Probability Theory II (G6106)

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http://stat.columbia.edu/∼porbanz/G6106S15.html

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Homework 8

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Homework submission: Please leave your solution in my postbox in the Department of Statistics, 10th floor SSW.

Problem 1 (Gaussian processes)

Let $\mathbb{T}$ be the set of finite subsets of $\mathbb{R}_+$, ordered by inclusion. Let $\mathbf{m} : \mathbb{R}_+ \to \mathbb{R}$ be a function and $\mathbf{k} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ a symmetric function (in the sense that $\mathbf{k}(u, v) = \mathbf{k}(v, u)$). Require that $\mathbf{k}$ has the property that, for every $t = \{u_1, \ldots, u_n\}$ of $\mathbb{R}_+$, the symmetric $n \times n$ matrix

$$\Sigma_t := \big(\mathbf{k}(u_i, u_j)\big)_{i,j \leq n} \tag{1}$$

is positive semidefinite. For every $t = \{u_1, \ldots, u_n\}$, define a probability measure P_t on $\mathbb{R}^t$ as the Gaussian distribution with mean vector $(\mathbf{m}(u_1), \ldots, \mathbf{m}(u_n))$ and covariance matrix Σ_t.

Question: Show that the family $(P_t)_{t \in \mathbb{T}}$ is projective with respect to product space projections.

Problem 2 (Pólya urns as inverse limits)

Fix two integers $n_0 \in \mathbb{N}$ and $w \in \{0, \ldots, n_0\}$. For each $n \in \mathbb{N}$, define $\mathbf{X}_n$ as the product space

$$\mathbf{X}_n := \prod_{j=0}^{n} \{0, \ldots, j + n_0\} \; . \tag{2}$$

Define a measure P_n on $\mathbf{X}_n$ as follows: Let P_0 be the measure on $\{0, \ldots, n\}$ with $P_0(w) = 1$. Let $\mathbf{p}_n$ be the probability kernel

$$\mathbf{p}_n(\{y\}, x) := \begin{cases} \frac{x}{n+n_0} & \text{if } y = x + 1 \\ 1 - \frac{x}{n+n_0} & \text{if } y = x \\ 0 & \text{otherwise} \end{cases} \; , \tag{3}$$

and define P_n by

$$P_{n+1}(dx_{n+1}, \ldots, dx_0) := \mathbf{p}_n(dx_{n+1}, x_n, \ldots, x_0) P_n(dx_n, \ldots, dx_0) \; . \tag{4}$$

Question (a): Show that the family (P_n) is projective with respect to product space projection.

By Theorem 5.15, the family (P_n) defines an inverse limit measure $P = \varprojlim(P_n)$. (Although we are in the product space case, I refer to Theorem 5.15 rather than Kolmogorov's extension theorem because the factors $\{0, \ldots, n + n_0\}$ differ for different values of n; otherwise, this is precisely the Kolmogorov setup.)

Question (b): Show that P describes the law of a Pólya urn which initially contains n_0 balls, of which w are white and $(n_0 - w)$ black (cf Chapter 1.8): If (X_n) is a sequence with law P, each element X_n is the number of white balls after n draws from the urn.

Problem 3 (Inverse limit constructions)

To get some practice with the inverse limit definition, we will show that continuous mappings can be constructed as inverse limits, similar to the construction of probability measures (though the proof is much easier than for probability measures). We consider two inverse limits, of families $(\mathbf{X}_t)$ and $(\mathbf{Y}_t)$ of spaces, indexed by the same index set.

Let $(\mathbb{T}, \preceq)$ be a directed index set. For each $t \in \mathbb{T}$, let $\mathbf{X}_t$ and $\mathbf{Y}_t$ be topological spaces, and whenever $s \preceq t$, let

$$\pi_{ts} : \mathbf{X}_t \to \mathbf{X}_s \qquad \text{and} \qquad \tau_{ts} : \mathbf{Y}_t \to \mathbf{Y}_s \tag{5}$$

be continuous, surjective mappings (i.e. they correspond to the mappings pr_{ts} in the class notes). Let π_t and τ_t be the respective mappings defined on the limit space by equation (5.4) in the class notes (corresponding to pr_t). Endow each of the two inverse limits space $\mathbf{X} := \varprojlim(\mathbf{X}_t)$ and $\mathbf{Y} := \varprojlim(\mathbf{Y}_t)$ with the respective inverse limit topologies.

Now suppose there are continuous mappings $f_t : \mathbf{X}_t \to \mathbf{Y}_t$ satisfying

$$f_s \circ \pi_{ts} = \tau_{ts} \circ f_t \qquad \text{whenever } s \preceq t\ . \tag{6}$$

Question: Show that there is a uniquely determined map $f : \varprojlim(\mathbf{X}_t) \to \varprojlim(\mathbf{Y}_t)$

$$f_t \circ \pi_t = \tau_t \circ f \qquad \text{for all } t \in \mathbb{T}\ , \tag{7}$$

and that f is again continuous.

Problem 4 (Transforming Brownian motion)

Let $(X_u)_{u \in \mathbb{R}_+}$ be Brownian motion.

Question (a): Show that, for any constant $c > 0$, $(c^{-1} X_{c^2 u})$ is Brownian motion.

Question (b): Show that

$$(1+u) X_{\frac{u}{1+u}} - u X_1 \qquad \text{for } u \in [0,1] \tag{8}$$

is Brownian motion on $[0,1]$.

Problem 5 (Brownian motion is not monotone)

Let $X = (X_u)_{u \in \mathbb{R}_+}$ be Brownian motion. Regard the paths of X as random functions $u \mapsto X_u$.

Question: Show that, for any interval $[s,t]$ with $s < t$, the path of X is almost surely not monotone on any interval $[s,t]$ with $s < t$.

Hint: Note a continuous function f is non-decreasing on an interval, say $[0,1]$, if and only if

$$f\Big(\frac{i+1}{n}\Big) - f\Big(\frac{i}{n}\Big) \geq 0 \qquad \text{for all } i < n \tag{9}$$

holds for every $n \in \mathbb{N}$. To quantify over all intervals, note that it is sufficient to consider only intervals of the form $[s,t]$ with $s, t \in \mathbb{Q}_+$.