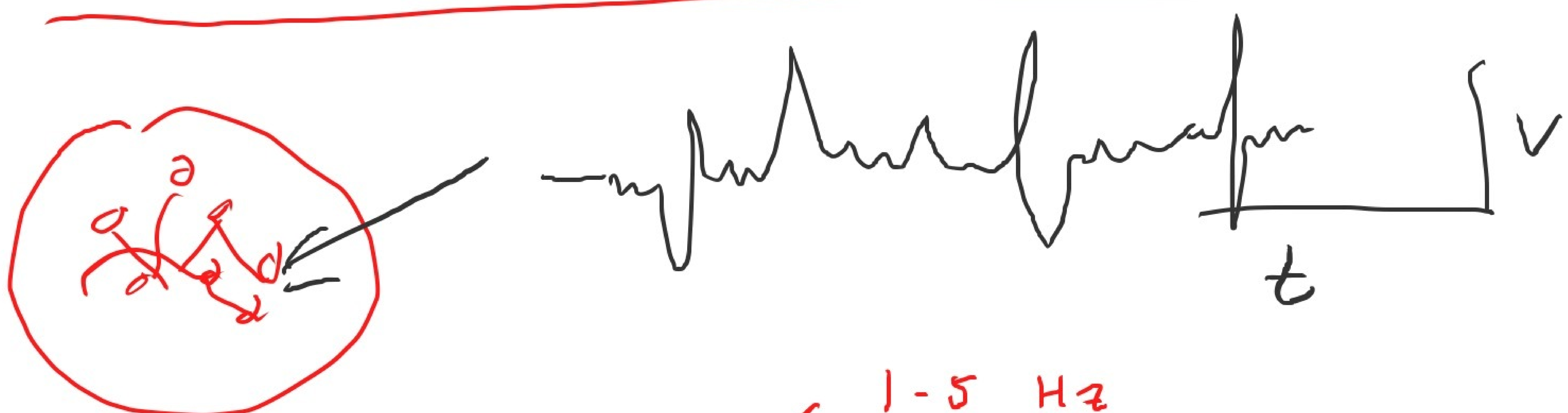
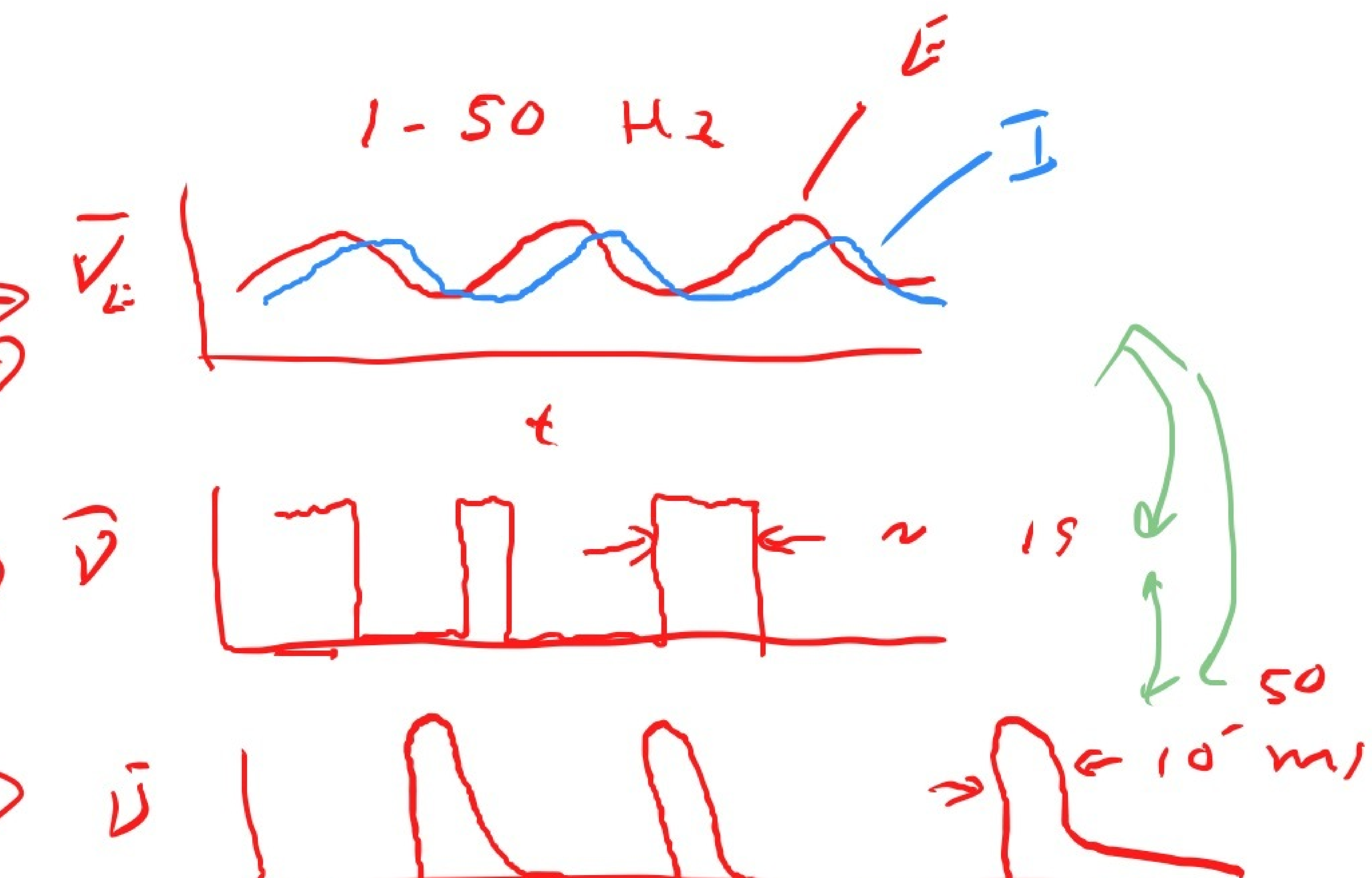
Networks of recurrently connected neurons

