Recap

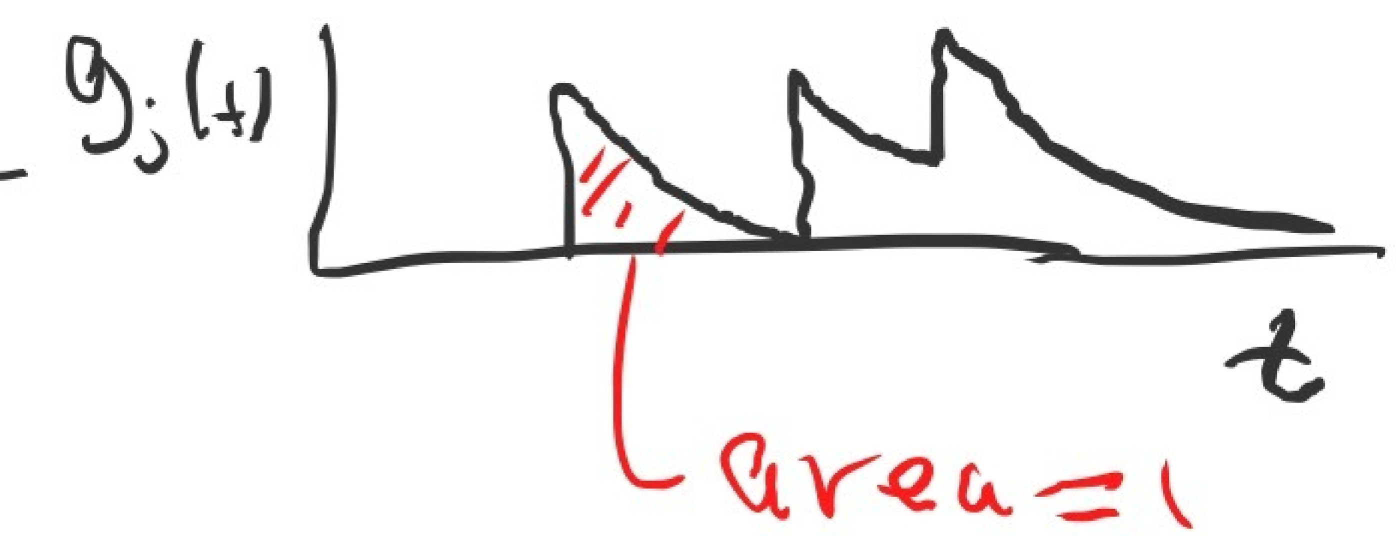
single neuron dynamics

$$\tau_m \frac{dV_i}{dt} = (\quad)_i - \sum_j A_{ij}(V_i - \mathcal{E}_j)\, g(t) + V_{ext,i}$$

conductance-based

$$\nu_i = \phi\left(\frac{1}{\sqrt{K}} \sum_j W_{ij} \nu_j + \dots\right)$$

an approximation



$$-A_{ij}(V_i - \mathcal{E}_j) \to \frac{\tau_m}{\sqrt{K}} \tilde{W}_{ij}$$

(current-based models)

units of voltage $\quad \frac{\tilde{W}_{ij}}{\sqrt{K}} \sim 0.5$ mV

$$\frac{1}{\sqrt{K}} \sum_j \tilde{W}_{ij}\, g_j(t)$$

$g_j(t) - \nu_j$

$$= \frac{1}{\sqrt{K}} \sum_j \left(\bar{W} + \delta \tilde{W}_{ij}\right)\left(\nu_j + \delta g_j(t)\right)$$

$$= \sqrt{K}\, \bar{w} \langle \nu \rangle + \frac{1}{\sqrt{K}} \sum_j \delta \tilde{W}_{ij} \nu_j + \frac{1}{\sqrt{K}} \sum_j \tilde{W}_{ij} \delta g_j(t)$$

$$\frac{1}{\sqrt{K}} \sum_j \tilde{w}_{ij}\, g_j(t)$$

$$= \underbrace{\sqrt{K}\ \bar{w}\langle\nu\rangle}_{\text{big, but we have E + I neurons}} + \underbrace{\frac{1}{\sqrt{K}} \sum_j \delta\tilde{w}_{ij}\, \nu_j}_{\hookrightarrow\ \sigma\, \xi_i} + \underbrace{\frac{1}{\sqrt{K}} \sum_j \tilde{w}_{ij}\, \delta g_j(t)}_{\gamma\, \eta(t)}$$

Spikes arrive at random times

$\xi_i \sim N(0,1)$

$\sigma^2 = \mathrm{Var}[w] \langle \nu^2 \rangle$

→ spread in firing rate

$\eta(t)$ Gaussian process, $O(1)$

$\gamma^2 \sim \langle \tilde{w}^2 \rangle \sim O(1)$

– fluctuations

$$\frac{1}{\sqrt{K}} \sum_j \tilde{w}_{ij}\, \delta g_j(t)$$

$$= \frac{1}{\sqrt{K}}\, \bar{w} \sum_j \delta g_j(t) + \frac{1}{\sqrt{K}} \sum_j \delta \tilde{w}_{ij}\, \delta g_j(t)$$

average over index these are small

$$= \sqrt{K}\, \bar{w} \left(\underbrace{\frac{1}{K} \sum_j \delta g_j(t)}_{G(t)} \right)$$

