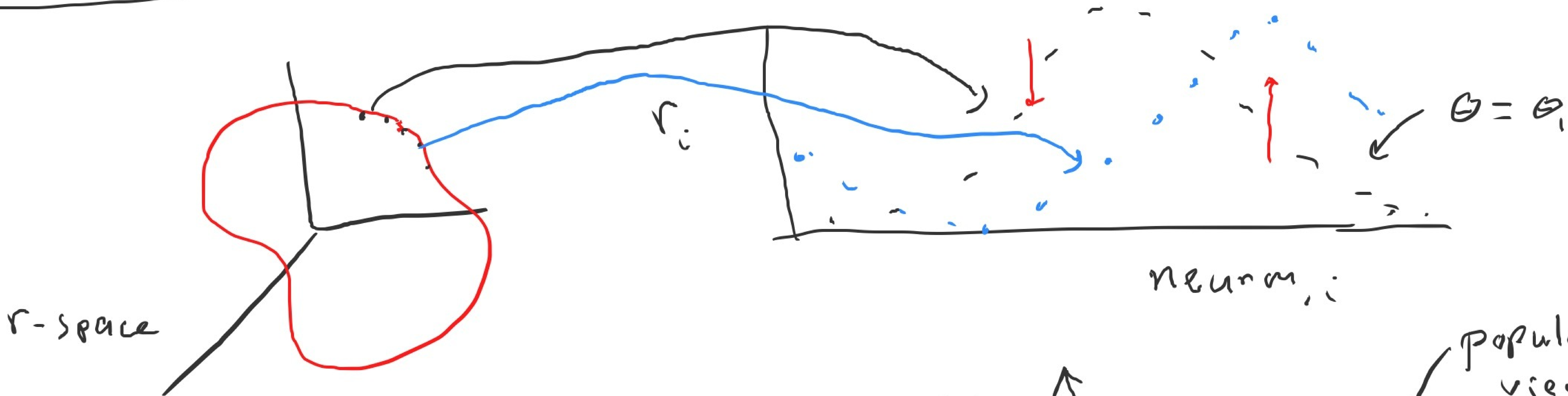
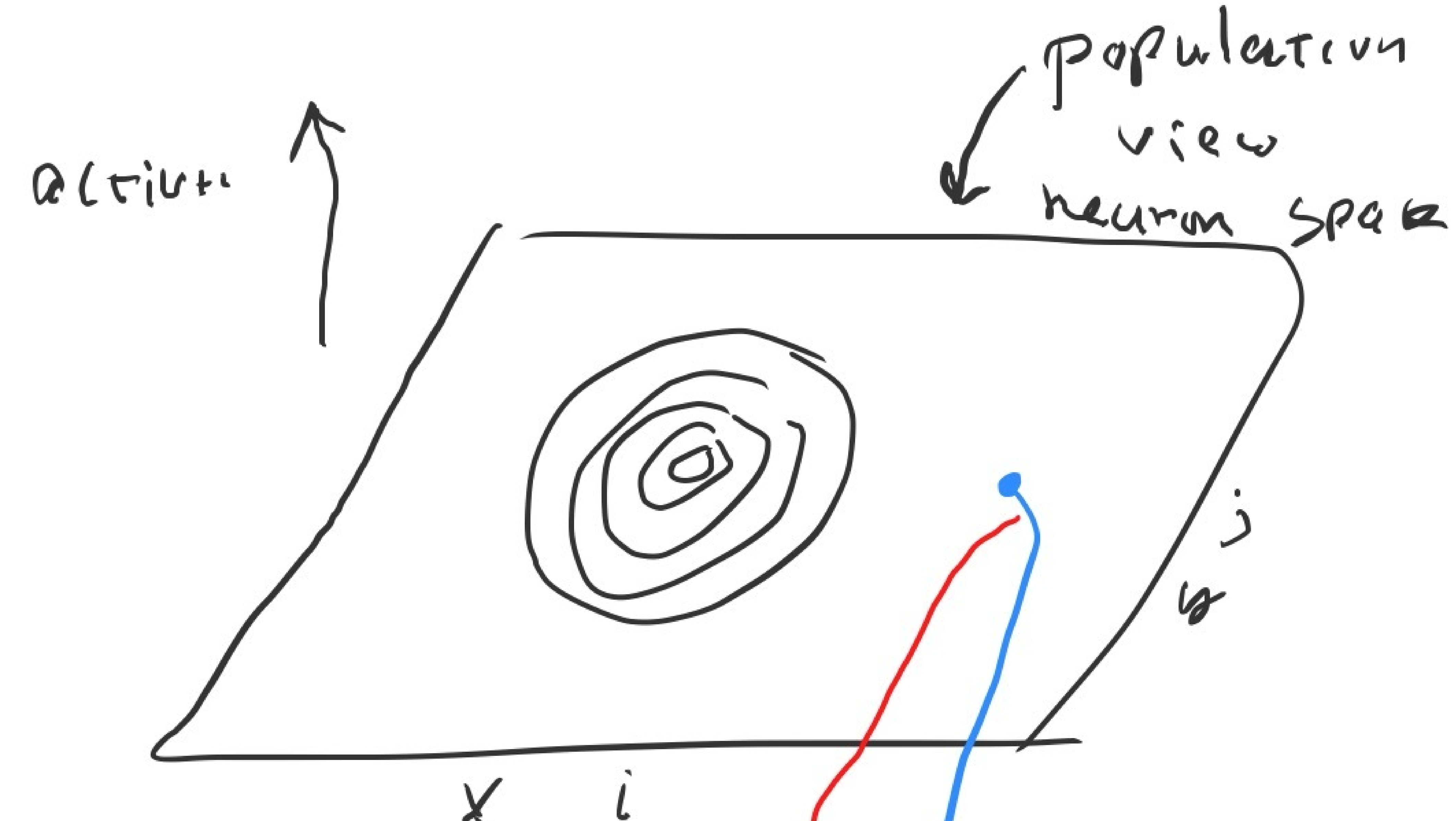
$$\dot{r}_i = \phi\left(\sum_j W_{ij} r_j + \sum_j^{k(t)} V_{ij} r_j + h_i^{(t)}\right) - r_i$$

