Probability Theory II (G6106)
Spring 2015
http://stat.columbia.edu/∼porbanz/G6106S15.html

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Homework 3

Due: 25 February 2014

Homework submission: We will collect your homework **at the beginning of class** on the due date. If you cannot attend class that day, you can leave your solution in my postbox in the Department of Statistics, 10th floor SSW, at any time before then.

Problem 1 (Warm-up)

Question: Show that every metrizable topological space is a Hausdorff space.

Problem 2 (Metric topologies)

Let $\mathcal{X}$ be a set and d a metric on $\mathcal{X}$. We had defined the metric topology on $\mathcal{X}$ as

$$\tau := \{A \subset \mathcal{X} \,|\, \overline{A} \text{ closed with respect to } d\} \;.$$

Question: Show that the set system τ above satisfies the properties of a topology, as required in Definition 2.1.

Problem 3 (Distance functions of sets are continuous)

For any subset A of a metric space $(\mathbf{X}, d)$, define the distance function of A as

$$d(x, A) := \inf_{y \in A} d(x, y) \;.$$

Question: Show that $x \mapsto d(x, A)$ is Lipschitz continuous with Lipschitz constant 1.

Problem 4 (Measures on the Borel sets are determined by open sets)

Lemma 2.7 states: If two measures defined on the Borel σ-algebra of $\mathbf{X}$ coincide on all open sets, or if they coincide on all closed sets, then they are identical.

Question: Please proof this results *without* assuming the result that $\cap$-stable generators determine measures; instead, use the monotone class theorem.

Problem 5 (Weak convergence of point masses)

Let P be a probability measure on a metrizable space $\mathbf{X}$, and (x_n) a sequence in $\mathbf{X}$ such that $\delta_{x_n} \xrightarrow{w} P$.

Question: Show that $P = \delta_x$ for some x.

Problem 6 (Tightness and countable additivity)

Let μ be a finitely additive set function on a measurable space $(\mathcal{X}, \mathcal{A})$, and finite (i.e. $\mu(\mathcal{X}) < \infty$).

Question (a): Show that μ is a measure (i.e. that it is countably additive) if and only if $\mu(A_n) \searrow 0$ whenever $A_n \searrow \emptyset$.

A family of sets $\mathcal{C} \subset \mathcal{A}$ is called a called a **compact class** if every sequence (C_n) of sets in $\mathcal{C}$ has the following property:

every finite subset of sets C_n in (C_n) has non-empty intersection $\quad \Rightarrow \quad$ (C_n) has non-empty intersection

The set function μ is called **tight** with respect to a compact class $\mathcal{C}$ if

$$\mu(A) = \sup\{\mu(C) | C \in \mathcal{C} \text{ and } C \subset A\} \; .$$

You will notice the similarity to the definition of inner regularity. Inner regularity is not a form of tightness, however, since the closed sets do not form a compact class. The prototypical compact class are the compact subsets of a Hausdorff space, and the term "tight measure", without further qualification, is usually used to refer to measures which are tight with respect to the compact sets.

Question (b): Show that, if μ is tight with respect to some compact class $\mathcal{C} \subset \mathcal{A}$, then it is countably additive on $\mathcal{A}$.

Note: Question (b) is a bit more difficult than the other problems in this homework.