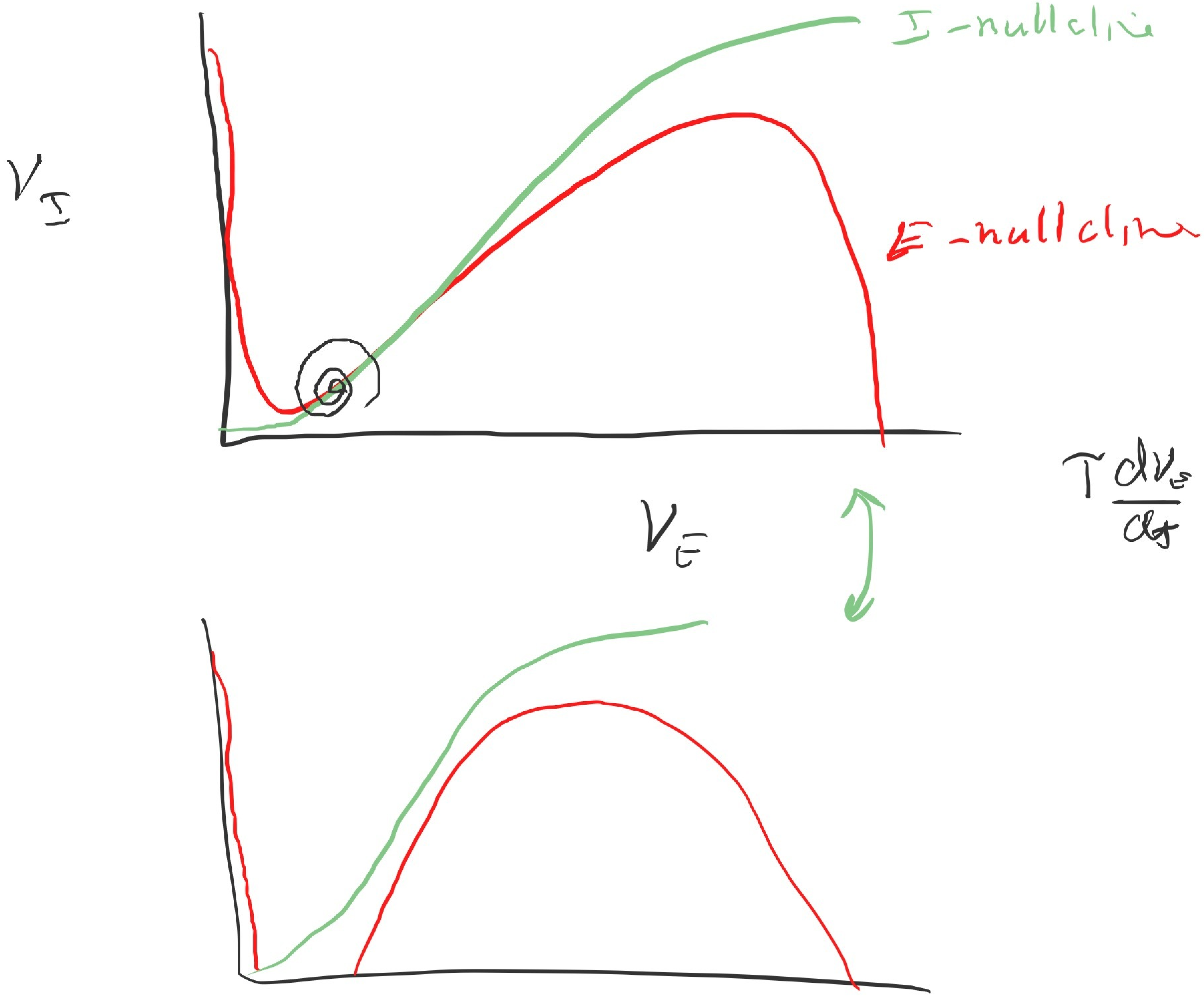
$$T \frac{\langle v_E \rangle}{dt} = (\quad) + \text{time-ind.}$$

$$+ T \sqrt{K} \left[\bar{w}_{EE} G_E(t) - \bar{w}_{EI} G_I(t) \right]$$

$$T \frac{d\langle v_E \rangle}{dt} = (\ ) + [+\sqrt{K}[\quad]]$$

$$+ T \frac{1}{K\sqrt{K}} \underbrace{\sum_{ij} \delta w_{ij}\, \delta g_j(t)}_{O(K)\ \cdot\ \text{time-dep}} \qquad O\left(\frac{1}{\sqrt{K}}\right)$$

$$+ \frac{1}{\sqrt{K}} \left(\text{time-dep. fluct.} \right)$$



smoothed gain function

$$T \frac{dV_E}{dt} = \phi_E\left(\sqrt{K}\left(w_{EE}V_E - w_{EI}V_I + h_E\right)\right) - V_E$$

Structured connectivity

K = average number of connections/neuron

$$\tau \dot{\nu}_{Ei} = \phi_E \left(\sqrt{K} \left(W_{EE} \nu_E - W_{EI} \nu_I + h_E \right) + \sigma_E \xi_i \right)$$

random $\qquad\qquad$ structure

$$\frac{1}{\sqrt{K}} W_{ij} + \frac{1}{K} J_{ij}$$

$$\underset{O(1)}{J_{ij} \sim u_i u_j} \qquad\qquad \underline{\underline{J}} = \sum_e \alpha_e \, \underset{\sim}{u}_e \, \underline{u}_e$$

$$\underline{\underline{J}} \cdot \underset{\sim}{\nu} = \sum_e \alpha_e \, \underline{u}_e \, \underbrace{\underline{u}_e \cdot \underline{\nu}}$$

$\underset{\sim}{\nu}$ is aligned with $\underline{u}_m$

$\Rightarrow \underline{u}_m \cdot \underline{\nu} \propto O(K)$

$$\frac{1}{K} \underline{\underline{J}} \cdot \underline{\nu} \sim \alpha_m \frac{\underline{u}_m \cdot \cancel{K}}{\cancel{K}} \sim O(1)$$

Hopfield networks (papers will be on web page)

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$$

$$\xi_i^\mu \begin{cases} 1 & \text{prob } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}$$

- all-all connectivity
- binary neurons
- discrete time updates

$$\underline{\underline{J}} = \frac{1}{N} \sum_{\mu} \underline{\xi}^\mu \underline{\xi}^\mu$$

$$S_i(t+1) = \text{sign}\left[\sum_j J_{ij} S_j(t) \right]$$

$S_i(t) = \pm 1$

+1 : neuron on

−1 : neuron off

goal: find stable states.

