Probability Theory II (G6106)
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http://stat.columbia.edu/∼porbanz/G6106S16.html

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Homework 3

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Problem 1

Let $X = (X_t)_{t\in\mathbb{N}}$ be a sequence of integrable random variables. Show that the Doob decomposition of X depends on the choice of filtration unless X is $\sigma(X_1)$-measurable almost surely.

Hint: Define $X'_t := (X_1, X_{t+1})$, and consider the filtrations $(\sigma(X_s, s \leq t))_{t\in\mathbb{N}}$ and $(\sigma(X'_s, s \leq t))_{t\in\mathbb{N}}$.

Problem 2

Give an example of an $\mathbf{L}_1$-bounded martingale that is not uniformly integrable.

Problem 3 (Recall your calculus class)

The purpose of this problem is to recall some very basic facts from analysis in $\mathbb{R}^m$, whose more general counterparts in metric spaces we will encounter in the coming weeks. We consider a function $f : \mathbb{R}^m \to \mathbb{R}$. We use $\lim_n f(x_n) = f(x)$ (for all convergent sequences $x_n \to x$) as the definition of continuity, and denote the Euclidean distance between x and y by $d(x, y)$. Recall that the preimage of a set $B \in \mathbb{R}$ under f is $f^{-1}(B) := \{x \in \mathbb{R}^m \mid f(x) \in B\}$.

Question (a): Show that f is continuous if and only if the preimage $f^{-1}(B)$ of every open set $B \subset \mathbb{R}$ is open.

Question (b): Recall that a subset A is called *compact* if every sequence consisting of points in A has a convergent subsequence with limit in A. Show that a set in $\mathbb{R}^m$ is compact iff it is closed and its diameter $\sup_{x,y\in A} d(x, y)$ is finite.

Question (c): Show that continuous images of compact sets are compact, but continuous preimages of compact sets need not be compact (both for functions $\mathbb{R}^m \to \mathbb{R}$).

Question (d): Show that the alternative distance functions

$$d'(x, y) := \sum_{i=1}^{m} |x_i - y_i| \qquad \text{and} \qquad d''(x, y) := \max_{i\leq m} |x_i - y_i|$$

are equivalent to d, in the sense that any sequence converges with respect to one of the three distances if and only if it converges with respect to all three.

Note that, if we replace vectors with d entries by infinite sequences in $\mathbb{R}$, the functions d, d' and d'' in the last question turn into the metrics ℓ_1, ℓ_2 and ℓ_∞, which are *not* equivalent.