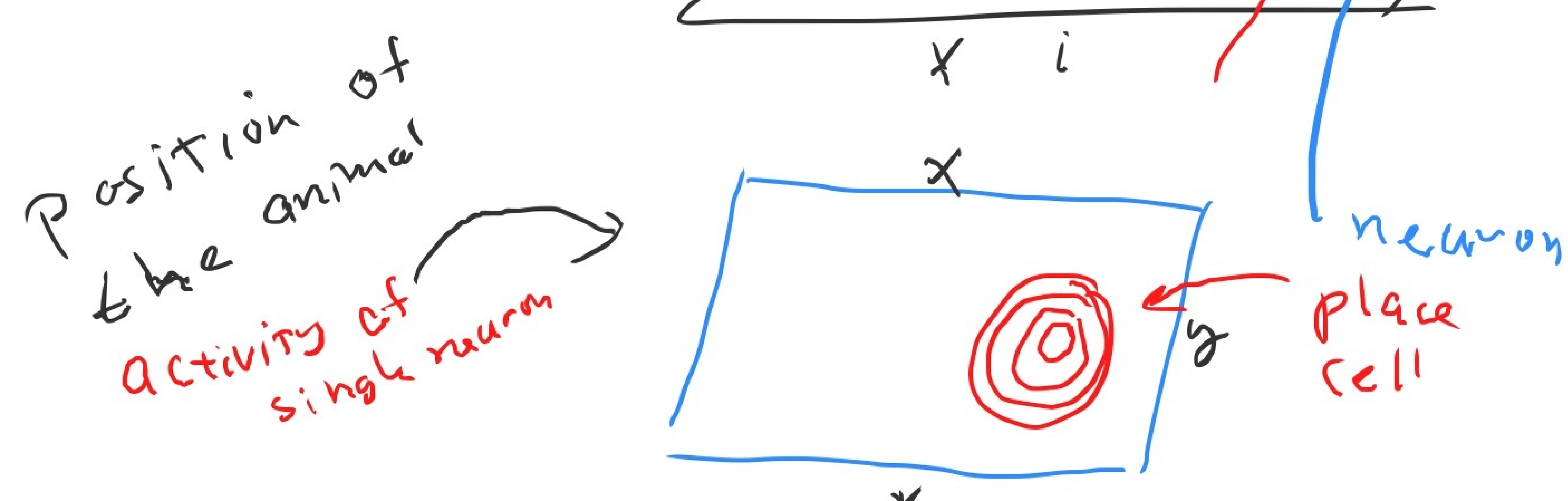
to move bump, external input must modify recurrent connectivity $V_{ij} = -V_{ji}$

$$f_i(\theta) = \phi\left(\sum_j W_{ij} f_j(\theta)\right) \longleftarrow \text{for a range of } \theta$$