if $J_{ii} = 0$ the $H = -\frac{1}{2} \sum_{ij} S_i J_{ij} S_j$

decreases or stays the same on every time step

$$S_i(t+1) = \mathrm{sign}\left[\frac{1}{n}\sum_j \sum_\mu \xi_i^\mu \xi_j^\mu S_j(t)\right]$$

guess for fixed points :

$$S_j(t) = \xi_j^\nu$$

$$\frac{1}{n}\sum_\mu \xi_i^\mu \underline{\xi}^\mu \cdot \underline{S}$$

$\underline{S} = \underline{\xi}^\nu \Rightarrow$

term $= 1$

$$\frac{1}{n}\sum_j \left(\sum_{\mu\neq\nu} \xi_i^\mu \xi_j^\mu \xi_j^\nu + \xi_i \underbrace{\xi_j^\nu \xi_j^\nu}_{1}\right)$$

$$= \sum_{\mu\neq\nu} \xi_i^\mu \frac{1}{n}\sum_j \xi_j^\mu \xi_j^\nu + \xi_i^\nu \underbrace{\frac{1}{n}\sum_j \xi_j^\nu \xi_j^\nu}_{1}$$

$$\xi_i^\nu \overset{?}{=} \mathrm{sign}\Big[\underset{\substack{\uparrow\\ \text{signal}}}{\xi_i^\nu} + \underbrace{\frac{1}{n}\sum_{\mu\neq\nu}^{P}\sum_{j=1}^{n} \xi_i^\mu \xi_j^\mu \xi_j^\nu}_{\text{noise}}\Big]$$

$n(P-1)$ terms

$\frac{1}{n}\sqrt{n(P-1)}\, z_i \quad z_i \sim N(0,1)$

$$\frac{1}{n}\sum_{\mu\neq\nu}\sum_{j=1}^{n}\underbrace{\xi_i^{\mu}\,\xi_j^{\mu}\,\xi_j^{\nu}}$$

$+1$ prob $\frac{1}{2}$

-1 prob $\frac{1}{2}$

$n\cdot(p-1)$

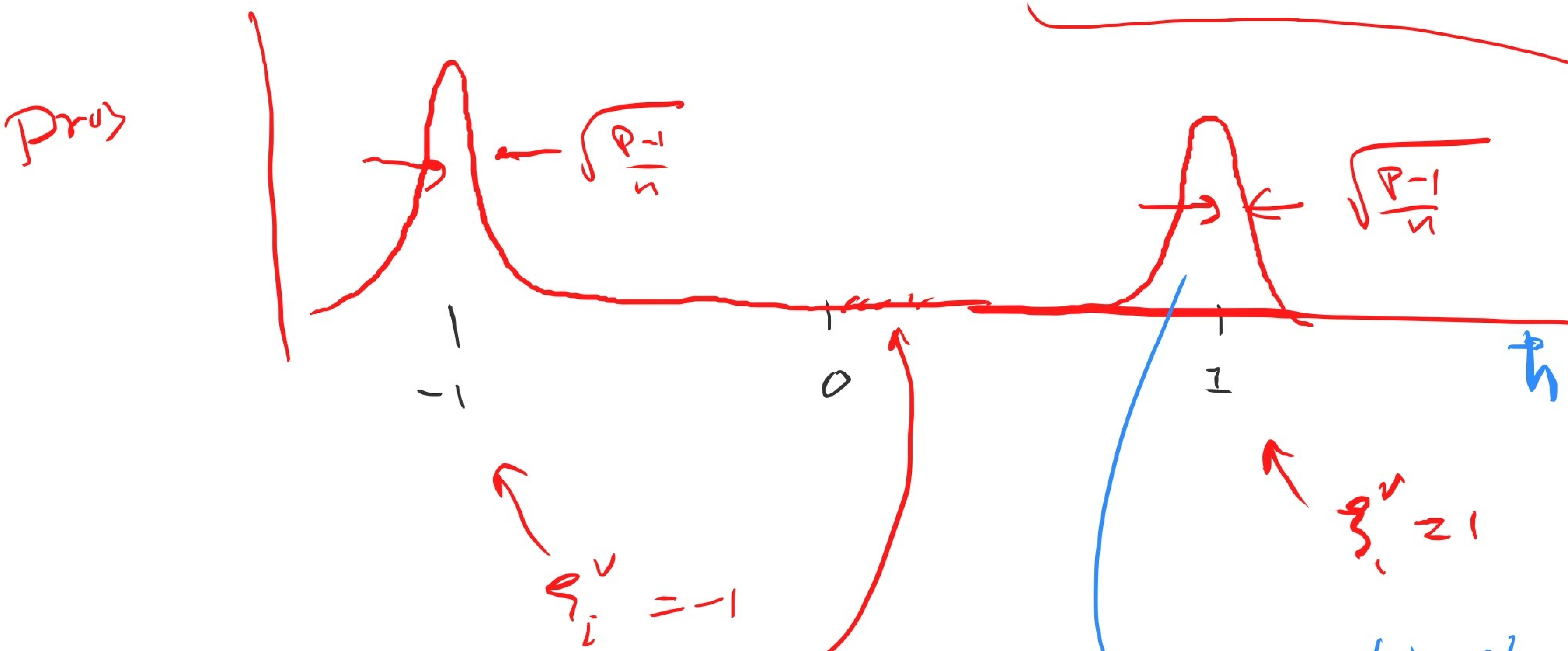
$\text{var} = n(p-1)$

$$\frac{1}{n}\sqrt{n(p-1)}\;z_i = \sqrt{\frac{p-1}{n}}\;z_i$$

$\uparrow N(0,1)$

$$\simeq \sqrt{\frac{p}{n}}\;z_i$$

$$\xi_i^\nu \stackrel{?}{=} \mathrm{sign}\left[\underbrace{\xi_i^\nu + \sqrt{\frac{p-1}{n}}\; z_i}\right]$$