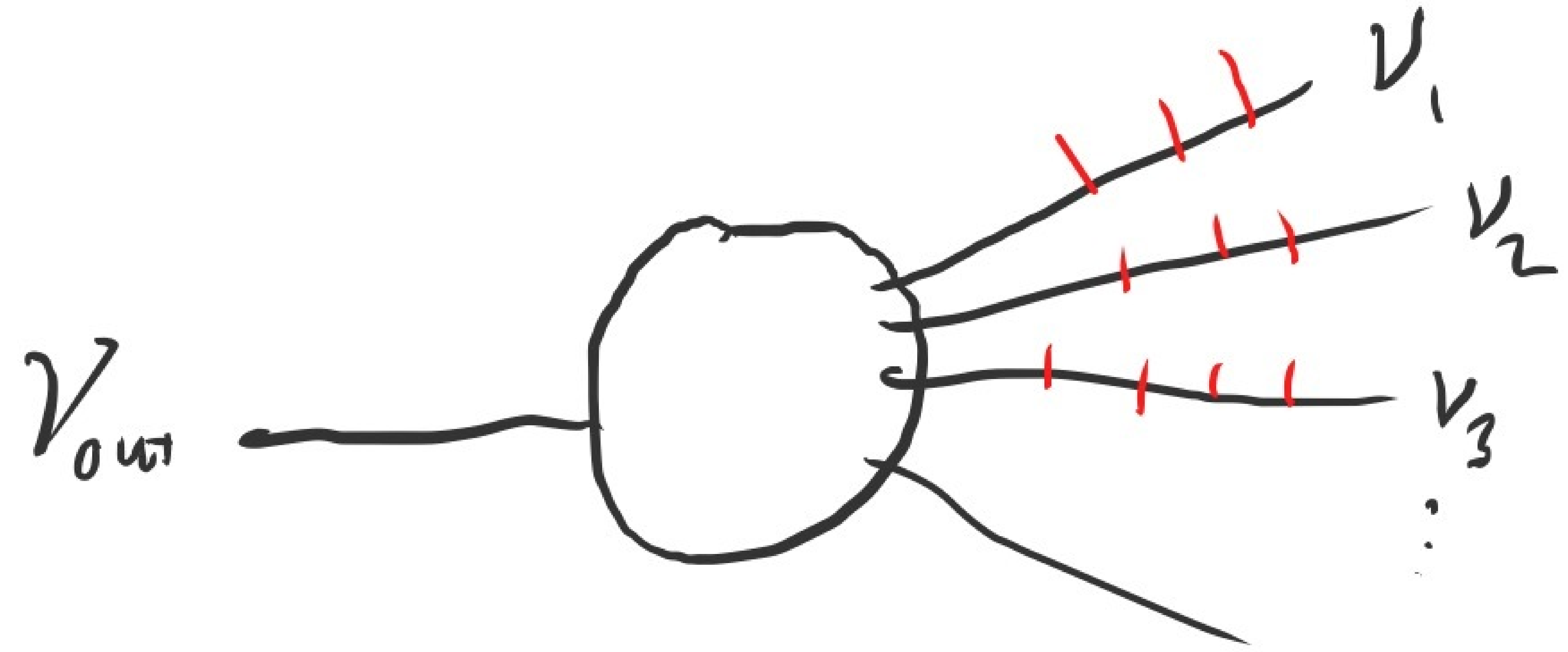


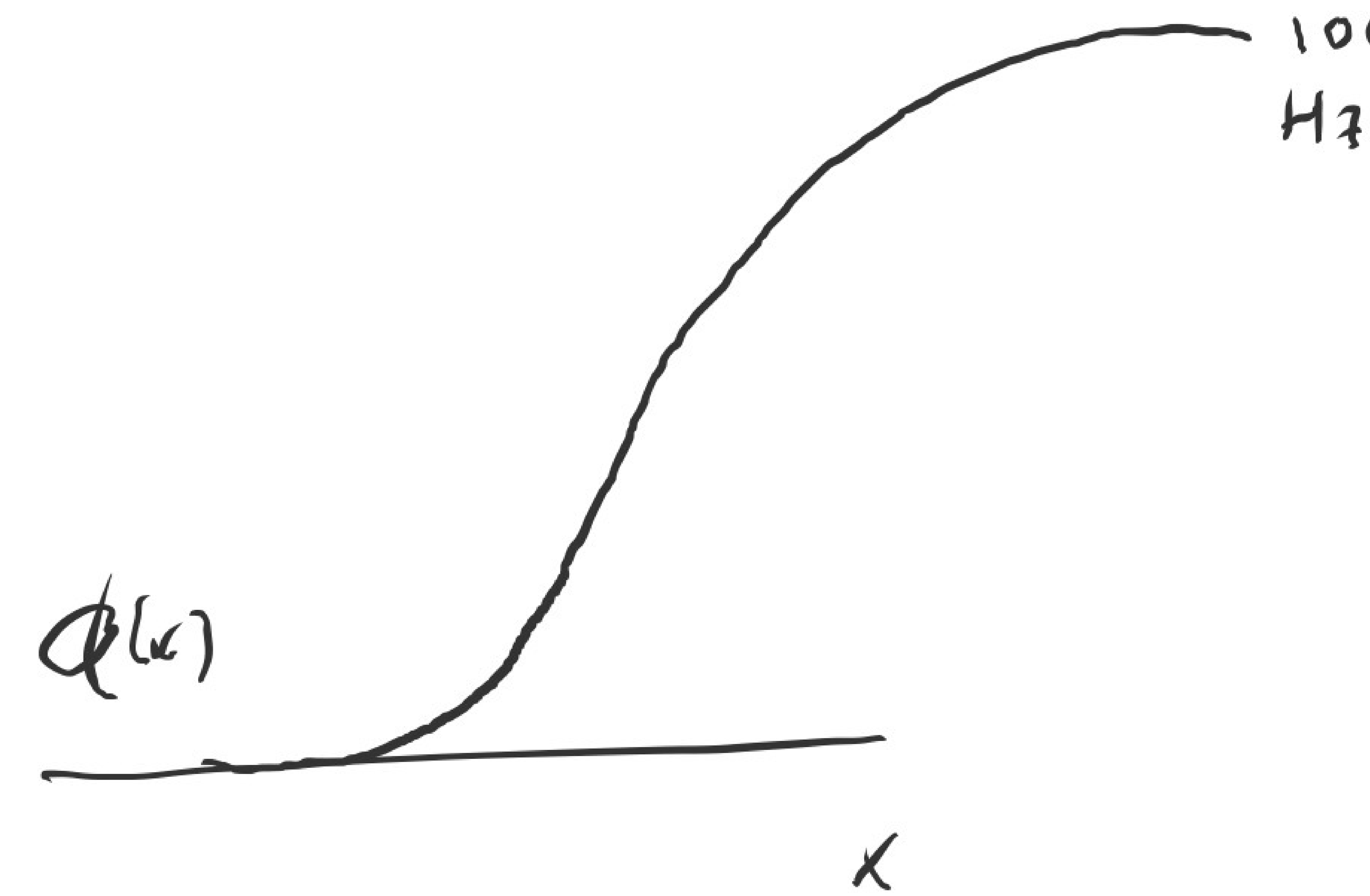
HH:

1-5 Hz   100 Hz

1. low firing rates compared to max.
2. broad distribution of firing rates – log normal
3. irregular activity
4. Oscillations
5. Excitatory leads Inhibitory
6. Up-down states
7. Bumps
8. State switching







Insanely simple model:

$$\nu_i = \phi\left(\sum_j W_{ij}\,\nu_j\right)$$

neurons are either excitatory or inhibitory

$$\nu_{Ei} = \phi_E\left(\frac{1}{\sqrt{K}}\sum_{j=1}^{N_E} W_{EE,ij}\,\nu_{Ej} - \frac{1}{\sqrt{K}}\sum_j W_{EI,ij}\,\nu_{Ij}\right)$$

$$\nu_{Ii} = \phi_I\left(\frac{1}{\sqrt{K}}\sum_j W_{IE,ij}\,\nu_{Ej} - \frac{1}{\sqrt{K}}\sum_j W_{II,ij}\,\nu_{Ij}\right)$$

$P(W_{EE,ij} \neq 0) = \frac{K}{N_E}$ (sparse connectivity)

K = average number of connections / neuron

$$W_{ij} = W_{ij}^{\text{random}} + W_{ij}^{\text{structured from learning}}$$

