

Mini-course on representation theoretical methods in ML

Lecture 1: Groups

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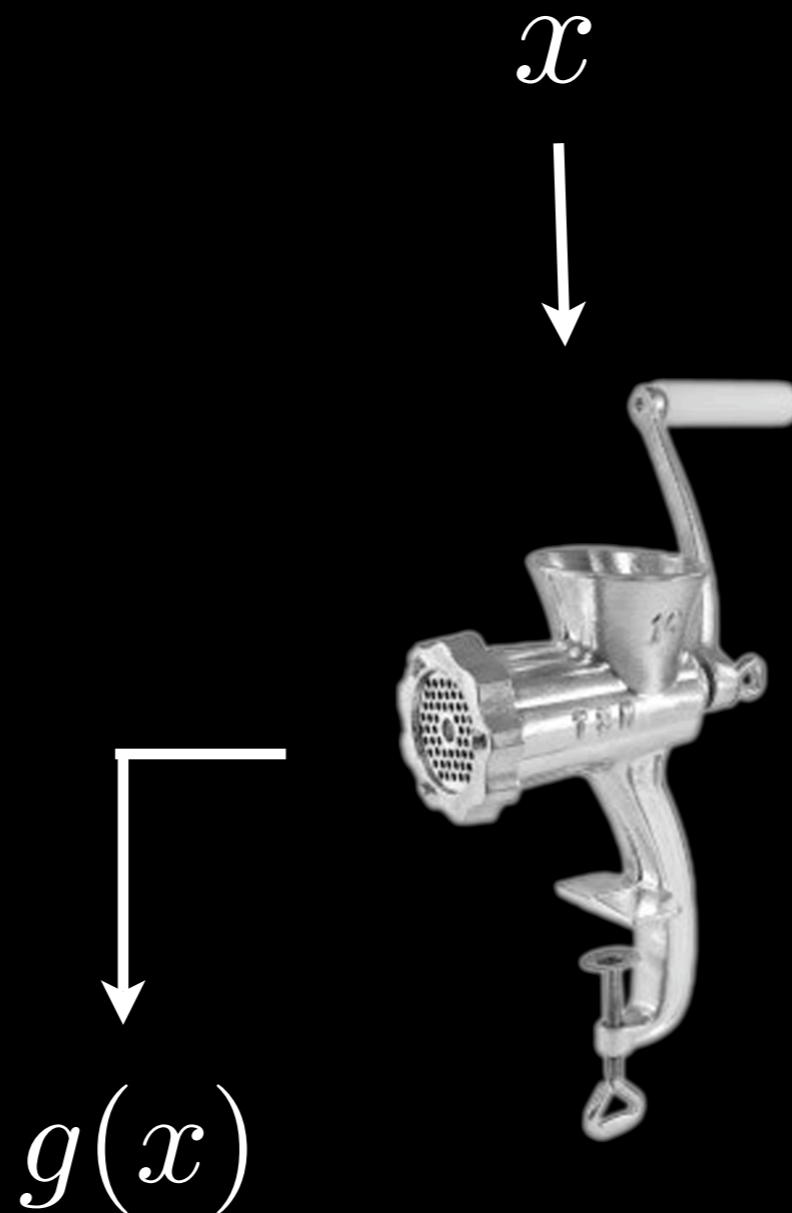
What is a group?