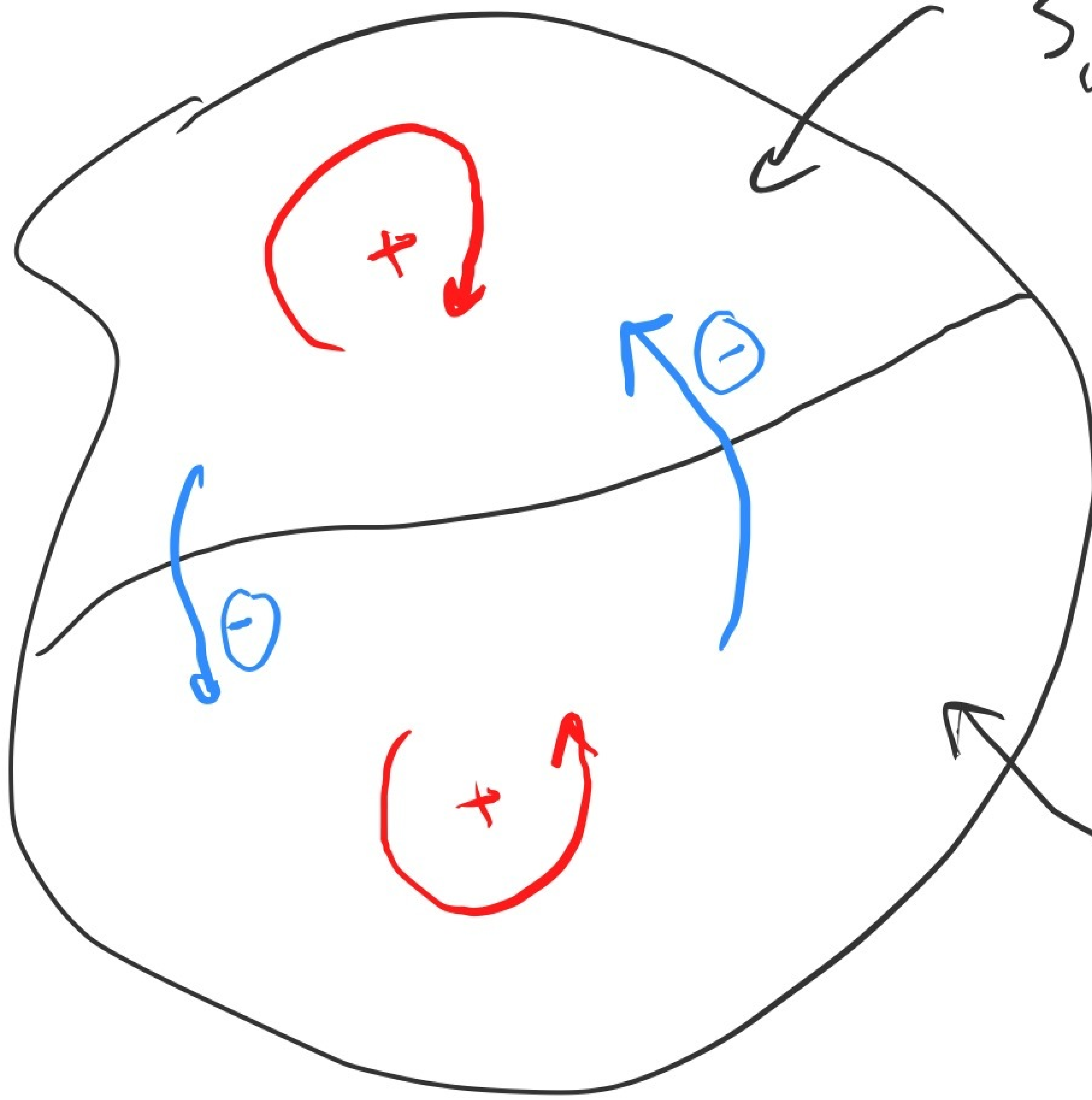


$$e^{-\frac{(h-1)^2}{2\left[(p-1)/n\right]}}$$

$P_{error} = \mathrm{Prob}(\text{tails of})$

$$\boxed{\begin{array}{l} p \ll n \\ P_{error} \text{ is exponentially small} \end{array}}$$

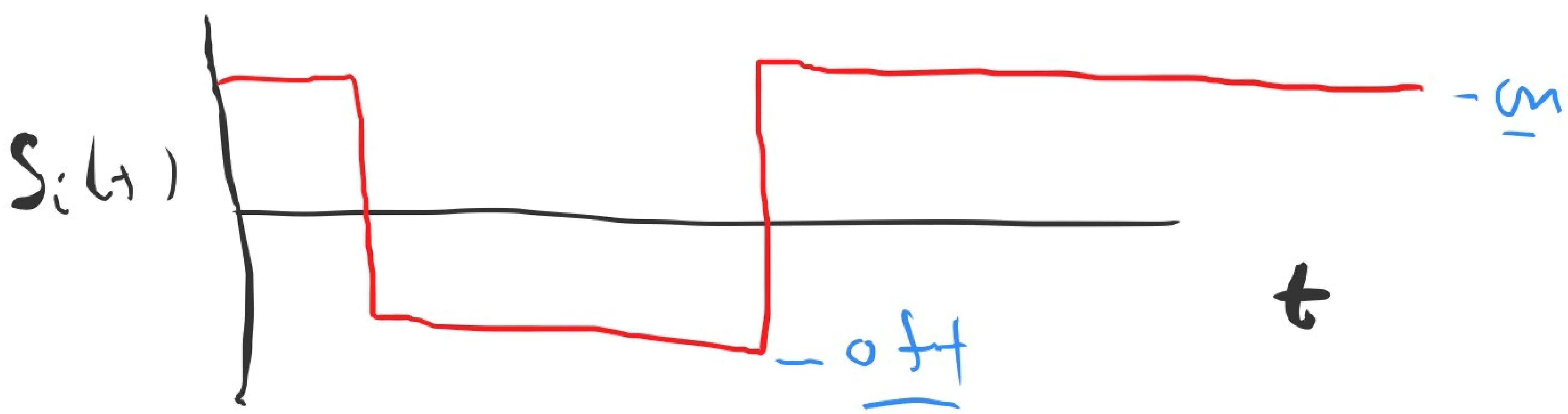
$\xi_i^\mu = +1$ $\qquad$ $S_i = -\xi_i^\mu$