- head direction cells
- place cells 2-d bump attractors



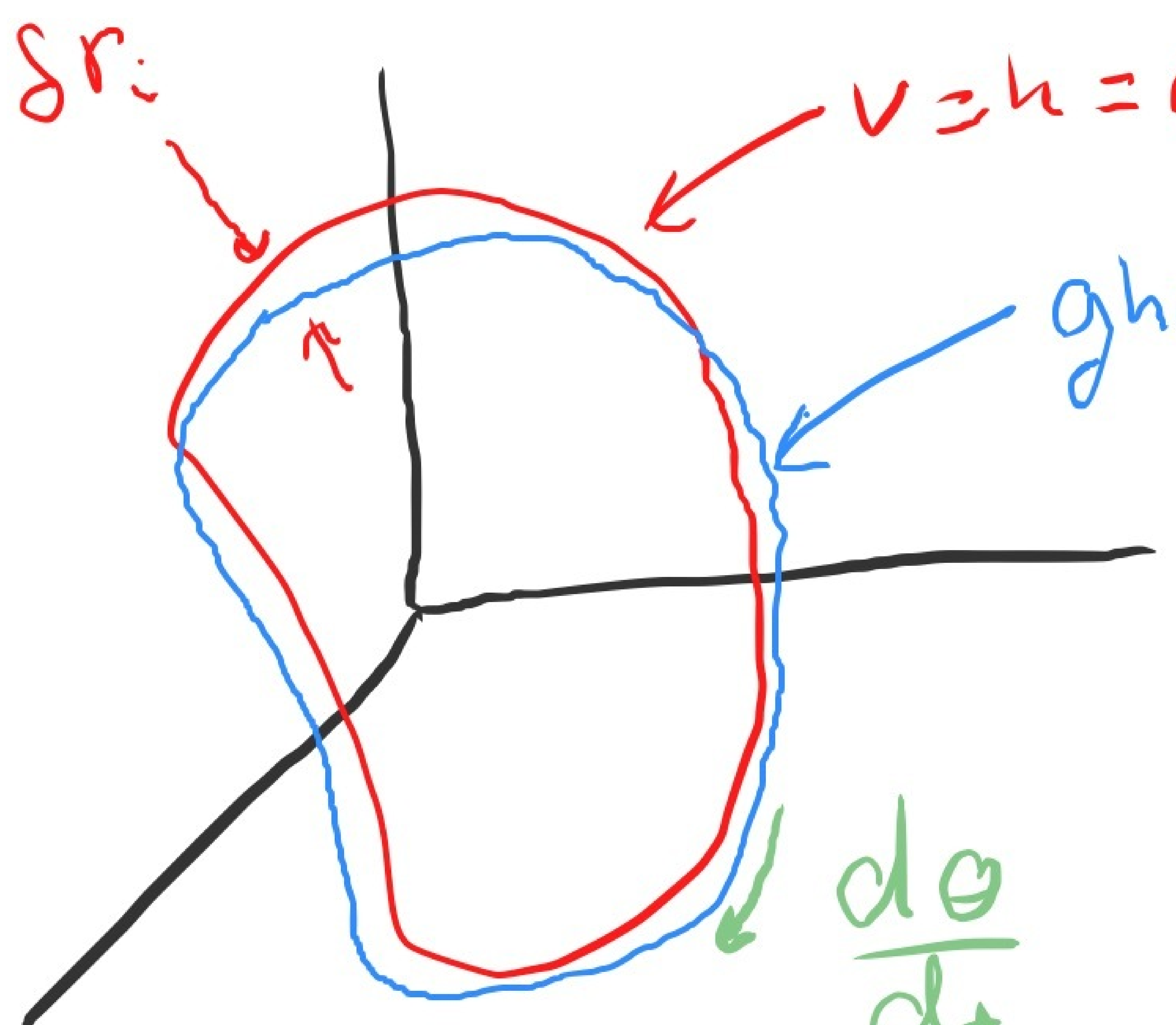


$$r_i = f_i(\theta) + \underbrace{\delta r_i(\theta)}_{\text{small}}$$

$\vdots$

$$\delta \dot{\underset{\sim}{r}} = \underline{\underline{J}} \cdot \delta \underline{r} + \underline{S}$$

- standard perturbation theory (v + h are small)
- slightly harder because we have a line attractor



$$\frac{d\theta}{dt} = \frac{\underline{V}_0^+ \cdot \underline{S}}{\underline{V}_0^+ \cdot \underset{\sim}{f}'(\theta)}$$

$$J_{ij} = \phi'_i W_{ij} - \delta_{ij}$$

$$\phi'_i \equiv \phi\left(\sum_j W_{ij} f_j(\theta)\right)$$

$$S_i = \phi'_i(\theta)\left[\sum_j v_{ij} f_j(\theta) + h_i\right]$$

$$\underline{\underline{J}} \cdot \underline{V}_0 = 0$$

$$\underline{V}_0^+ \cdot \underline{\underline{J}} = 0$$

$$\underline{V}_0 \propto \underset{\sim}{f}'(\theta)$$

$$f_i(\theta) = \phi\left(\sum_j W_{ij} f_j(\theta)\right)$$

$$f_i'(\theta) = \sum_j \phi_i' W_{ij} f_j'(\theta)$$

$$\Rightarrow \sum_j (\phi_i' W_{ij} - \delta_{ij}) \underbrace{f_j'(\theta)}_{\propto\, \underline{V}_0} = 0$$

$$\sum_i V_{0_i}^{+} \left[\phi_i' W_{ij} - \delta_{ij}\right] = 0$$

$$\boxed{\text{assume } W_{ij} = W_{ji}} \Longrightarrow \boxed{V_{0_i}^{+} = \frac{f_i'(\theta)}{\phi_i'(\theta)}}$$

$$\phi_j' \sum_i \left[W_{ji} \phi_i' - \delta_{ji}\right] V_{0_i}^{+} = 0$$

$$\sum_i \overbrace{(\phi_j' W_{ji} - \delta_{ji})}^{J_{ji}} \underbrace{\phi_i' V_{0_i}^{+}}_{\propto\, f_i'(\theta)} = 0$$

$$\dot{\Theta} = \frac{\sum_i \frac{f_i'(\Theta)}{\phi_i'} s_i}{\sum_i \frac{f_i'(\Theta)^2}{\phi_i'(\Theta)}} = \frac{\sum_i f_i'(\Theta)\left[\sum_j V_{ij} f_j + h_i\right]}{Z(\Theta)}$$

$$\longrightarrow Z(\Theta)$$

$\Theta \in [0, 2\pi]$, $\Delta\Theta = \frac{2\pi}{N}$

$$W_{ij} = W_{i-j}$$

equally spaced angle

$$f_i(\Theta) \equiv f(\Theta_i - \Theta)$$

$$\phi_i'(\Theta) = \phi'(\Theta_i - \Theta)$$

$$V_{ij} = V(\Theta_i, \Theta_j)$$

$$h_i = h(\Theta_i - \Theta_0)$$

$$Z(\Theta) = \sum_i \frac{f'(\Theta_i - \Theta)^2}{\phi'(\Theta_i - \Theta)}$$

$$\rightarrow N \int d\psi \, \frac{f'(\psi - \Theta)^2}{\phi'(\psi - \Theta)}$$

$\psi \to \psi + \Theta$

$$= N \int d\psi \; f'(\psi)^2 / \phi'(\psi)$$

$$\sum_i f_i' \left[\sum_j V_{ij} f_j + h_i \right]$$

$$= \sum_i f'(\theta_i - \theta) \left[\sum_j V(\theta_i, \theta_j) f(\theta_j - \theta) + h(\theta_i - \theta_0) \right]$$

$$\to N^2 \int d\psi\, d\psi'\ f'(\psi - \theta) V(\psi, \psi') f(\psi' - \theta)$$

$$+ N \int d\psi\ f'(\psi - \theta) h(\psi - \theta_0)$$

for now, $V = 0$

goal: external input should move us along the line attractor

external signal

$$h_i(\psi - \theta_0) = \underline{g(t)}\, h_0$$

$$N g(t) h_0 \int d\psi\ f'(\psi - \theta) = N g(t) h_0 \left. f(\psi - \theta) \right|_{\psi_{min}}^{\psi_{max}}$$

periodic: $\psi_{min} = \psi_{max}$

$$\dot{\Theta} = \frac{N^2 \int d\psi\, d\psi'\, f'(\psi-\theta)\, V(\psi,\psi')\, f(\psi-\theta)}{2}$$

