Probability Theory II (G6106)

Spring 2015
http://stat.columbia.edu/∼porbanz/G6106S15.html

Peter Orbanz
porbanz@stat.columbia.edu

Morgane Austern
ma3293@columbia.edu

Homework 5

Due: 25 March 2015

Homework submission: We will collect your homework **at the beginning of class** on the due date. If you cannot attend class that day, you can leave your solution in my postbox in the Department of Statistics, 10th floor SSW, at any time before then.

Problem 1

For each n, define the σ-algebra

$$\mathcal{F}_n := \sigma\Big(\Big[\frac{i-1}{2^n}, \frac{i}{2^n}\Big] \,\Big|\, 1 \leq i \leq 2^n\Big) \tag{1}$$

and a random variable

$$X_n(\omega) := \begin{cases} 2^n & \text{if } \omega \leq 2^{-n} \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

1. Show that $(X_n, \mathcal{F}_n)$ is a martingale with $\sup_n \mathbb{E}[X_n] < \infty$.
2. Show that (X_n) does not converge in $\mathbf{L}_1$.

Problem 2 (Conditional probabilities define measures)

Let Y be a random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with values in a measurable space $(\mathcal{Y}, \mathcal{A}_Y)$. Recall that we define the conditional probability of a given set $A \in \mathcal{A}$ as

$$\mathbb{P}(A|Y = y) := \mathbb{E}[\mathbb{I}_A|Y = y] . \tag{3}$$

Question (a): Show that, for any $A \in \mathcal{A}$ and $C \in \mathcal{A}_Y$,

$$\mathbb{P}(A \cap \{Y \in C\}) = \int_C \mathbb{P}(A|Y = y)P_Y(dy) , \tag{4}$$

where P_Y denotes the law of Y.

Question (b): Show that, for any fixed value $y \in \mathcal{Y}$, the function $A \mapsto \mathbb{P}(A|Y = y)$ is P_Y-almost surely a probability measure on $(\Omega, \mathcal{A})$.

Problem 3 (Independence)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$ two sub-σ-algebras. We again use the definition

$$\mathbb{P}(A|\mathcal{C})(\omega) := \mathbb{E}[\mathbb{I}_A|\mathcal{C}](\omega) \tag{5}$$

for the conditional probability of A given $\mathcal{C}$. Show that the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ are independent if and only if

$$\forall B \in \mathcal{B} : \qquad \mathbb{P}(B|\mathcal{C}) = \mathbb{P}(B) . \tag{6}$$

Note: Recall the definition of independent σ-algebras from [Probability I, Chapter 10].

Problem 4 (Conditional densities)

Let X and Y be random variables with values $\mathbb{R}$, with joint law P, and let λ denote Lebesgue measure on $\mathbb{R}$. Let $\mathbf{p}$ be a version of the conditional distribution of X given Y, that is, $\mathbf{p}(A, y) = \mathbb{P}(X \in A | Y = y)$ almost surely. Suppose $f(x, y)$ is a density of the joint distribution P with respect to $\lambda \otimes \lambda$, and $f(y) := \int_{\mathbb{R}} f(x, y)\lambda(dx)$.

Question: Show that, if $f(y) > 0$ for all $y \in \mathbb{R}$,

$$f(x|y) := \frac{f(x, y)}{f(y)} \tag{7}$$

is a density of $\mathbf{p}(\bullet, y)$ with respect to λ for all y.