n neurons

$\xi_i^\mu = -1$

$J_{ij} = \xi_i^\mu \xi_j^\mu$



Hopfield network

- binary
- discrete time
- all-all connectivity

$$S_i(t+1) = \mathrm{sign}\left[\frac{1}{n}\sum_j \left(\sum_\mu \xi_i^\mu \xi_j^\mu\right) S_j(t)\right]$$

$S_j(t)$

$$S_i(t) = \xi_i^\nu$$

smaller

$\mathcal{N}(0, \frac{p}{n})$

$\Rightarrow$ true if $\xi_i^\nu = \mathrm{sign}\left[\xi_i^\nu + \text{"noise"}_i\right]$

$p \ll n$, true (exponentially likely to be true)

as $p/n \sim O(1)$, start getting bit flips

$$\frac{1}{n}\xi_i^\nu \sum_j \xi_j^\nu \xi_j^\nu + \frac{1}{n}\sum_{\mu\neq\nu}\sum_j \xi_i^\mu \xi_j^\mu \xi_j^\nu$$

Q: how big can we make p before things fall apart?

$$\frac{p}{n} < 0.138$$ (Sompolinsky) (replica method)

$\boxed{\text{\# memories you can store} \sim n, \text{ number of neurons in the network}}$

assume K connections / neuron, on average

$$J_{ij} = \frac{1}{K} \sum_{\mu=1}^{p} C_{ij} \xi_i^\mu \xi_j^\mu$$

$$C_{ij} = \begin{cases} 1 & \text{prob } K/n \\ 0 & \text{prob } 1-K/n \end{cases}$$

guess: $S_i(t) = \xi_i^\nu$

$$\xi_i^\nu \stackrel{?}{=} \operatorname{sign}\Big[\underbrace{\frac{1}{K} \sum_j C_{ij} \overbrace{\xi_i^\nu \xi_j^\nu \xi_j^\nu}^{1}}_{= \xi_i^\nu \frac{1}{K}\sum_j C_{ij} \cong \xi_i^\nu} + \underbrace{\frac{1}{K} \underbrace{\sum_{\mu \neq \nu}}_{p-1 \text{ terms}} \underbrace{\sum_j}_{\sim K \text{ nonzero terms}} C_{ij} \xi_i^\mu \xi_j^\mu \xi_j^\nu}_{\frac{\sqrt{(p-1)K}\, z_i}{K} = \sqrt{\frac{p-1}{K}}}\Big]$$

$$\frac{1}{K} \times \underbrace{\big((p-1) \times K \text{ random terms}\big)}_{\text{var} = (p-1)K}$$

$$\boxed{\xi_i^\nu = \operatorname{sign}\left[\xi_i^\nu + \sqrt{\frac{p}{K}}\, z_i\right]}$$

$p \gg 1 \Rightarrow p-1 \cong p$

$$\xi_i^\nu \stackrel{?}{=} \mathrm{sign}\left[\xi_i^\mu + \sqrt{\frac{P}{K}}\, z_i\right]$$

