Probability Theory II (G6106)

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http://stat.columbia.edu/∼porbanz/G6106S15.html

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Homework 1

Due: 4 February 2015

Homework submission: We will collect your homework **at the beginning of class** on the due date. If you cannot attend class that day, please leave your solution in my postbox in the Department of Statistics, 10th floor SSW, at any time before then.

Problem 1 (Directed sets)

Question (a): Let $\mathcal{X}$ be a set and $x \in \mathcal{X}$. Recall that a **neighborhood** of x is any set $A \subset \mathcal{X}$ which contains x. Let $\mathbb{T}$ be the set of all neighborhoods of a fixed point x, ordered by reverse inclusion, i.e. $A \preceq B$ iff $A \supset B$. Show that $(\mathbb{T}, \preceq)$ is a directed set.

Question (b): Let $(\mathbb{T}_1, \preceq_1)$ and $(\mathbb{T}_1, \preceq_1)$ be directed sets. Show that the Cartesion product $\mathbb{T}_1 \times \mathbb{T}_2$ is directed in the partial order defined by $(s_1, s_2) \preceq (t_1, t_2)$ iff $s_1 \preceq_1 t_1$ and $s_2 \preceq_2 t_2$.

Problem 2 (A martingale indexed by partitions)

Let $(\Omega, \mathcal{A})$ be a measurable space. A **finite measurable partition** $s = (A_1, \dots, A_n)$ of Ω is a subdivision of Ω into a finite number of disjoint measurable sets A_i whose union is Ω. We say that a partition $t = (B_1, \dots, B_m)$ is a **refinement** of another partition $s = (A_1, \dots, A_n)$ if every set B_j in t is a subset of some set A_i in s; in words, t can be obtaine from s by splitting sets in s further, without changing any of the existing set boundaries in s.

Let $\mathbb{T}$ be the set of all finite measurable partitions of Ω, and defined as binary relation $\preceq$ as

$$s \preceq t \qquad \Leftrightarrow \qquad t \text{ is a refinement of } s\ .$$

Question (a): Show that $\preceq$ is a partial order on $\mathbb{T}$.

Question (b): Show that the partially ordered set $(\mathbb{T}, \preceq)$ is directed.

Later on in the lecture, we will use this construction to prove the Radon-Nikodym theorem on the existence of densities. We anticipate a part of the proof in this problem (you can find the proof in Chapter 1.9 of the class notes, but you are *not* required to read ahead to solve this problem). The proof idea is to "discretize" the density f of a measure μ with respect to a probability measure P on finite partitions s as above. To this end, let $s \in \mathbb{T}$, so s is of the form $s = (A_1, \dots, A_n)$ for some $n \geq 2$. Define a finite σ-algebra

$$\mathcal{F}_s := \sigma(s) = \sigma(A_1, \dots, A_n)\ .$$

Now let μ be a measure and P a probability measure, both defined $(\Omega, \mathcal{A})$. For each s, we define the function

$$Y_s(x) := \sum_{j=1}^{n} f_s(A_j)\mathbb{I}_{A_j}(x) \qquad \text{where } f_s(A_j) := \begin{cases} \frac{\mu(A_j)}{P(A_j)} & P(A_j) > 0 \\ 0 & P(A_j) = 0 \end{cases}\ .$$

Note that Y_s is a real-valued, measurable function defined on a probability space $(\Omega, \mathcal{A}, P)$, and hence a real-valued random variable (even though it may not seem particularly random).

Question (c): Show that $(Y_s, \mathcal{F}_s)_{s \in \mathbb{T}}$ is a martingale.

Problem 3 (A martingale workout)

Let $(\mathbb{T}, \preceq)$ be a directed set and $\mathcal{F} = (\mathcal{F}_s)_{s \in \mathbb{T}}$ a filtration. For each $i = 1, \ldots, n$, let $(X_s^i, \mathcal{F}_s)_{s \in \mathbb{T}}$ be a martingale.

Question: Show that $(\max_{i \leq n} X_s^i, \mathcal{F}_s)$ is a submartingale.

Problem 4 (...and another one.)

Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be a supermartingale, and assume there is a random variable X_∞ and a σ-algebra $\mathcal{F}_\infty$ such that $(X_n, \mathcal{F}_n)_{n \in \mathbb{N} \cup \{\infty\}}$ is again a supermartingale.

Question: Show that $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ converges almost surely, and that $\lim X_n =_{\text{a.s.}} X_\infty$.

Hint: Show that $Y_n := X_n - \mathbb{E}[X_\infty | \mathcal{F}_n]$ defines a positive supermartingale on the same filtration, and apply a suitable result from Probability I, Chapter 27, to (Y_n).