$$V_{ij} = V_{i-j}\, K(t)$$

if $\quad V(\psi,\psi') = K(t)\, V(\psi-\psi')$

$\psi \to \psi + \theta$

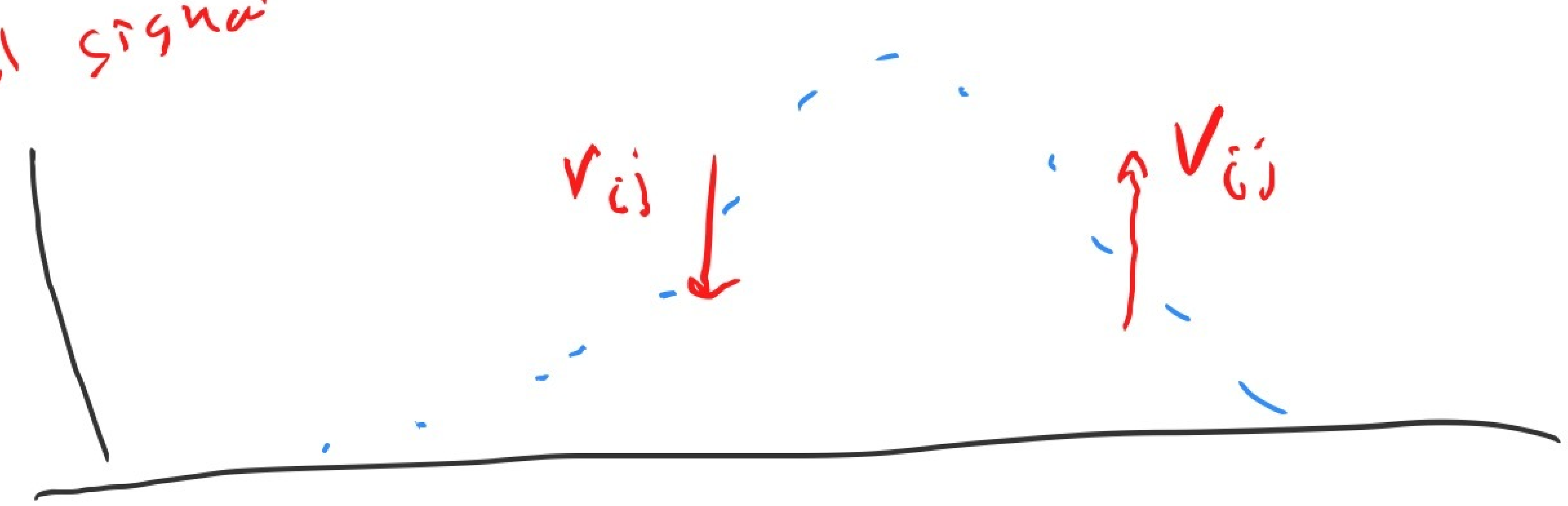
$\psi' \to \psi' + \theta$

odd function

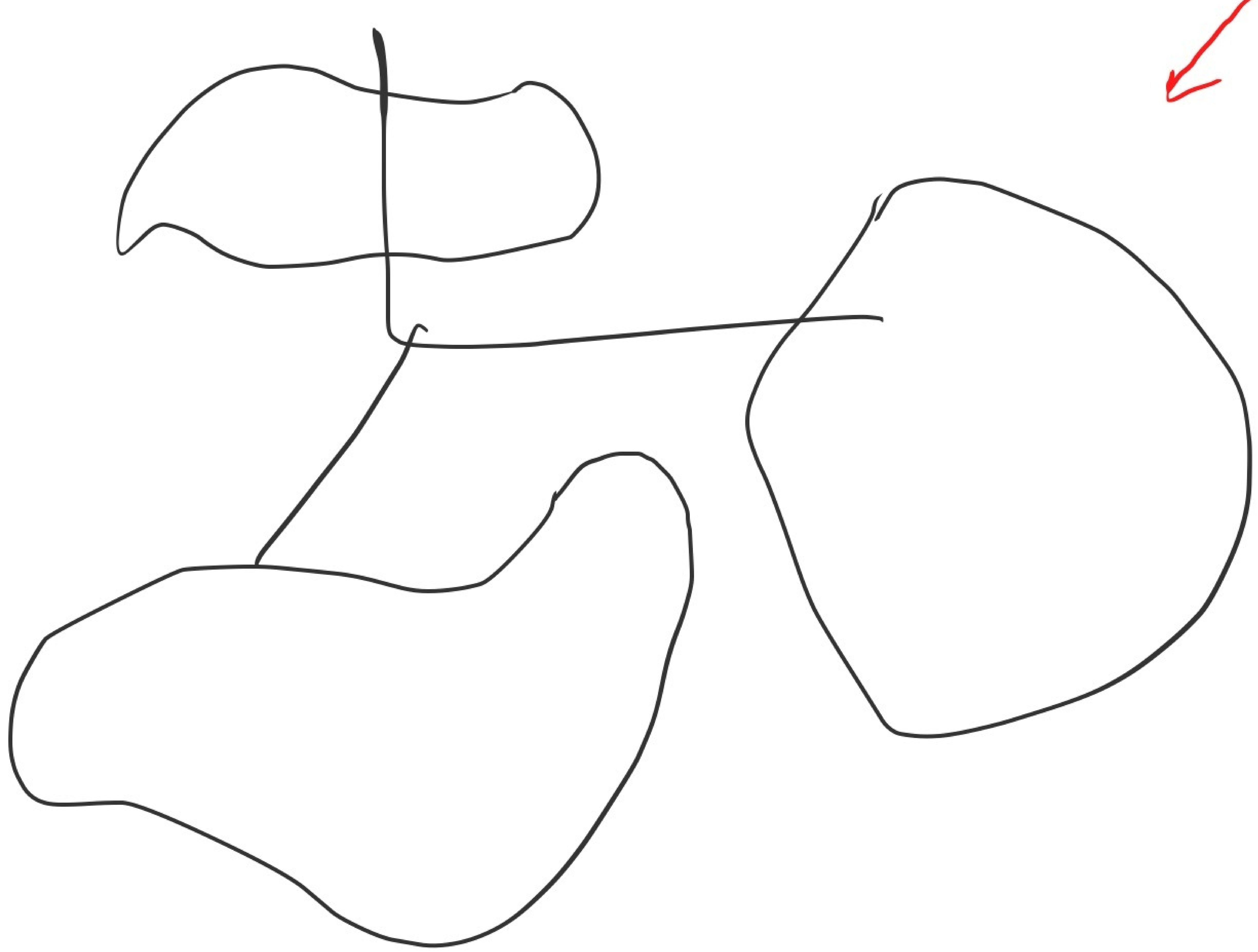
$$\dot{\Theta} = \frac{N^2}{2}\, \underline{K(t)} \int \underline{d\psi\, d\psi'\, f'(\psi)\, V(\psi-\psi')\, f(\psi)}$$

external signal

ind. of θ



$$\dot{r}_i = \phi\left(\sum_{\mu} \sum_{j} W_{ij}^{\mu} r_j \right)$$



Hard to embed lots of line attractors because they're structurally unstable!