$N(0,1)$

$\Rightarrow$ you can store approx. K memories

$\sim 1{,}000$

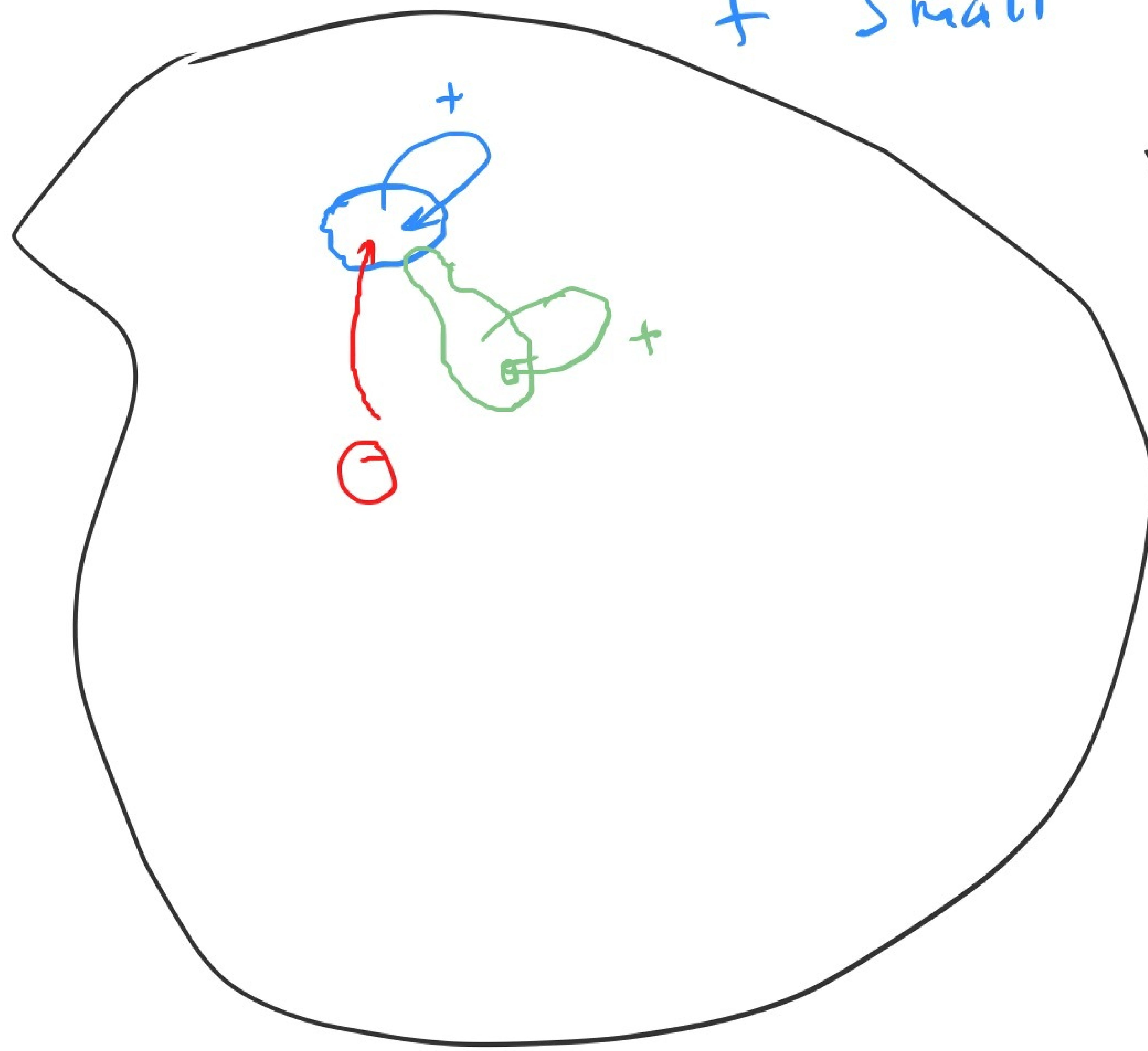
$(0.138)K \approx 100$ memories

looks bad for Hopfield networks

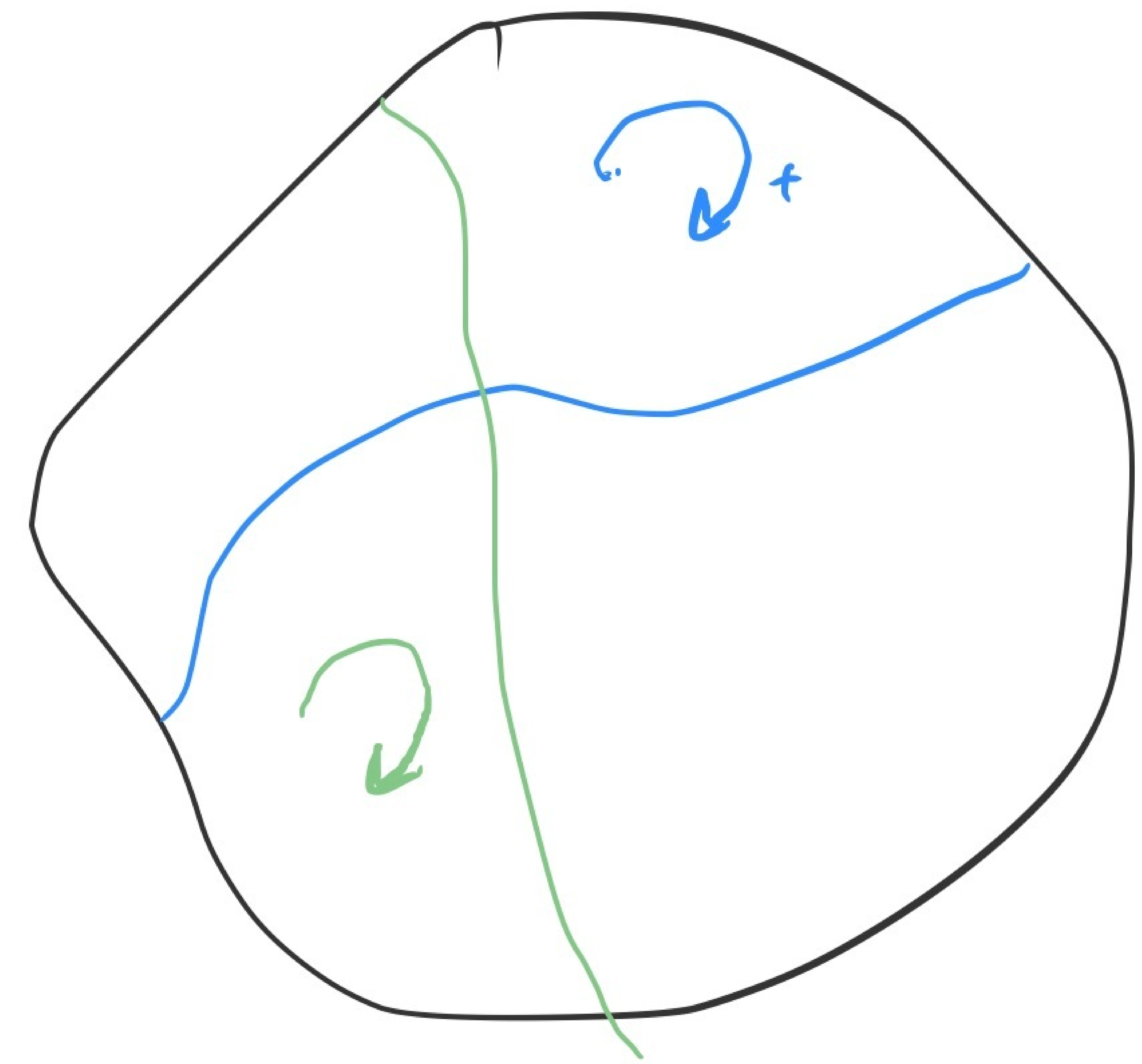
$$\xi_i^\mu = \begin{cases} 1 & \text{prob } f \\ -1 & \text{prob } \underbrace{1-f}_{\text{big}} \end{cases} \qquad f \ll \frac{1}{2}$$

$$\longrightarrow \begin{cases} 1 & \text{prob } f \\ 0 & \text{prob } 1-f \end{cases}$$

f small



n



$\#\ \text{memories} \sim \dfrac{K}{f\,|\log 1/f|}$ $\qquad f \sim \dfrac{K}{n}$

$\Rightarrow$ can store n memories.

more realistic

- continuous firing
- sparse connectivity
- random background connectivity
  - E-I network
- state when no memories are active ✓
- what's the capacity?