consider only random components

$$W_{QR,ij} \sim P_{QR}(w) \quad \text{iid}$$

E, I

$$\langle W_{QR,ij} \rangle = W_{QR}$$

Solve $\longrightarrow$ want to compute the distribution over firing rates.

$$\nu_i = \phi\Big(\underbrace{\frac{1}{\sqrt{N}}\sum_{j=1}^{N} W_{ij}\,\nu_j}_{\text{random variable with respect to index } i.}\Big)$$

$W_{ij} \sim p(w)$ iid $\qquad$ $p(w)$ is independent of N

goal: compute $p(\nu)$ $\qquad$ large N limit

$$\xi_i = \underbrace{\frac{1}{\sqrt{N}}\sum_j W_{ij}\,\nu_j}_{\text{Gaussian random variable}}$$

$$\langle \xi \rangle = \frac{1}{i}\sum_{i=1}^{n} \frac{1}{\sqrt{N}}\sum_j W_{ij}\,\nu_j$$

$$= \frac{1}{\sqrt{n}}\sum_j \nu_j \; \frac{1}{n}\sum_{j=1}^{n} W_{ij}$$

$$P(\xi) \sim \mathcal{N}\Big(\underset{\langle \nu \rangle}{\mu(\nu)},\ \underbrace{\overset{\langle \nu^2 \rangle}{\sigma^2(\nu)}}_{\text{moments of } \nu}\Big)$$