RNNs that learn a task

$$\tau \frac{dx_i}{dt} = \sum_{j=1}^{n} W_{ij} \underbrace{\phi(x_j)}_{\text{firing rate}} - x_i$$

x_i ~ membrane potential

in the past

$$r_i = \phi\left(\sum_j W_{ij} r_j\right) - r_i$$

$$z_\mu = \sum_j A_{\mu j} \phi(x_j)$$

target function

Goal: adjust weights so that $z_\mu(t) = z^*_\mu(t)$

supervised learning

learn: mapping from $u_\nu(t)$ to $z^*_\mu(t)$

add control signal

often random

$$\tau \frac{dx_i}{dt} = \sum_j W_{ij} \phi(x_j) + \sum_\nu C_{i\nu} u_\nu(t) + \sum_\mu J_{i\mu} z_\mu - x_i$$

$\boxed{\text{fix } W, C, J, \text{ learn } A \text{ to output } z^*_\mu}$

- do we need recurrent (random) connections?

it would seem no

- set $W_{ij} = 0$

$$\tau \frac{dx_i}{dt} = \sum_{\nu} C_{i\nu} u_\nu(t) + \sum_{\mu, j} J_{i\mu} A_{\mu j} \phi(x_j) - x_i$$

low rank (p)

$i = 1, \ldots, n$ n-dim system

$\mu = 1, \ldots, p$ $p \ll n$

- we should be able to reduce to p-dim system
- we really want an equation for z_μ

$$\mathcal{L} x_i = \left(\tau \frac{d}{dt} + 1\right) x_i \qquad \mathcal{L}^{-1} f(t) = \int_{-\infty}^{t} \frac{dt'}{\tau} e^{-(t-t')} f(t') \quad (?)$$

$$\mathcal{L} x_i = \sum_{\nu} C_{i\nu} u_\nu + \sum_{\mu} J_{i\mu} z_\mu$$

$$z_\mu = \sum_j A_{\mu j} \phi(x_j)$$

$$x_i = \sum_{\nu} C_{i\nu} \mathcal{L}^{-1} u_\nu + \sum_{\mu} J_{i\mu} \mathcal{L}^{-1} z_\mu$$

$$z_\mu = \sum_j A_{\mu j} \phi\left(\sum_\nu C_{j\nu} \mathcal{L}^{-1} u_\nu + \sum J_{j\nu} \mathcal{L}^{-1} z_\nu\right)$$

$$\tilde{z}_\mu = \mathcal{L}^{-1} z_\mu \qquad \tilde{u}_\nu = \mathcal{L}^{-1} u_\nu$$

$$\boxed{\mathcal{L} \tilde{z}_\mu = \sum_j A_{\mu j} \phi\left(\sum_\nu J_{j\nu} \tilde{z}_\nu + \sum C_{j\nu} \tilde{u}_\nu\right)} = \tau \frac{d\tilde{z}_\mu}{dt} + \tilde{z}_\mu$$

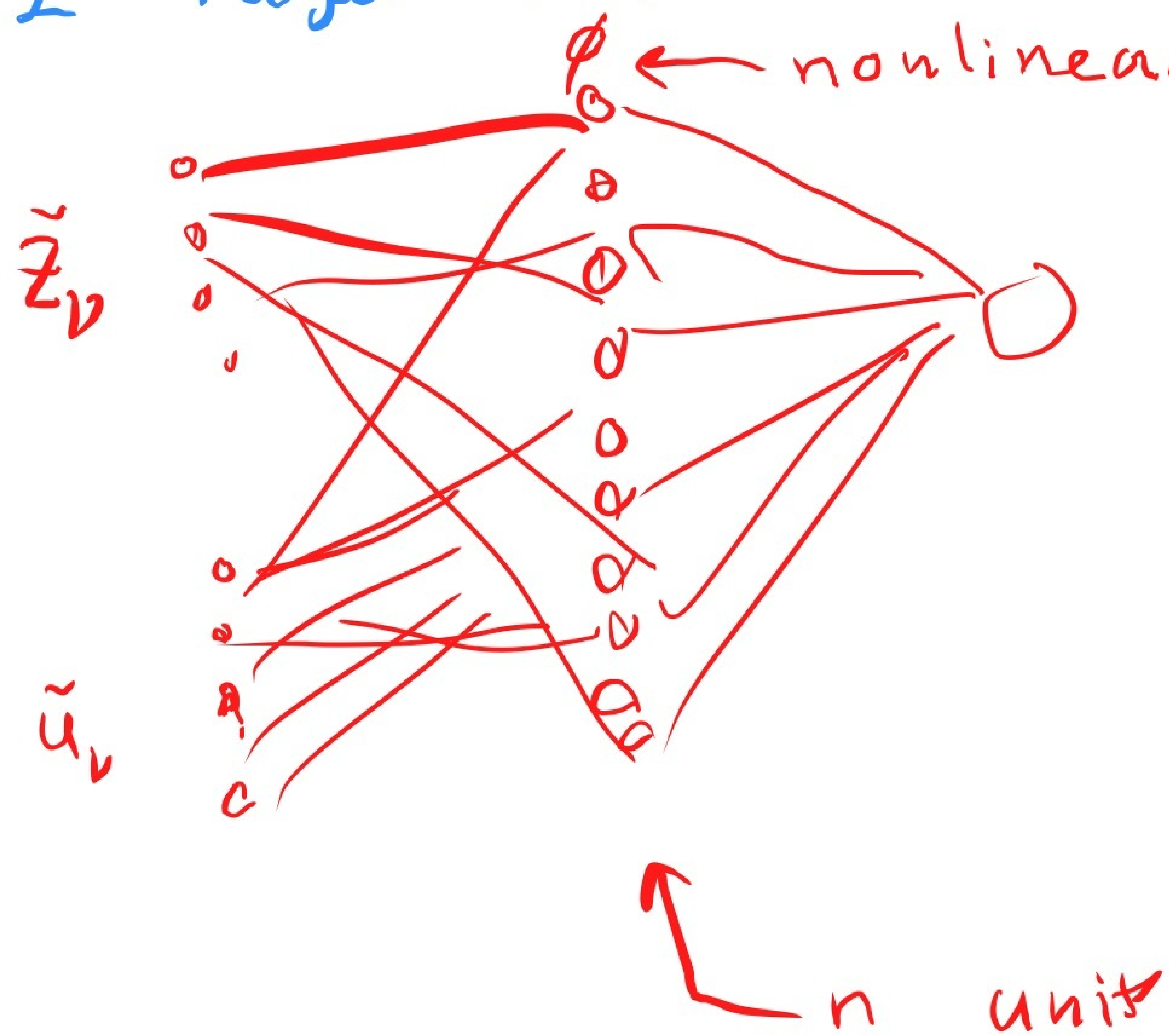
$$\tau \frac{d\tilde{z}_\mu}{dt} = \sum_{j=1}^{n} A_{\mu j}\, \phi\Big(\sum_\nu J_{j\nu} \tilde{z}_\nu + \sum_\nu C_{j\nu} \tilde{u}_\nu\Big) - \tilde{z}_\mu$$