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connectivity matrix $\frac{1}{\sqrt{n}} W_{EE,ij} + \frac{\beta}{n f(1-f)} \sum_{\mu} \eta_i^{\mu} (\eta_j^{\mu} - f)$

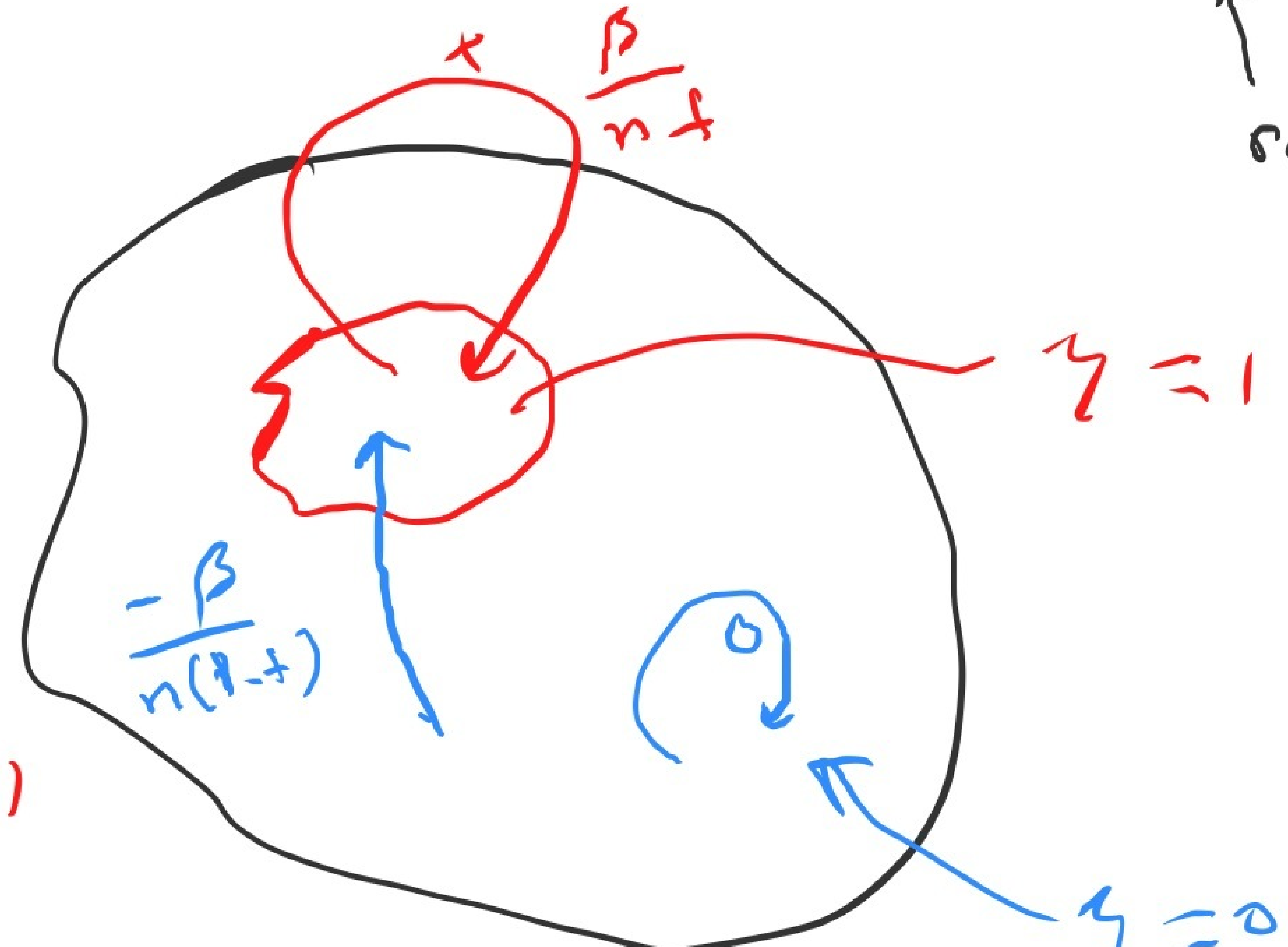
↑ random

$$\eta_i = \begin{cases} 1 & \text{prob } f \\ 0 & \text{prob } 1-f \end{cases}$$

all neurons have the same firing rate:

inh: $(\cancel{nf})\left(-\frac{\beta}{\cancel{nf}}\right)$

exc: $(n(\cancel{1-f}))\frac{\beta}{n(\cancel{1-f})}$



$$m = \frac{1}{nf(1-f)} \sum_j (\eta_j - f) V_{Ej}$$

$V_{Ej} = \text{const}$ $\frac{1}{n} \sum_j (\eta_j - f) \cong 0$

$\frac{1}{n} \sum_i \eta_i \cong f$

random

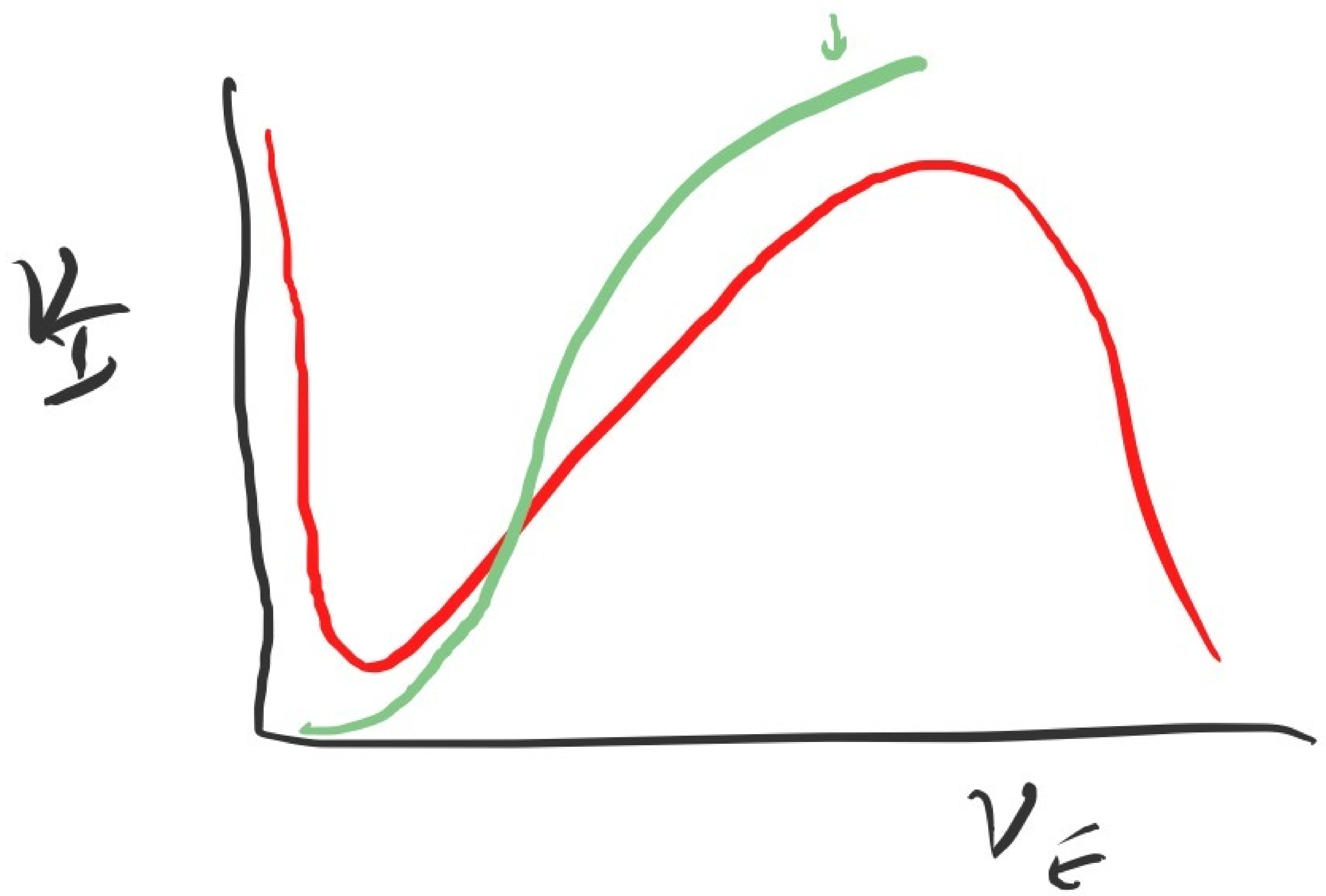
$$\tau_E \frac{d\nu_{Ei}}{dt} = \phi_E\left(\nu_E + (i\text{-dependent terms}) + \frac{\beta}{nf(1-f)} \sum_j \eta_i (\eta_j - f)\nu_{Ej} \; , \; \nu_I \right) - \nu_I$$

$$\tau_I \frac{d\nu_{Ii}}{dt} = \phi_I(\nu_E, \nu_I) - \nu_I$$

simplify: $\tau_I \to 0 \Rightarrow \nu_I = f(\nu_E)$

$\phi_E(w_{EE}\nu_E - w_{EI}\nu_I)$

$\to \phi_E(\nu_E, \nu_I)$



- One memory
- ignored spread in firing rate

$$\phi_E\left(\nu_E + \frac{\beta}{nf(1-f)}\eta_i \sum_j (\eta_j - f)\nu_{Ej},\ f(\nu_E)\right)$$

$$\tau \frac{d\nu_{Ej}}{dt} = \phi_E\left(\nu_E,\ \frac{\beta}{nf(1-f)}\eta_i \sum_j (\eta_j - f)\nu_{Ej}\right) - \nu_{Ej}$$

define $$m = \frac{1}{nf(1-f)}\sum_j (\eta_j - f)\nu_j$$

overlap

$$\tau \dot{\nu}_E = \frac{1}{n} \sum_i \phi_E(\nu_E, \beta \eta_i m) - \nu_E$$

$$\tau \dot{m} = \frac{1}{nf(1-f)} \sum_i (\eta_i - f)\, \phi_E(\nu_E, \beta \eta_i m) - m$$

$$f\, \phi_E(\nu_E, \beta m) + (1-f)\, \phi_E(\nu_E, 0)$$

$$\frac{1}{f\cancel{(1-f)}}\left[f\cancel{(1-f)}\, \phi_E(\nu_E, \beta m) - \cancel{(1-f)} f\, \phi_E(\nu_E, 0) \right]$$

$$= \phi_E(\nu_E, \beta m) - \phi_E(\nu_E, 0)$$

$$\tau \dot{\nu}_E = \phi_E(\nu_E, 0) + f\,[\phi_E(\nu_E, \beta m) - \phi_E(\nu_E, 0)] - \nu_E$$

$$\tau \dot{m} = \underbrace{\phi_E(\nu_E, \beta m) - \phi_E(\nu_E, 0)}_{\Delta\phi_E} - m$$

equations for dynamics of one memory