$$\nu_i = \phi(\xi_i)$$

$$\langle \nu \rangle = \frac{1}{N}\sum_i \phi(\xi_i)$$

$$\to \int d\xi\; p(\xi)\,\phi(\xi)$$

depends on $\langle \nu \rangle, \langle \nu^2 \rangle$

$$\langle \nu^2 \rangle = \int d\xi\; p(\xi)\,\phi^2(\xi)$$

$$W_{ij} = \bar{w} + \delta W_{ij}$$

$$\xi_i = \frac{1}{\sqrt{N}} \sum_j \left(\bar{w} + \delta W_{ij} \right) \nu_j$$

$$= \frac{1}{\sqrt{N}} \bar{w} \sum_{j=1}^{n} \nu_j + \frac{1}{\sqrt{N}} \sum_j \delta W_{ij} \nu_j$$

$$= \sqrt{n}\, \bar{w}\, \bar{\nu} + \underbrace{\frac{1}{\sqrt{N}} \sum_j \delta W_{ij} \nu_j}_{\delta \xi_i}$$

$$\langle \xi \rangle = \sqrt{n}\, \bar{w}\, \bar{\nu} + \frac{1}{\sqrt{n}} \frac{1}{n} \underbrace{\sum_i \sum_j \delta W_{ij} \nu_j}_{\sqrt{n^2} \; (\;)}$$

$$\frac{1}{\sqrt{n}} \frac{1}{n} n \sim \frac{1}{\sqrt{n}}$$

$$\mathrm{Var}(\xi) = \frac{1}{n} \sum_i \left(\xi_i - \langle \xi \rangle \right)^2$$

$$= \frac{1}{n} \sum_i \left(\frac{1}{\sqrt{n}} \sum_j \delta W_{ij} \right)^2$$

$$= \frac{1}{n} \sum_i \frac{1}{n} \sum_{jj'} \delta W_{ij} \delta W_{ij'} \nu_j \nu_{j'}$$

$$= \frac{1}{n} \sum_{jj'} \nu_j \nu_{j'} \frac{1}{n} \sum_i \delta W_{ij} \delta W_{ij'}$$

$$X = \sum_{e=1}^{m} z_e \qquad \frac{1}{m}$$

$$z_e \sim \underline{P(z)} \quad \text{iid}$$

$$\langle x \rangle = 0$$

$$\mathrm{Var}(x) = \left\langle \left(\sum_{e=1}^{m} z_e \right)^2 \right\rangle$$

$$= \sum_{e=1}^{m} \sum_{e'=1}^{m} \langle z_e z_{e'} \rangle$$

$$\llcorner 0 \quad e \neq e'$$

$$\mu, \sigma^2 \qquad = \sum_{e=1}^{m} \langle z_e^2 \rangle$$

$$= m\, \sigma_z^2 \qquad \nu W_{ij}$$

$$X = \frac{\sqrt{m}\, \sigma_z}{m}$$

$$\mathrm{Var}(\xi) = \frac{1}{n}\sum_{jj'} \gamma_j \gamma_{j'} \underbrace{\frac{1}{n}\sum_i dW_{ij}\, dW_{ij'}}$$

zero on average

$$\left(\frac{1}{n}\sum_i dW_{ij}^2 \;\; \delta_{jj'} + \frac{1}{n}\sum_i dW_{ij}\, \cancel{dW_{ij'}} \, (1-\delta_{jj'}) \right)$$

$$\delta_{jj'} = \begin{cases} 1 & j = j' \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma_w^2 + \delta(dW_{ij}^2)$$

$$= \frac{1}{n}\sum_j \gamma_j^2 \underbrace{\frac{1}{n}\sum_i dW_{ij}^2}_{\mathrm{Var}[w]} + (\qquad)$$

$$= \sigma_w^2 \langle \gamma^2 \rangle$$

$$\boxed{\begin{aligned} &\frac{1}{m}\sum_{e=1}^{m} (\bar{z} + z_e) \\ &= \bar{z} + \underbrace{\frac{1}{m} \underbrace{\sum_{e=1}^{m} z_e}_{O(\sqrt{m})}}_{\frac{1}{\sqrt{m}} \mathrm{Var}(z)} \\ &\cong \bar{z} \end{aligned}}$$

$$\langle \xi \rangle = \sqrt{n}\ \bar{w}\ \langle v \rangle$$

$$\mathrm{var}[\xi] = \sigma_w^2 \langle v^2 \rangle$$

$$v_i = \phi\Big(\underbrace{\frac{1}{\sqrt{n}} \sum_j w_{ij} v_j}_{\xi_i}\Big)$$

