Probability Theory II (G6106)

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http://stat.columbia.edu/∼porbanz/G6106S16.html

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Homework 4

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Problem 1 (Warm-up)

Question: Show that every metrizable topological space is a Hausdorff space.

Problem 2 (Metric topologies)

Let $\mathcal{X}$ be a set and d a metric on $\mathcal{X}$. We had defined the metric topology on $\mathcal{X}$ as

$$\tau := \{A \subset \mathcal{X} \,|\, \overline{A} \text{ closed with respect to } d\} \ .$$

Question: Show that the set system τ above satisfies the properties of a topology, as required in Definition 2.1.

Problem 3 (Distance functions of sets are continuous)

For any subset A of a metric space $(\mathbf{X}, d)$, define the distance function of A as

$$d(x, A) := \inf_{y \in A} d(x, y) \ .$$

Question: Show that $x \mapsto d(x, A)$ is Lipschitz continuous with Lipschitz constant 1.

Problem 4 (Measures on the Borel sets are determined by open sets)

Lemma 2.7 states: If two measures defined on the Borel σ-algebra of $\mathbf{X}$ coincide on all open sets, or if they coincide on all closed sets, then they are identical.

Question: Please prove this result *without* assuming the result that $\cap$-stable generators determine measures; instead, use the monotone class theorem.

Problem 5 (Weak convergence of point masses)

Let P be a probability measure on a metrizable space $\mathbf{X}$, and (x_n) a sequence in $\mathbf{X}$ such that $\delta_{x_n} \xrightarrow{w} P$.

Question: Show that $P = \delta_x$ for some x.

Problem 6 (Tightness and countable additivity)

Let μ be a non-negative, finitely additive set function on a measurable space $(\mathcal{X}, \mathcal{A})$, and finite (i.e. $\mu(\mathcal{X}) < \infty$).

Question (a): Show that μ is a measure (i.e. that it is countably additive) if and only if $\mu(A_n) \searrow 0$ whenever $A_n \searrow \varnothing$.

A family of sets $\mathcal{C} \subset \mathcal{A}$ is called a called a **compact class** if every sequence (C_n) of sets in $\mathcal{C}$ has the following property:

every finite subset of sets C_n in (C_n) has non-empty intersection $\quad \Rightarrow \quad$ (C_n) has non-empty intersection

The set function μ is **tight** with respect to a compact class $\mathcal{C}$ if

$$\mu(A) = \sup\{\mu(C) | C \in \mathcal{C} \text{ and } C \subset A\} \; .$$

You will notice the similarity to the definition of inner regularity. Inner regularity is not a form of tightness, however, since the closed sets do not form a compact class. The prototypical compact class are the compact subsets of a Hausdorff space, and the term "tight measure", without further qualification, is usually used to refer to measures which are tight with respect to the compact sets.

Question (b): Show that, if μ is tight with respect to some compact class $\mathcal{C} \subset \mathcal{A}$, then it is countably additive on $\mathcal{A}$.

Note: Question (b) is a bit more difficult than the other problems in this homework.