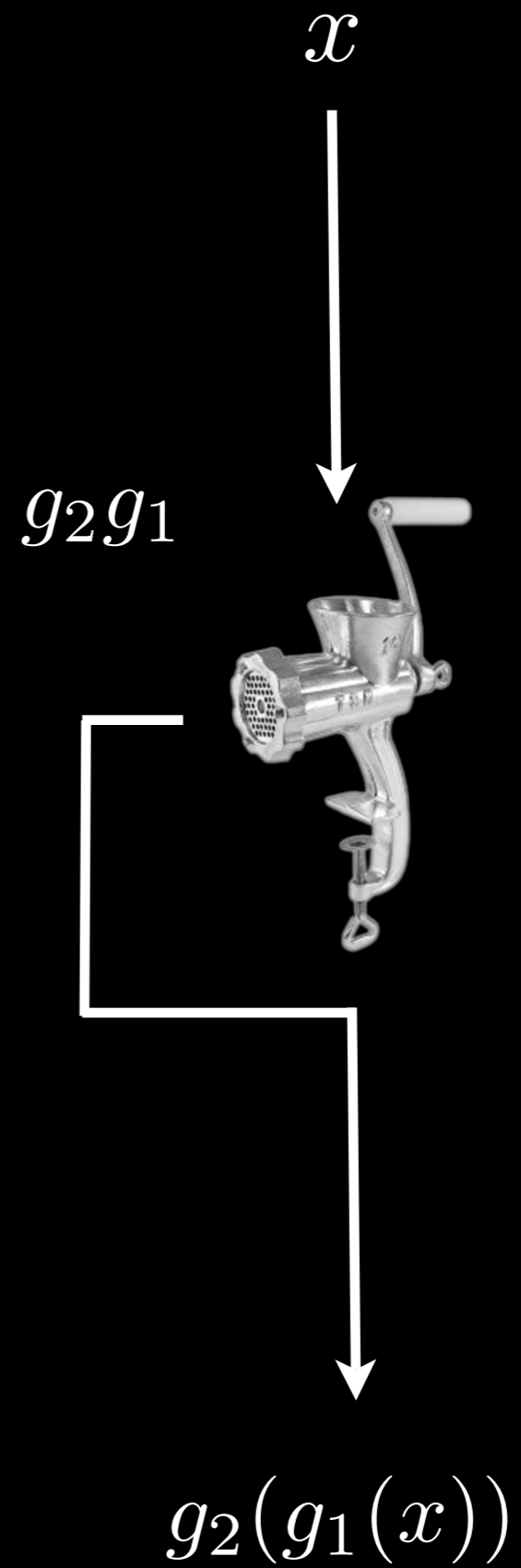
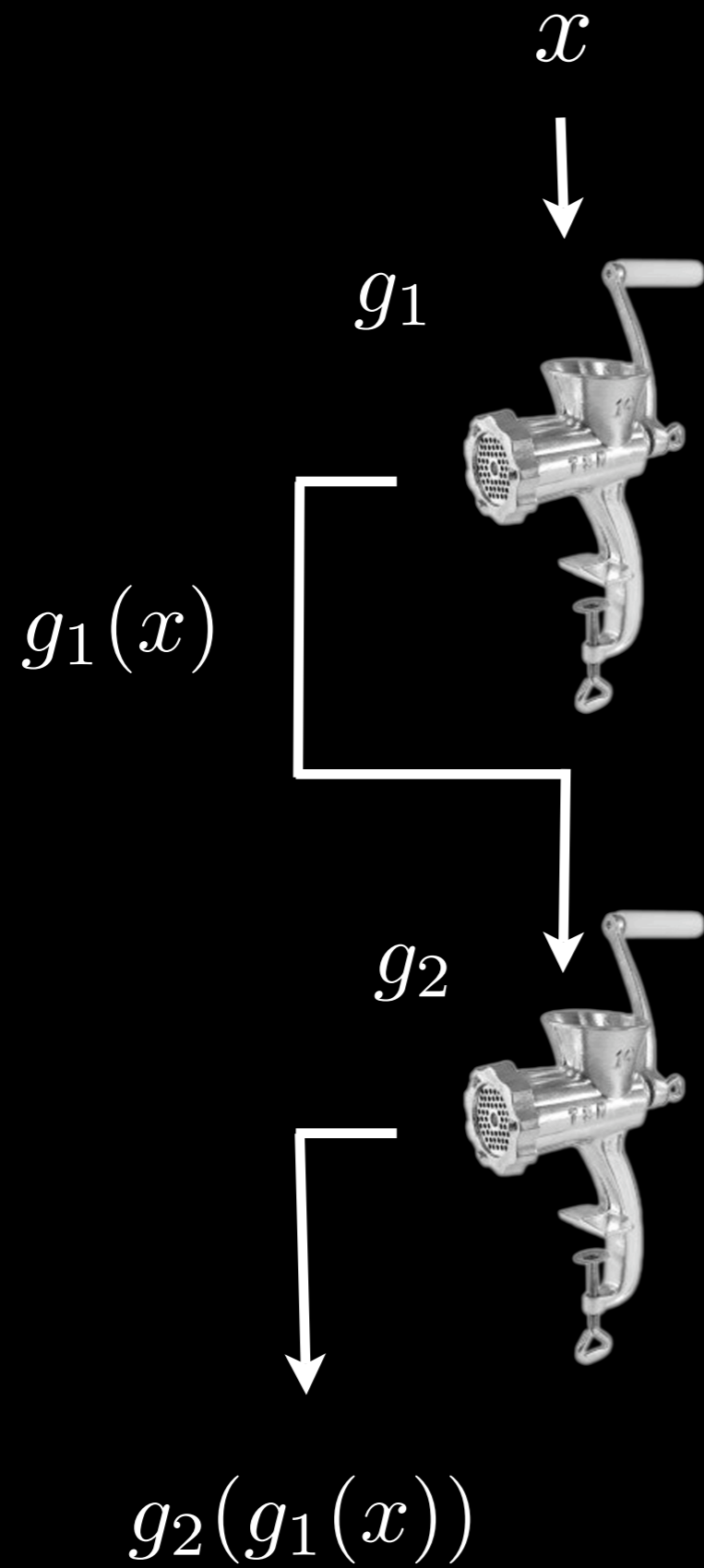
G is a **group** if for any $x, y, z \in G$