$$v_i = \phi(\xi_i)$$

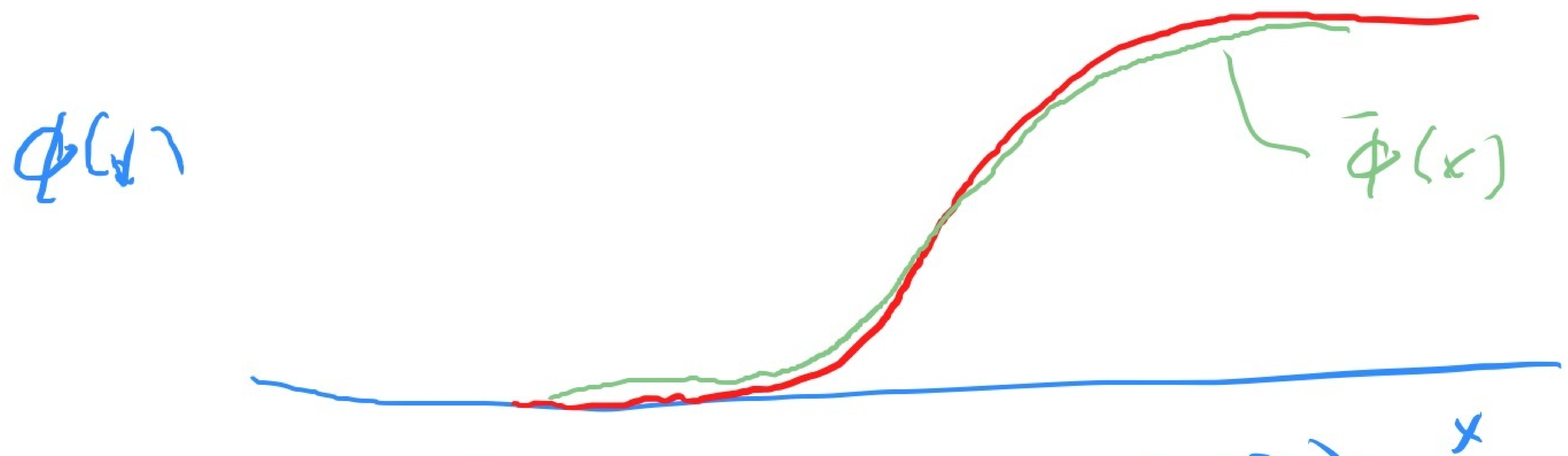
$$\langle v \rangle = \frac{1}{n} \sum_i \phi(\xi_i) \longrightarrow \int d\xi\, \rho(\xi)\, \phi(\xi)$$

$$\langle v^2 \rangle = \frac{1}{n} \sum_i \phi^2(\xi_i) \longrightarrow \int d\xi\, \rho(\xi)\, \phi^2(\xi)$$

$$\xi \sim N(\sqrt{n}\,\bar{w}\langle v \rangle,\ \sigma_w^2 \langle v^2 \rangle)$$

$$\boxed{\langle v^2 \rangle = \int dz\, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, \phi^2\Big(\underbrace{\sqrt{n}\,\bar{w}\langle v \rangle}_{\text{large}} + \underbrace{\sigma_w \sqrt{\langle v^2 \rangle}\, z}_{\text{small}}\Big)}$$

$$\bar\phi^{l} = \int dz \, \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, \phi\left(\sqrt{n}\,\bar w \langle v\rangle + \sigma_w \sqrt{\langle v^2\rangle}\, z\right)$$

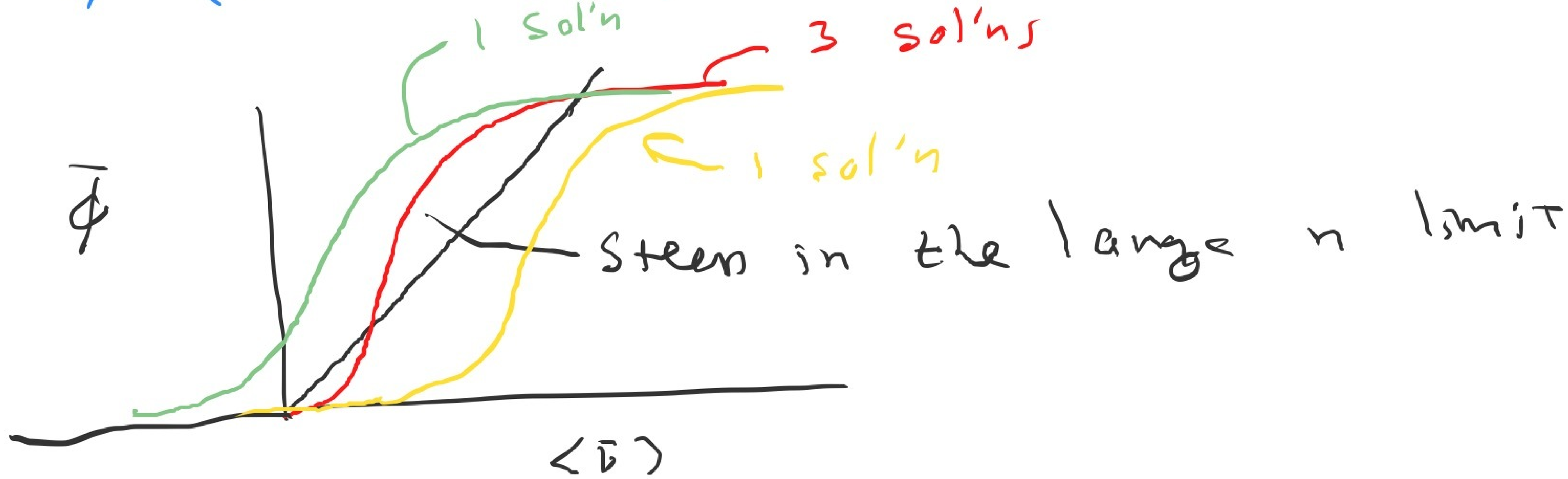


solve for $\langle v^2\rangle$ in terms of $\langle v\rangle$

$$\langle v^2\rangle = \bar\phi^2\left(\sqrt{n}\,\bar w \langle v\rangle, \langle v^2\rangle\right)$$

weak

$$\langle v\rangle = \bar\phi\left(\sqrt{n}\,\bar w \langle v\rangle, \langle v^2\rangle(\langle v\rangle)\right)$$



make up dynamics

ν_i

big

$\nu_i \uparrow$

$$\tau \frac{d\nu_i}{dt} = \phi\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} W_{ij}\nu_j\right) - \nu_i$$