$\mu = 1, \dots, p$

$n \gg 1$

$p \sim 1-5-10$

- 2 layer network



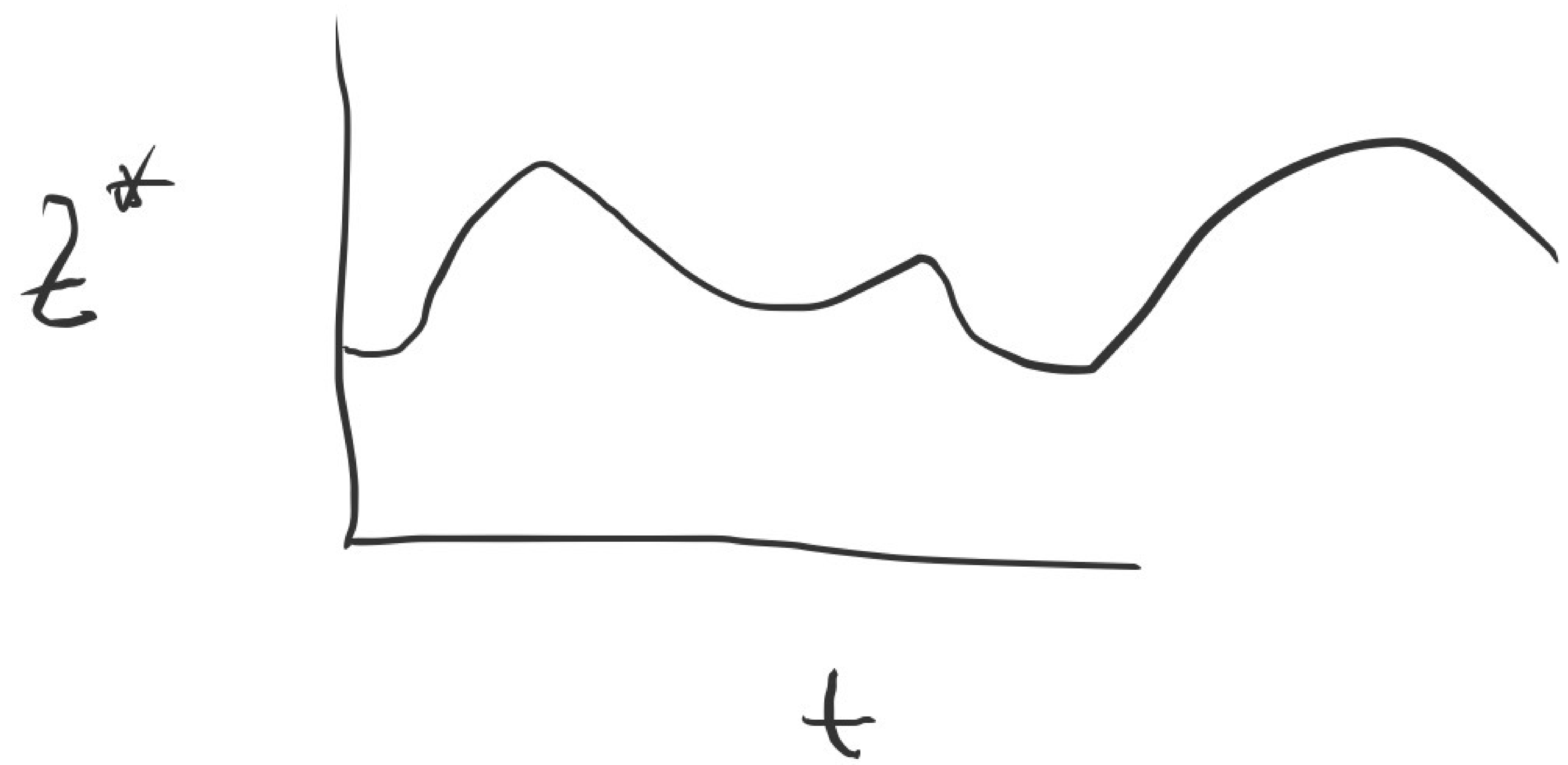
$= \tau \frac{d\tilde{z}_\mu}{dt} + \tilde{z}_\mu$

