Assignment 6
Theoretical Neuroscience

Due November 24, 2006

1. **Stability of equilibria**

   Consider Wilson-Cowan equations of the form

$$\begin{aligned} \tau\dot{\nu}_E &= \phi_E(\nu_E,\nu_I) - \nu_E && \text{(1a)} \\ \tau\dot{\nu}_I &= \phi_I(\nu_I,\nu_E) - \nu_I && \text{(1b)} \end{aligned}$$

   where the gain functions, ϕ_E and ϕ_I, are increasing functions of ν_E and decreasing functions of ν_I (e.g, $\phi_E(\nu_E,\nu_I) \sim 1 + \tanh(W_{EE}\nu_E - W_{EI}\nu_I + \theta_E)$).

   Nullclines for Eq. (1) are sketched in the figure below. Show that equilibria A and C are unstable, D is stable, and B may or may not be stable. Give conditions for the stability of equilibrium B in terms of the derivatives of the gain functions evaluated at the equilibrium.

   Hint: This problem is relatively hard, in the sense that it requires a somewhat deep understanding of nullclines and their construction, and also strong familiarity with linear stability analysis in two dimensions. On the other hand, the answer doesn't require huge amount of algebra – only a few lines. The main insight you need is that you can compute the slopes of the nullclines in terms of derivatives of the gain functions. Once you do that, the rest should be easy (ish).

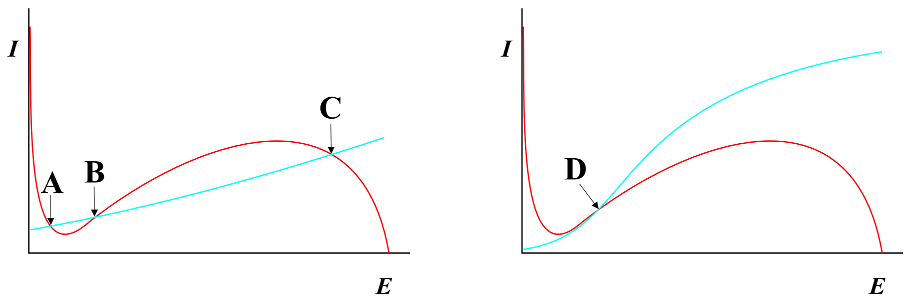

Figure 1: Two possible sets of nullclines. In both figures, the red curve is the excitatory nullcline and the blue curve is the inhibitory one.

2. **Binary weights**

Assume a large network of interconnected neurones with weights that take two values,

$$\nu_i = \phi\left(\overline{\nu} + \sum_{j=1}^{N} W_{ij}\nu_j\right) \tag{2a}$$

$$W_{ij} = \frac{\sigma}{\sqrt{Np(1-p)}} \begin{cases} 1-p & \text{with probability } p \\ p & \text{with probability } 1-p \end{cases} \tag{2b}$$

where N is the number of neurons, $\overline{\nu} \equiv N^{-1}\sum_i \nu_i$, and below you will also need $\overline{\nu^2} \equiv N^{-1}\sum_i \nu_i^2$.

2a. Write down the mean field equations for $\overline{\nu}$ and $\overline{\nu^2}$.

2b. Let $\phi(x) = \nu_0\Theta(x - \nu_0\theta_0)$ where $\Theta(x)$ is the Heaviside function and $\nu_0 > 0$. Using the mean field equations from part 2a, solve for $\overline{\nu^2}$ in terms of $\overline{\nu}$, and then find a single equation for $\overline{\nu}$. This equation should have the form

$$\overline{\nu} = f(\overline{\nu})\,.$$

Sketch $f(\overline{\nu})$ for $\theta_0 > 0$ and $\theta_0 < 0$.

2c. Is it possible to have three equilibria when $\theta_0 < 0$?

2d. Compute, and sketch, $P(\nu)$, the probability distribution of the firing rates, when $\theta_0 > 0$. Choose any θ_0 you want, but be sure to let us know what it is.