(made up)

$\langle \delta\langle \nu\rangle\rangle \sim \frac{1}{\sqrt{n}}$

$\delta\langle \nu^2\rangle \sim O(1)$

evolves slow

$$\tau \frac{d\sqrt{n}\langle r\rangle}{dt} = \sqrt{n}\left[\bar{\Phi}\left(\sqrt{n}\,\bar{w}\langle r\rangle, \langle r^2\rangle\right) - \langle r\rangle\right]$$

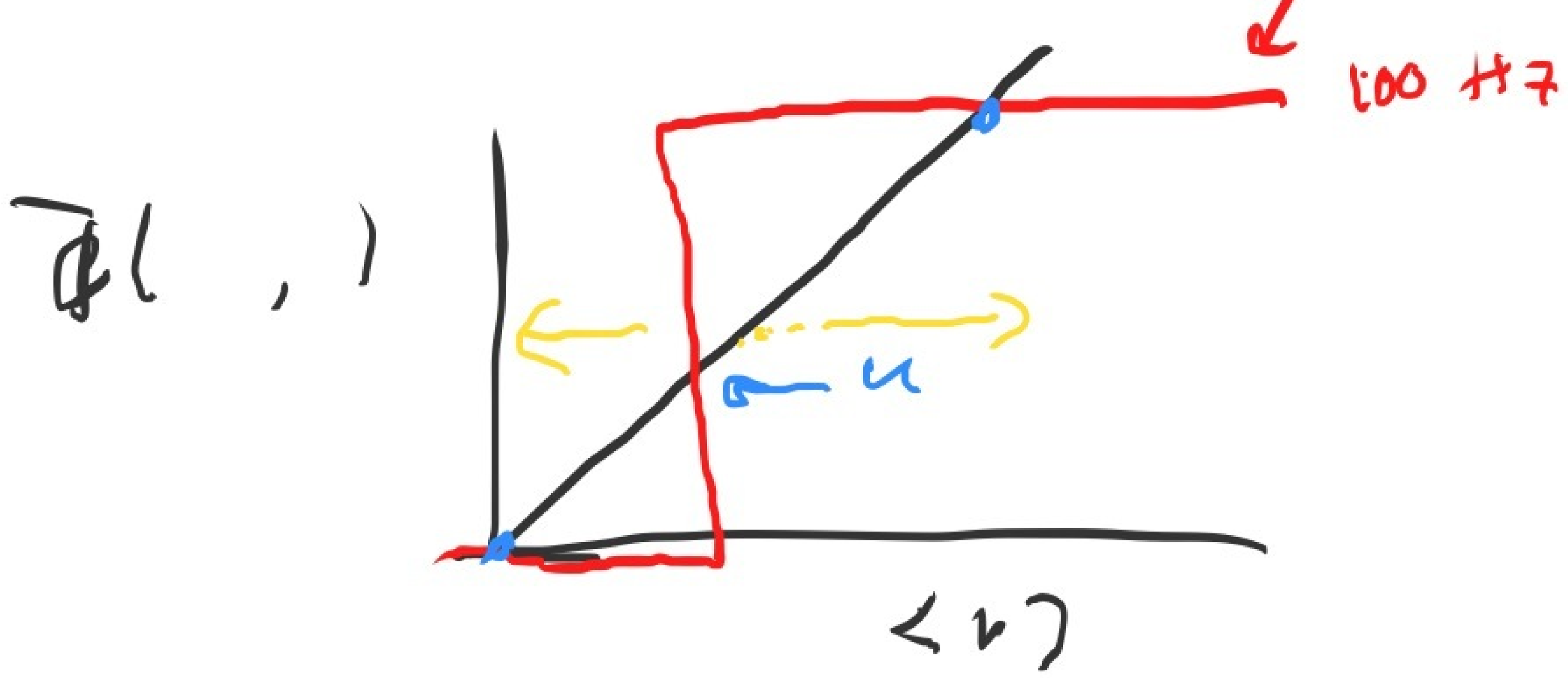
$$\tau \frac{d\langle r^2\rangle}{dt} = \bar{\Phi}^2\left(\sqrt{n}\,\bar{w}\langle r\rangle, \langle r^2\rangle\right) - \langle r^2\rangle$$

fix at some value

care about:

$$\tau \frac{d\langle r\rangle}{dt} = \bar{\Phi}\left(\sqrt{n}\,\bar{w}\langle r\rangle, \langle r^2\rangle\right) - \langle r\rangle$$

fix $\langle r^2\rangle$



large n, all E network;

$\langle r\rangle$ is either 0 or r_{max}

$$\nu_{Qi} = \phi_Q\left(\frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n} W_{QE,ij}\,\nu_{Ej} - \sum_{j=1}^{n} W_{QI,ij}\,\nu_{Ij}\right)\right)$$

$Q = E, I$

$$\xi_{QR,i} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} W_{QR,ij}\,\nu_{Rj} \sim N\left(\sqrt{n}\,\bar{W}_{QR}\langle \nu_R\rangle,\ \sigma_{QR}^2\langle \nu_R^2\rangle\right)$$

$$\boxed{\nu_{Qi} = \phi_Q\left(\sqrt{n}\left(W_{QE}\langle\nu_E\rangle - W_{QI}\langle\nu_I\rangle\right) + z_i\sqrt{\sigma_{QE}^2\langle\nu_E^2\rangle + \sigma_{QI}^2\langle\nu_I^2\rangle}\right)}$$

