Probability Theory II (G6106)
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http://stat.columbia.edu/∼porbanz/G6106S15.html

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Homework 6

Due: 1 April 2015

Homework submission: We will collect your homework **at the beginning of class** on the due date. If you cannot attend class that day, you can leave your solution in my postbox in the Department of Statistics, 10th floor SSW, at any time before then.

Problem 1 (Compact classes)

Let $\mathbf{X}$ be a Hausdorff space. Let $\mathcal{K}$ be the set of all compact sets in $\mathbf{X}$.

Question: Show $\mathcal{K}$ is a compact class.

Problem 2 (Random variables contract under conditioning)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be a random variable in $L_2(\mathbb{P})$ (that is, whose $L_2(\mathbb{P})$-norm is $\|X\|_2 < \infty$).

Question: Show that $\|\mathbb{E}[X|\mathcal{C}]\|_2 \leq \|X\|_2$.

Remark: This is in fact true for every L_p-norm with $p > 1$, and is related to the well-known phenomenon that conditioning reduces variance (as e.g. in the Rao-Blackwell theorem), and more loosely to the fact that conditioning reduces entropy.

Problem 3 (Pairs of random variables)

Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$, where λ is Lebesgue measure.

Question: Give an example of real-valued random variables X, X' and Y on $[0,1]$ such that X and X' are identically distributed, but (X', Y) and (X, Y) are not.

Hint: This is every bit as easy as it seems—no horseshoes.

Problem 4 (Bayes' theorem)

You will have encountered Bayes' theorem before. In this problem, we ask you to prove the formal version of this result, using the existence theorem for conditional densities.

Question (a): Read Section 3.7 of the class notes; you will need Theorem 3.26.

The formal statement of the theorem is as follows:

Theorem 1 *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $\Theta : \Omega \to \mathbf{T}$ a random variable taking values in a Borel space $\mathbf{T}$, with law Q. Let $X_i : \Omega \to \mathbf{X}$, for $i \in \mathbb{N}$, be random variables with values in a Borel space $\mathbf{X}$, which are conditionally iid, that is: All X_i have identical conditional distribution*

$$\mathbf{p}(\bullet, \theta) =_{\text{a.s.}} \mathbb{P}[X \in \bullet | \Theta = \theta] , \tag{1}$$

where $\mathbf{p} : \mathbf{T} \to \mathbf{PM}(\mathbf{X})$ is a probability kernel, and the joint conditional distribution factorizes as

$$\mathbb{P}[X_1 \in A_1, \ldots, X_n \in A_n | \Theta = \theta] =_{\text{a.s.}} \prod_{i=1}^{n} \mathbf{p}(A_i, \theta) \ . \tag{2}$$

Require that there exists a σ-finite measure μ on $\mathbf{X}$ such that the absolute continuity relation

$$\mathbf{p}(\bullet, \theta) \ll \mu \qquad \text{for all } \theta \in \mathbf{T} \tag{3}$$

is satisfied. Then conditional distribution of Θ given $X_1, \ldots, X_n$ is given by

$$Q[d\theta | X_1 = x_1, \ldots, X_n = x_n] = \frac{\prod_{i=1}^{n} p(x_i|\theta)}{p(x_1, \ldots, x_n)} Q(d\theta) \ , \tag{4}$$

where $p(x|\theta)$ is the conditional density of $\mathbf{p}$ guaranteed by Theorem 3.26, and

$$p(x_1, \ldots, x_n) := \int_{\mathbf{T}} \prod_{i=1}^{n} p(x_i|\theta) Q(d\theta) \ . \tag{5}$$

Moreover, $\mathbb{P}\{p(X_1, \ldots, X_n) \in \{0, \infty\}\} = 0$.

Question (b): Prove Bayes' theorem.

Remark: Bayes' theorem is often stated in terms of densities as $p(\theta|x) = \frac{p(x|\theta)}{p(x)} p(\theta)$ (for $n = 1$), which is perfectly safe if, for example, X and Θ both take values in Euclidean space and have smooth distributions. In general, we have to be a bit more careful: Equation (4) above is a representation of the conditional law $\mathcal{L}(\Theta|X_1, \ldots, X_n)$ by a density with respect to $\mathcal{L}(\Theta)$. To ensure that such a density exists, we have to verify absolute continuity of $\mathcal{L}(\Theta|X_1, \ldots, X_n)$ with respect to $\mathcal{L}(\Theta)$. The theorem shows that whether this absolute continuity is satisfied depends only on the conditional distribution of X, via (3).

Problem 5 (Rejection Sampling)

Let P and Q be two probability measures on a countable discrete space $\mathbf{X}$. Suppose there is a constant $c > 0$ such that

$$f(x) := \frac{Q(\{x\})}{P(\{x\})} \leq c \qquad \text{for all } x \in \mathbf{X} \text{ with } P(\{x\}) > 0 \ . \tag{6}$$

Let $X_1, X_2, \ldots$ be i.i.d. random variables with law P, and $U_1, U_2, \ldots$ i.i.d. uniform variables in $[0, 1]$. Now define an integer random variable N as the smallest value of n such that

$$U_n \leq \frac{f(X_n)}{c} \ . \tag{7}$$

Question: Show that the random variable X_N has law Q.