3. Fourier transforms and the symmetric group
1. \( G \) acts on a vector space \( U \) via the linear operators
\[
T_g : U \to U \quad g \in G
\]
which must satisfy \( T_{g_1} T_{g_2} = T_{g_1 g_2} \).

2. Equivalent to a system of matrices \( \rho : G \to \mathbb{C}^{d \times d} \) satisfying
\[
\rho(g_1) \rho(g_2) = \rho(g_1 g_2).
\]

3. Notion of equivalence and reducibility
\[
\rho_1(x) = T^{-1} \rho_2(x) T \quad \rho(x) = T^{-1} \begin{pmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{pmatrix} T
\]
leads to complete set of inequivalent irreducible representations \( \mathcal{R} \).

4. Any \( G \)-module reduces in the form
\[
U = W_1 \oplus W_2 \oplus \ldots \oplus W_k
\]
\[
T_g(f) = [T_g]_1 ([f]_1) \oplus \ldots \oplus [T_g]_k ([f]_k)
\]
1. \( G = \mathbb{R} \) \( \rho_k(x) = e^{ikx} \)

2. \( G = SO(3) \)

\[ [\rho_l(\theta, \phi, \psi)]_{m,m'} = e^{-il\psi} Y_{lm}(\theta, \phi) \quad l = 2k + 1 \]
5. A specific type of $G$–module:

$$U = L(X) = \{ f: X \to \mathbb{C} \}$$

where $G$ acts on $X$ by $x \mapsto g(x)$, and by extension

$$f \mapsto f^g \quad f^g(x) = f(g^{-1}(x)).$$

6. Now what about taking $X = G$ and $g_1(g_2) = g_1g_2$?

$$f \mapsto f^y \quad f^y(x) = f(y^{-1}x).$$
Transformations in Quantum Mechanics:

\[ |\psi\rangle \mapsto U(g)|\psi\rangle \quad g \in G \]

unitary representation of \( G \)

symmetry group \( \rightarrow \) embed in \( SU(n) \) \( \rightarrow \) look at irreps
Example: spin

\[
\text{SO}(3) \hookrightarrow \text{SU}(2) \quad \text{double cover!}
\]

\[
\text{SU}(2) \quad \text{has one irrep for each} \quad d \in \mathbb{N}
\]

Fermions:

\[
d = 2s \quad m = -(s - 1/2)\hbar, -(s - 3/2)\hbar, \ldots, (s - 1/2)\hbar
\]

Bosons:

\[
d = 2s + 1 \quad m = -s\hbar, -(s - 1)\hbar, \ldots, s\hbar
\]
back to Fourier transforms
$\sqrt{\heartsuit} = {?}$  $\cos\heartsuit = {?}$

$\frac{d}{dx} \heartsuit = {?}$  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \heartsuit = {?}$

$F\{\heartsuit\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{it\heartsuit} dt = {?}$

*My normal approach is useless here.*
$G$ acts on $L(G)$ by $f \mapsto f^y$, $f^y(x) = f(y^{-1}x)$. 
Theorem (Peter-Weyl). The decomposition of $L(G)$ into irreducible $G$-modules contains each irreducible $G$-module with multiplicity equal to its degree.

\[ L(G) = \bigoplus_{\rho \in \mathcal{R}} \bigoplus_{i=1}^{d_{\rho}} W_{\rho} \]
The **Fourier transform** on a group is

\[ \hat{f}(\rho) = \sum_{x \in G} f(x) \rho(x) \quad \rho \in \mathcal{R} \]
\[ \hat{f}(\rho) = \sum_{x \in G} f(x) \rho(x) \quad f(x) = \frac{1}{|G|} \sum_{\rho \in \mathcal{R}} d_\rho \text{tr} \left[ \hat{f}(\rho) \rho(x^{-1}) \right] \]

1. Linearity: \[ \hat{f} + \hat{g} = \hat{f} + \hat{g} \]

2. Unitarity: \[ \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \]

3. Left-translation: \[ \hat{f}^z(\rho) = \rho(z) \hat{f}(\rho) \]

4. Convolution: \[ \hat{f} \ast \hat{g}(\rho) = \hat{f}(\rho) \hat{g}(\rho) \]

5. The individual components correspond to different levels of smoothness.
What about the isotypics?

Recall the group algebra \( \mathbb{C}G \):

\[
f \cdot g = f \ast g \quad f \ast g(x) = \sum_{y \in G} f(xy^{-1}) g(y)
\]

The isotypics are just the irreducible sub-algebras.

\[
\mathbb{C}G = \bigoplus_{\rho \in \mathcal{R}} \text{GL}(\mathbb{C}^{d_{\rho}})
\]
\[ CG = V_1 \oplus V_2 \oplus \ldots \oplus V_k \]

isotypics

\[ V_i = W_1 \oplus W_2 \oplus \ldots \oplus W_{d_{\rho_i}} \]

irreducible G-modules
Part 2

The symmetric group
$S_n$ is the group of bijections

$$\sigma: \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$$

under composition of maps.

Clearly, $|S_n| = n!$. 
\[ \sigma(1) = 3 \]
\[ \sigma(2) = 2 \]
\[ \sigma(3) = 6 \]
\[ \sigma(4) = 5 \]
\[ \sigma(5) = 4 \]
\[ \sigma(6) = 1 \]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 2 & 6 & 5 & 4 & 1
\end{pmatrix}
\]
Cycle notation

Example:

\[ \sigma = (163)(45)(2) \]

Cycle type:

\[ \sigma = (3, 2, 1) \]
Generators

Transpositions \((i, j)\) generate the whole group.

In fact, adjacent transpositions are sufficient, since (assuming \(i < j\))

\[
(i, j) = (i, i + 1) \ldots (j - 2, -1)(j - 1, j) \ldots (i + 1, i + 2)(i, i + 1)
\]
Subgroups

Cayley’s theorem:

Any finite group \( G \) is a subgroup of \( S_{|G|} \).
Subgroups

\[ S_k < S_n \] permutes \( \{1, 2, \ldots, k\} \)

\[ S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_k} < S_n \] permutes \( \{1, 2, \ldots, \lambda_1\}, \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}, \ldots, \{n - \lambda_k, \ldots, n\} \)
Normal subgroups

\[ A_n = \{ \sigma \in S_n \mid \text{sgn} (\sigma) = 1 \} \]

For \( n \geq 5 \) this is the only one!