ϕ_E or ϕ_I $\qquad\qquad$ $N(0,1)$

$$\langle \nu_E^{\ell} \rangle = \int dz \; \phi_E^{\ell}\left(\sqrt{n}\left(\bar{W}_{EE} \langle \nu_E \rangle - \bar{W}_{EI} \langle \nu_I \rangle \right) + z\,\sigma_E \right) \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

$$\langle \nu_I^{\ell} \rangle = \int dz \qquad I \qquad I \qquad I \qquad I$$

4 eqns,
4 unknowns

$$\sigma_E^2 = \sigma_{EE}^2 \langle \nu_E^2 \rangle + \sigma_{EI}^2 \langle \nu_I^2 \rangle$$

$$\uparrow \; \mathrm{var}(W_{EE,ij})$$

$$\sigma_I^2 \qquad I \qquad I$$

fix, solve in terms of $\langle \nu_E \rangle, \langle \nu_I \rangle$

$$\langle \nu_E \rangle = \bar{\phi}_E\left(\sqrt{n}\left(\bar{W}_{EE} \langle \nu_E \rangle - \bar{W}_{EI} \langle \nu_I \rangle \right), \underline{\langle \nu_E^2 \rangle, \langle \nu_I^2 \rangle} \right)$$

$$\langle \nu_I \rangle = \bar{\phi}_I\left(\sqrt{n}\left(\bar{W}_{IE} \langle \nu_I \rangle - \bar{W}_{II} \langle \nu_I \rangle \right), \underline{\langle \nu_E^2 \rangle, \langle \nu_I^2 \rangle} \right)$$

ignore

$$\tau_E \frac{d\bar{\nu}_E}{dt} = \bar{\phi}_E\left(\sqrt{n}\left(\bar{w}_{EE}\bar{\nu}_E - \bar{w}_{EI}\bar{\nu}_I\right)\right) - \bar{\nu}_E$$

$$\tau_I \frac{d\bar{\nu}_I}{dt} = \bar{\phi}_I\left(\sqrt{n}\left(\bar{w}_{IE}\bar{\nu}_I - \bar{w}_{II}\bar{\nu}_I\right)\right) - \bar{\nu}_I$$

first "derived" by Wilson + Cowan 1972

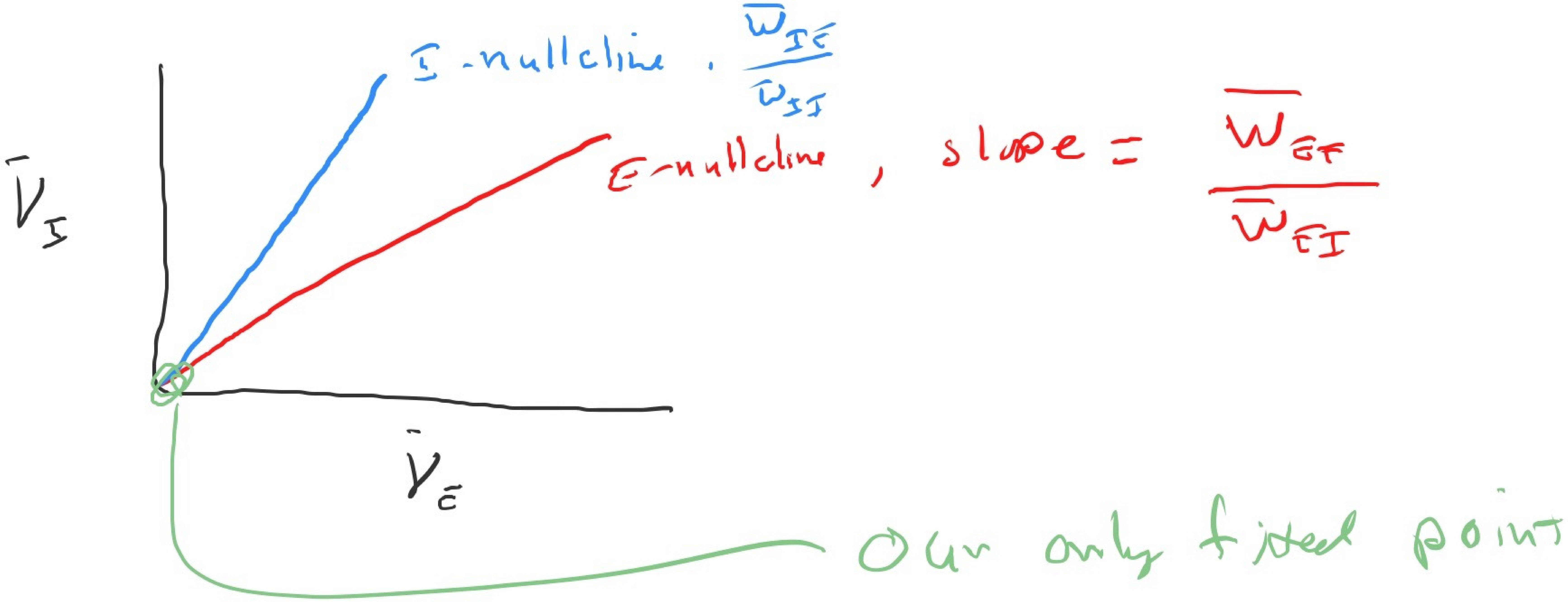
$\boxed{\text{2-D dynamics}}$

$$\tau \frac{dV_E}{dt} = \Phi_E\left(\sqrt{n}\left(\bar{W}_{EE}\bar{V}_E - \bar{W}_{EI}\bar{V}_I\right) + h_E\right) - V_E$$