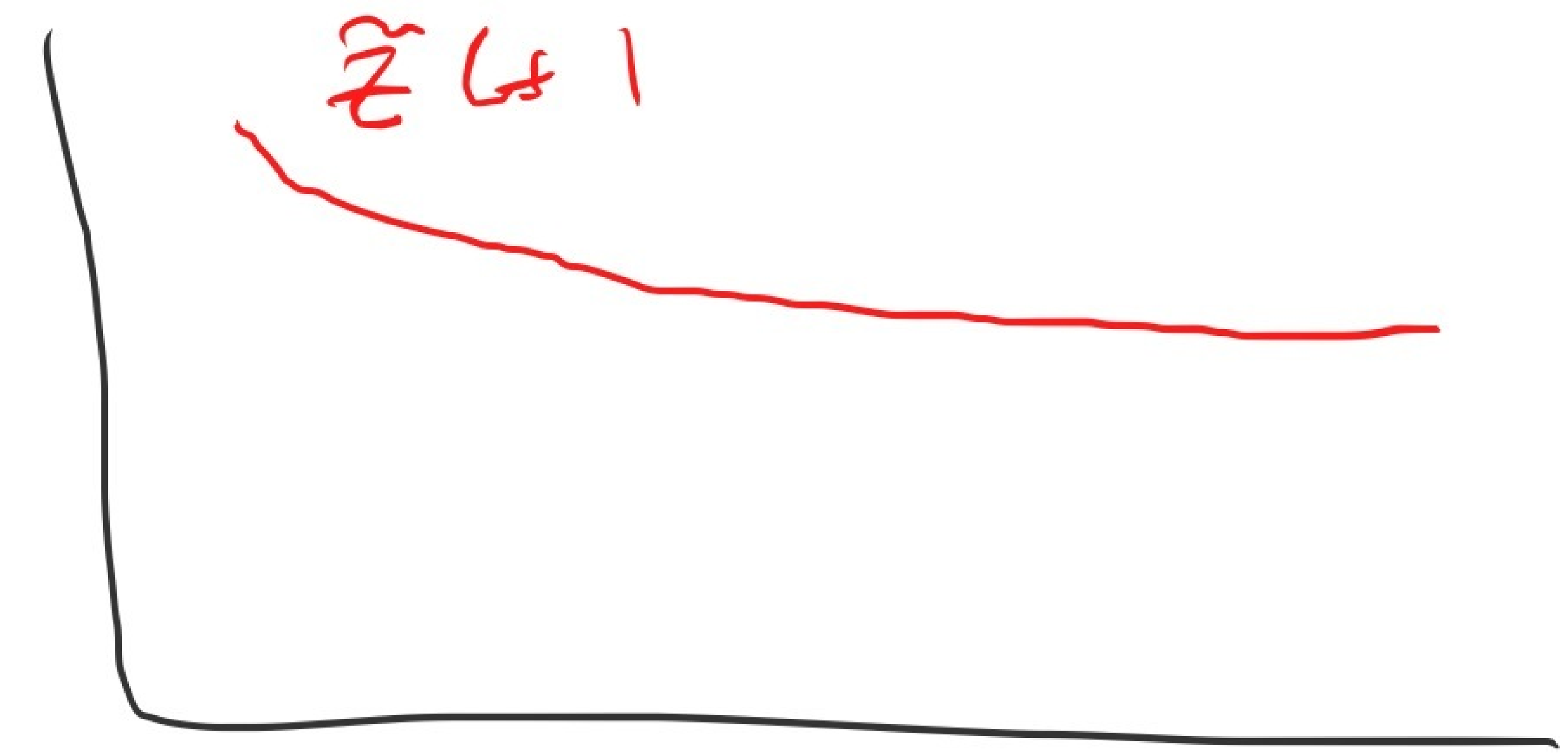
- Universal approximator

construct any function

$$\tau \frac{d\tilde{z}_\mu}{dt} + \tilde{z}_\mu = f(\underline{\tilde{z}}, \underline{\tilde{u}})$$

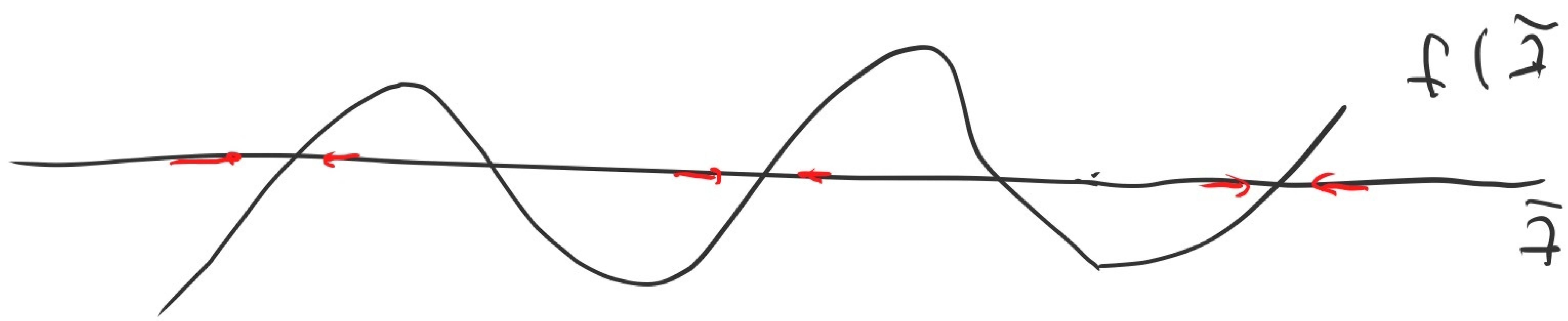
- Suppose z is 1-D $(p=1)$
- no control signal (just initial conditions : fixes $\underline{x}(t=0)$)
- target function





without recurrent connectivity

$$\tau \frac{d\tilde{z}}{dt} + \tilde{z} = f(\tilde{z})$$ ← only has fixed points



Conventional wisdom:

- choose W_{ij} so that the system is close to the edge of chaos

- you want the network to be able to generate a rich set of time-dependent functions

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Summarize:

- if z^*_μ can be generated from $\tau \frac{d\tilde{z}_\mu}{dt} + \tilde{z}_\mu = f(\underline{\tilde{z}}, \underline{\tilde{u}})$, W_{ij} is a nuisance

