

where ν_i is the firing rate of neuron i, h_i is the "synaptic drive," W_{ij} is the weight matrix, and ϕ is the gain function, which, as usual, is more or less sigmoidal (its precise shape won't be important).

We'll consider sparse connectivity, for which it is convenient to define

$$W_{ij} \equiv J_{ij}C_{ij}$$

where the J_{ij} are drawn *iid* (independent and identically distributed) from some distribution $P(J)$, and C_{ij} is either 0 or 1. Importantly, the C_{ij} are not necessarily independent. We will, however, assume sparse connectivity,

$$p(C_{ij} = 1) = \frac{K}{N} \sim 0.1\,.$$

The distribution of C_{ij} will be specified below. Until we specify it, we assume only that it is in some sense statistically homogeneous. What that means (loosely) is that averages over the distribution of C don't have much dependence on index; for instance, $\langle C_{1,2}\, C_{1,3}\rangle = \langle C_{2,7}\, C_{2,9}\rangle$.

As usual, we interpret K as the average number of connections per neuron.

For all questions, we'll consider the large N limit, and make the approximation that W_{ij} and ν_j are independent.

1. Why is the approximation that W_{ij} and ν_j are independent reasonable?

   (5 marks)

2. Show that h_i may be written

   $$h_i = K\langle J\rangle\overline{\nu} + \sum_j \delta W_{ij}\nu_j$$

   where $\delta W_{ij} = W_{ij} - (K/N)\langle J\rangle$, $\langle J\rangle$ is the mean value of J with respect to the distribution $P(J)$, and

   $$\overline{\nu} \equiv \frac{1}{N}\sum_j \nu_j\,.$$

   (3 marks)

3. We are interested in the variance of h, defined to be

$$\text{Var}[h] = \frac{1}{N}\sum_i \left\langle \Big(\sum_j \delta W_{ij}\nu_j\Big)^2 \right\rangle .$$

Here the angle brackets indicate an average over the distribution of W_{ij}. Show that

$$\text{Var}[h] = N\big(\text{Var}[W] - \langle \delta W_{ij}\delta W_{ij'}\rangle\big)\overline{\nu^2} + N^2\langle \delta W_{ij}\delta W_{ij'}\rangle\overline{\nu}^2$$

where $j \neq j'$, $\text{Var}[W] = \langle W_{ij}W_{ij}\rangle - \langle W_{ij}\rangle^2$, and

$$\overline{\nu^2} \equiv \frac{1}{N}\sum_j \nu_j^2 .$$

Note that because the angle brackets represent an average over the probability distribution of W, the covariance, $\langle \delta W_{ij}\delta W_{ij'}\rangle$, is independent of index (so long as $j \neq j'$).

(7 marks)

4. Show that

$$\begin{aligned}\text{Var}[W] &= \frac{K}{N}\big[\text{Var}[J] + (1 - K/N)\langle J\rangle^2\big]\\ \langle \delta W_{ij}\delta W_{ij'}\rangle &= \langle J\rangle^2\,\langle \delta C_{ij}\delta C_{ij'}\rangle\end{aligned}$$

where $\langle \delta C_{ij}\delta C_{ij'}\rangle \equiv \langle C_{ij}C_{ij'}\rangle - \langle C_{ij}\rangle^2$.

(5 marks)

5. Assume that each neuron receives *exactly* K connections, meaning

$$\sum_j C_{ij} = K \tag{2}$$

(recall that C_{ij} is either 0 or 1). When $i \neq k$ and $j \neq l$, C_{ij} and C_{kl} are uncorrelated. Show that

$$\langle \delta C_{ij}\delta C_{ij'}\rangle = -\frac{K(N-K)}{N^2(N-1)} .$$

(8 marks)

6. What is the effect of this small covariance on the distribution of the firing rates? That is, does it make it broader or more narrow?

(2 marks)

7. Now we'll consider correlations between the h_i on different neurons. Show that if $i \neq i'$,

$$\text{Covar}[h_i, h_{i'}] = N\overline{\nu^2}\langle \delta W_{ij}\delta W_{i'j}\rangle .$$

(3 marks)

8. Show that when each neurons receives exactly K connections (as in Eq. (**??**) above), h_i and $h_{i'}$ are uncorrelated when $i \neq i'$. But when each neuron *makes* exactly K connections,

$$\sum_i C_{ij} = K \,,$$

(where, again, when $i \neq k$ and $j \neq l$, C_{ij} and C_{kl} are uncorrelated), show that if $i \neq i'$,

$$\mathrm{Covar}[h_i, h_{i'}] = -\overline{\nu^2} \, \langle J \rangle^2 \frac{K(N-K)}{N(N-1)} \,.$$

(7 marks)

3 Coding

A one-dimensional growth cone is growing in a concentration gradient $c(x) = c_0(1 + mx)$, where c_0 is the strength of the gradient and m is its (very small) rate of change. It has n receptors, with receptor i, at location x_i on the cone, having probability of being bound of

$$p_i = \frac{c(x_i)/K_d}{1 + c(x_i)/K_d}$$

where K_d is the dissociation constant. Assume that different receptors bind independently, and they are evenly distributed about 0, so $\sum_i x_i = 0$. The cone knows c_0 and K_d, but not m.

1. Write down the likelihood for the sets $\mathcal{B}$ and $\mathcal{U}$ of bound and unbound receptors as a function of m. (5 points)
2. Expand the likelihood to second order in m about $m = 0$. (8 points)
3. Assuming a flat prior over m; what is the mean and variance of the Laplace approximation to the posterior over m? (8 points)
4. What is the Fisher information in $\mathcal{B};\mathcal{U}$ about m? (3 points)
5. If the cone has to decide whether $m \overset{<}{>} 0$, what test should it apply under this approximation? (4 points)
6. Write down an expression for the probability that it will make the correct decision? (4 points)
7. If the cone's extent is $x = \pm 1$, where should it place its receptors to maximize this probability. Why? (4 points)
8. It is possible that the cone only knows the location of receptors when they are bound. Will this affect its ability to discriminate the sign of m? Justify your answer. (4 points)

4 Learning

A rat hops at a rate of ν hops/second sequentially from position $p = 0$ to $p = 1$ via $p = 1/n, 2/n, \ldots$, and gets a reward of $r = 1$ at $p = 1$. At the end of the episode, it is then teleported back to $p = 0$ and resumes hopping. The discount factor is γ.

1. define a table look-up representation of spatial location. If the rat has such a representation, what is the optimum value function $V(p)$ for all values of $p \in \{0, 1/n, \ldots, 1\}$? (8 points)
2. Sam Gershman has suggested that it might instead have a uni-dimensional representation $x(p)$, so $V(p) = x(p)w$ for a weight w. Write down the temporal difference (TD) learning rule for w in going from $p = i/n$ to $p = (i+1)/n$. (2 points)
3. if the weight changes are accumulated through the whole episode, and are only applied during the teleportation at the end, write down the summed change to the weights implied by TD. (5 points)
4. write down an expression for the asymptotic value of the weight w, assuming convergence. (8 points)
5. if $x(p) = e^{-(1-p)}$, calculate the asymptotic weight w^*. (8 points)
6. using this weight, what will the form of the TD prediction error be across space. (6 points)
7. is this uni-dimensional representation credible for a rat? (3 points)