Probability Theory II (G6106)
Spring 2015
http://stat.columbia.edu/∼porbanz/G6106S15.html

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Homework 4

Due: 6 March 2015 (Friday)

Homework submission: Please submit **either** in class on Wednesday 4 February **or** until Friday 6 March in my postbox in the Department of Statistics, 10th floor SSW.

Problem 1 (Products of Polish spaces)

Let $\mathbf{X}_n$, for $n \in \mathbb{N}$, be Polish spaces.

Question: Show that $\prod_n \mathbf{X}_n$ is Polish in the product topology.

Problem 2 (Measurable sets of continuous functions)

Let $\mathbf{C}([0,1])$ be the set of continuous functions $[0,1] \to \mathbb{R}$, equipped with the supremum norm metric

$$d_{\mathbf{C}}(f,g) := \sup_{x \in [0,1]} |f(x) - g(x)| \ .$$

The metric space $(\mathbf{C}([0,1]), d_{\mathbf{C}})$ is Polish. Let D be the (dense) subset $D := \mathbb{Q} \cap [0,1]$. Define the projection map at x as

$$\mathrm{pr}_x : f \mapsto f(x) \qquad x \in [0,1], f \in \mathbf{C}[0,1] \ .$$

Question: Show that the Borel σ-algebra on $(\mathbf{C}([0,1]), d_{\mathbf{C}})$ is the smallest σ-algebra which makes the family $\{\mathrm{pr}_x | x \in D\}$ of mappings measurable.

Problem 3 (The ball σ-algebra)

Let $\mathbf{X}$ be a metric space (not necessarily separable).

Question: Show that every closed set which has a dense countable subset is ball-measurable.

Hint: Define F^δ as we have in class and note $F = \bigcap_n F^{1/n}$ for any closed set F.

Problem 4 (The Lévy-Prokhorov metric deserves its name)

Let $\mathbf{X}$ be a metrizable space. For any two probability measures P and Q on $\mathbf{X}$, define the **Lévy-Prokhorov metric** as

$$d_{\text{LP}}(P,Q) := \inf\{\delta > 0 \,|\, P(A) \leq Q(A^\delta) + \delta \text{ for all } A \in \mathcal{B}(\mathbf{X})\} \ .$$

Question: Show that d_{LP} is indeed a metric on the set of probability measures on $\mathbf{X}$.

Note: We are only asking you to verify the properties of a metric, *not* that d_{LP} metrizes the weak topology.

Problem 5 (Evaluation maps are measurable)

Let $\mathbf{X}$ be a metrizable space and $\mathbf{PM}(\mathbf{X})$ the set of probability measures on $\mathbf{X}$, endowed with the weak topology. For every Borel set A in $\mathbf{X}$, we define the **evaluation map**

$$\phi_{\mathsf{A}} : \mathbf{PM}(\mathbf{X}) \to [0,1] \qquad \text{as} \qquad \phi_{\mathsf{A}}(\mu) := \mu(A)\ .$$

Question: Show that ϕ_{A} is Borel measurable for every $A \in \mathcal{B}(\mathbf{X})$.

You can use the following fact: If $\mathbf{X}$ is metrizable, then for every closed set F in $\mathbf{X}$ and any $r > 0$, the subset of $\mathbf{PM}(\mathbf{X})$ defined by

$$\{\mu | \mu(F) \geq r\}$$

is closed in $\mathbf{PM}(\mathbf{X})$. Similarly, the sets of the form

$$\{\mu | \mu(G) > r\}$$

are open.