Peter Orbanz
porbanz@stat.columbia.edu
Morgane Austern
ma3293@columbia.edu

Probability Theory II (G6106)
Spring 2015
http://stat.columbia.edu/∼porbanz/G6106S15.html

Homework 2

Due: 11 February 2014

Homework submission: We will collect your homework **at the beginning of class** on the due date. If you cannot attend class that day, you can leave your solution in my postbox in the Department of Statistics, 10th floor SSW, at any time before then.

Problem 1 (Uniform integrability)

Let $\mathcal{X}$ be a family of non-negative random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying $\sup_{X \in \mathcal{X}} \mathbb{E}[X] < \infty$. Consider the following criterion:

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \qquad \sup_{X \in \mathcal{X}} \int_A X d\mathbb{P} \leq \varepsilon \qquad \text{for all } A \in \mathcal{A} \text{ with } \mathbb{P}(A) \leq \delta \,. \tag{1}$$

Question (a): Show that uniform integrability of $\mathcal{X}$ implies (1).

Question (b): Show that (1) implies uniform integrability of $\mathcal{X}$.

Problem 2 (Tail bounds)

Let $X_1, X_2, \dots$ be zero-mean, i.i.d. random variables with values in $[-1, 1]$, and define $T_n := \sum_{i=1}^n X_i$.

Question (a): Use the Azuma-Hoeffding inequality to prove the Hoeffding-like bound

$$\mathbb{P}(|T_n| \geq \varepsilon) \leq 2 \exp\Big(-\frac{\varepsilon^2}{2n}\Big) \,.$$

Question (b): Show that $\mathbb{P}(|T_n| \geq \delta_n)$ converges for $\delta_n := x\sqrt{n}$ (using the central limit theorem), and compare the limit you obtain to the bound in Question (a).

Problem 3 (Potentials)

Let $(X_n, \mathcal{F}_n)$ be a positive, discrete-time supermartingale.

Question (a): Show that $\lim_n \mathbb{E}[X_n] = 0$ implies $X_n \to 0$ almost surely and in $\mathbf{L}_1$.

Supermartingales with this property are sometimes called *potentials*.

Problem 4 (Recall your calculus class)

The purpose of this problem is to recall some very basic facts from analysis in $\mathbb{R}^m$, whose more general counterparts in metric spaces we will encounter in the coming weeks. We consider a function $f : \mathbb{R}^m \to \mathbb{R}$. We use $\lim_n f(x_n) = f(x)$ (for all convergent sequences $x_n \to x$ as the definition of continuity, and denote the Euclidean distance between x and y by $d(x, y)$. Recall that the preimage of a set $B \in \mathbb{R}$ under f is $f^{-1}(B) := \{x \in \mathbb{R}^m \mid f(x) \in B\}$.

Question (a): Show that f is continuous if and only if the preimage $f^{-1}(B)$ of every open set $B \subset \mathbb{R}$ is open.

Question (b): Recall that a subset A is called *compact* if every sequence consisting of points in A has a convergent subsequence with limit in A. Show that a set in $\mathbb{R}^m$ is compact iff it is closed and its diameter $\sup_{x,y \in A} d(x,y)$ is finite.

Question (c): Show that continuous images of compact sets are compact, but continuous preimages of compact sets need not be compact (both for functions $\mathbb{R}^m \to \mathbb{R}$).

Question (d): Show that the alternative distance functions

$$d'(x,y) := \sum_{i=1}^{m} |x_i - y_i| \qquad \text{and} \qquad d''(x,y) := \max_{i \leq m} |x_i - y_i|$$

are equivalent to d, in the sense that any sequence converges with respect to one of the three distances if and only if it converges with respect to all three.

Note that, if we replace vectors with d entries by infinite sequences in $\mathbb{R}$, the functions d, d' and d'' in the last question turn into the metrics ℓ_1, ℓ_2 and ℓ_∞, which are *not* equivalent.