$$\tau \frac{dV_I}{dt} = \Phi_I\left(\sqrt{n}\left(\bar{W}_{IE}\bar{V}_E - \bar{W}_{II}\bar{V}_I\right) + h_I\right) - V_I$$

nullclines in large n limit (large connectivity limit)

$$\sqrt{n}\Big(\underbrace{\bar{W}_{EE}\bar{V}_E - \bar{W}_{EI}V_I}_{\text{pinned to } 0 \text{ as } n\to\infty} + h_E\Big) \sim O(1)$$



$$\bar{W}_{EE}\bar{\nu}_E - \bar{W}_{EI}\bar{\nu}_I + h_E = 0$$

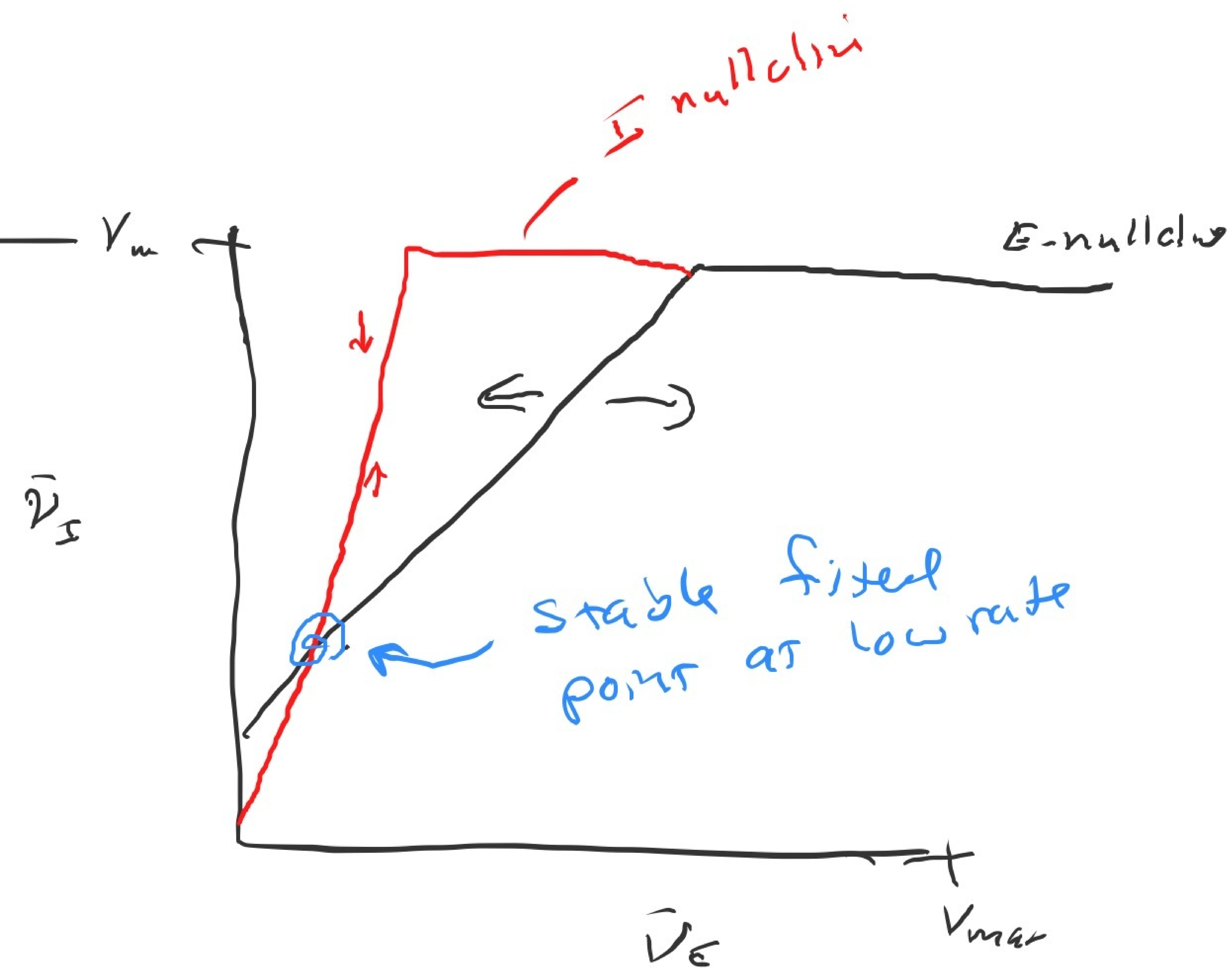
$$W_{IE}\bar{\nu}_E - W_{II}\bar{\nu}_I + h_I = 0$$

$$\bar{\nu}_E = \bar{\nu}_E(n=\infty) + O\left(\cancel{\frac{1}{\sqrt{n}}}\right)$$

ignore

$$E: \quad \bar{\nu}_I = \frac{h_E + \bar{W}_{EE}\bar{\nu}_E}{\bar{W}_{EI}}$$

$$I: \quad \bar{\nu}_I = \frac{h_I + \bar{W}_{IE}\bar{\nu}_E}{\bar{W}_{II}}$$



next time:

1. derive dynamics
2. relax large n assumption