- if it can't W_{ij} is essential

$\dot{z} = f(z)$

$\dot{z}_1 = -z_2$

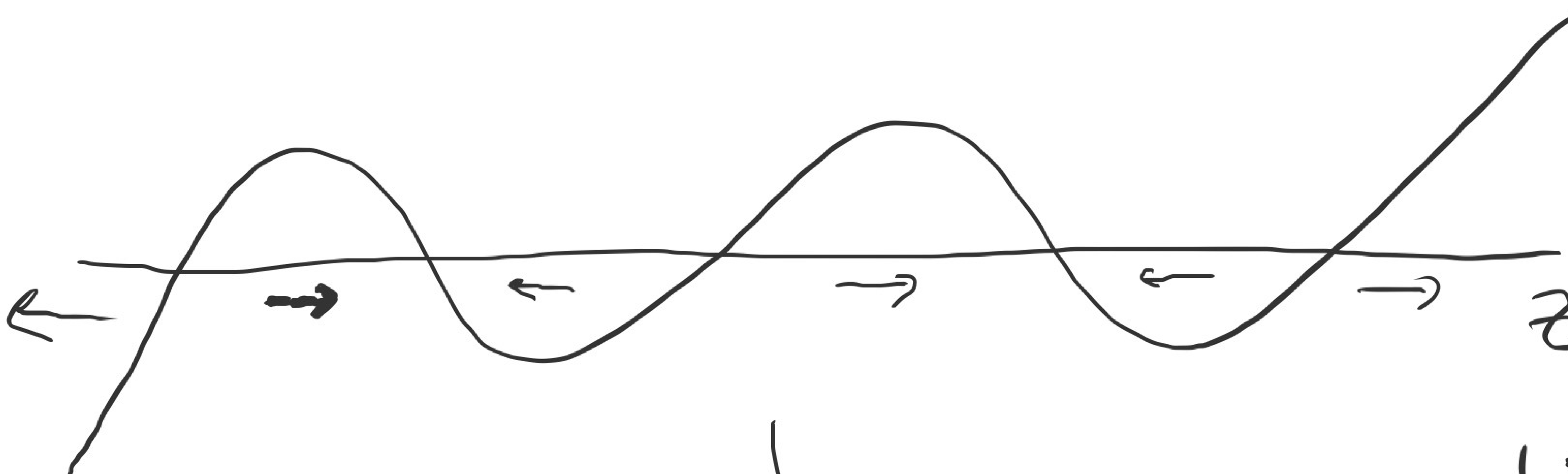
$\dot{z}_2 = z_1$

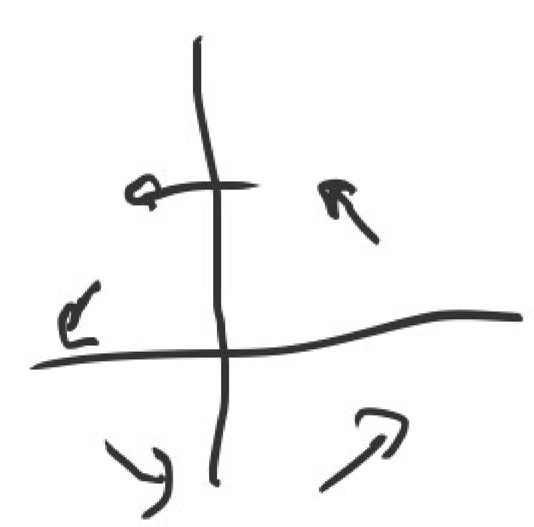
or

$\dot{z} = \ldots$

random recurrent connectivity

Sine





$$\tau \frac{dx_i}{dt} = g \sum_{j=1}^{n} W_{ij} \tanh(x_j) + J_i z$$

$z = 0$

$$z = \sum_j A_j \tanh(x_j)$$

$W_{ij} \sim$ iid $\quad \mathrm{var}[W_{ij}] = \frac{1}{n}$

$\Rightarrow \quad g < 1 \quad x_i = 0$ fixed points
$\quad\quad g > 1 \quad$ dynamics is chaotic

$\Big\}\ z = 0$

- for lots of tasks, $g \cong 1.5$ seems good

FORCE

$$\tau \frac{dx_i}{dt} = \sum_j W_{ij}\,\phi(x_j) + J_i z$$

$$z = \sum_i A_i\,\phi(x_i) \qquad \text{target: } z^*$$

train A_i only

recursive least squares

minimize loss $= \mathcal{L} = \frac{1}{2}\sum_{t=0}^{T}\left(z(t) - z^*(t)\right)^2 + \frac{\lambda}{2}\sum_i A_i^2$

$$\mathcal{L}(T) = \frac{1}{2}\sum_{t=0}^{T}\left(\sum_i A_i\,\phi(x_i(t)) - z^*(t)\right)^2 + \frac{\lambda}{2}\sum_i A_i^2$$

← linear regression
$n >$ # points in sum over t,
multiple solutions

$$\frac{\partial \mathcal{L}}{\partial A_i} = \sum_t \phi(x_i(t))\left(\sum_j \phi(x_j(t))\,A_j - z^*(t)\right) + \lambda A_i$$

— ridge regression

$$\sum_j \underbrace{\left(\sum_{t=1}^{T} \phi(x_i(t))\, \phi(x_j(t)) + d_{ij} \right)}_{P^{-1}_{ij}(T)} A_j = \underbrace{\sum_i \phi(x_i(t))\, z^*(t)}_{\Psi_i}$$

$$\underline{A} = \underline{\underline{P}} \cdot \underline{\Psi}$$ standard ridge regression

$$\boxed{\begin{aligned} P^{-1}_{ij}(T+\Delta t) &= P^{-1}_{ij}(T) + \phi(x_i(t+\Delta t))\, \phi(x_j(t+\Delta t)) \\ P_{ij}(T+\Delta t) &= P_{ij}(T) + \ldots \end{aligned}}$$

rank 1 update

a bunch of easy terms

FORCE learning

nothing to do with the brain!