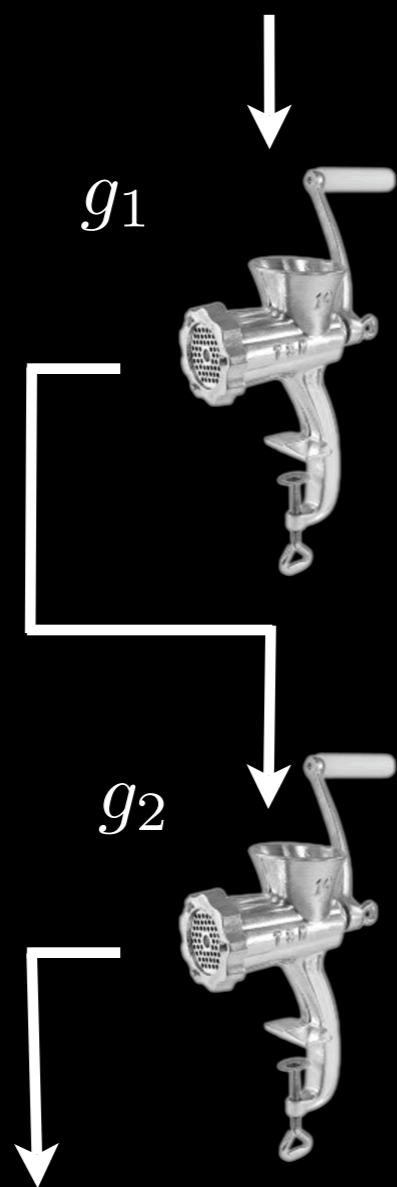
1. $xy \in G$,
2. $x(yz) = (xy)z$,
3. there is an $e \in G$ such that $ex = xe = x$,
4. there is an $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e$.

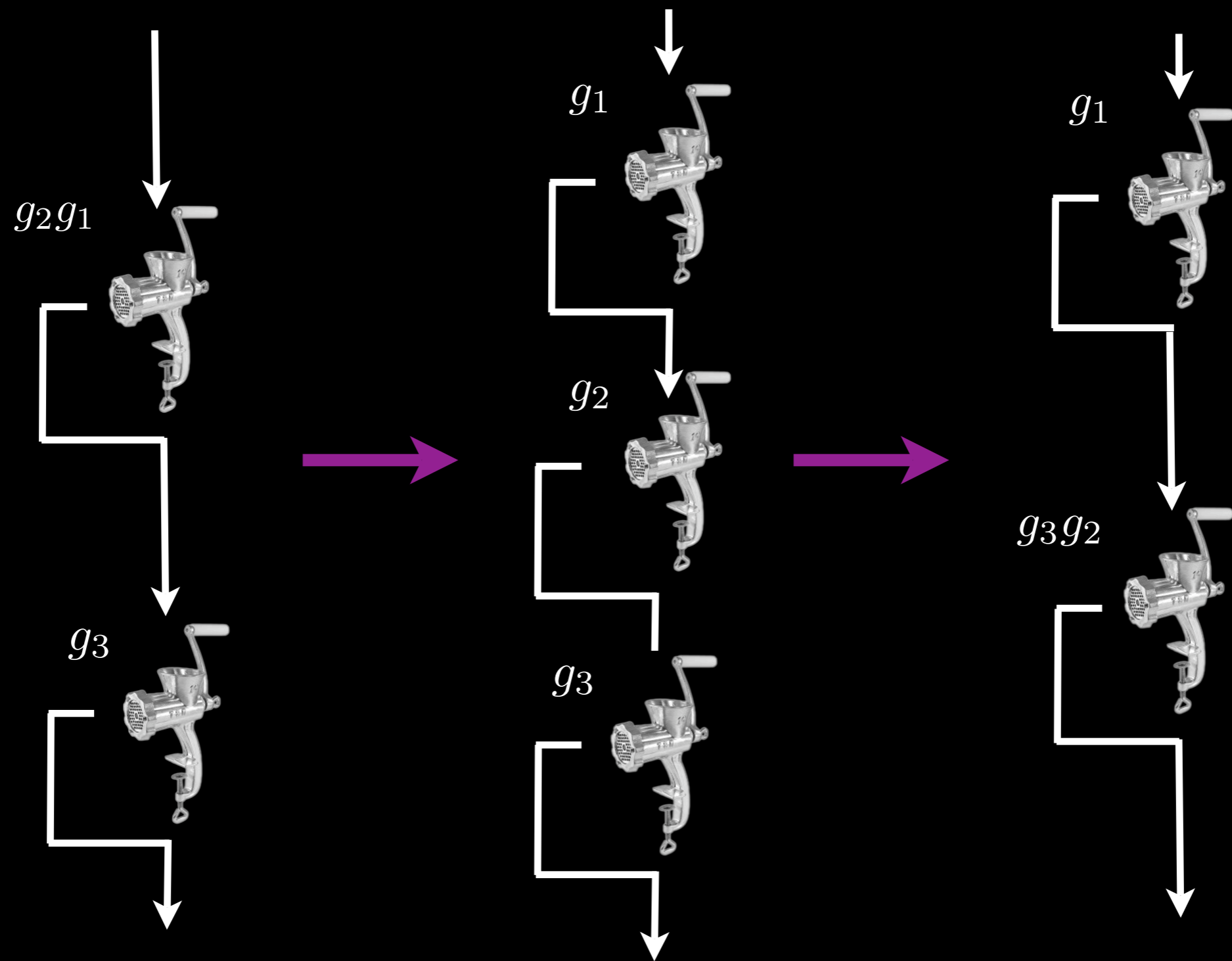
Why should we care?

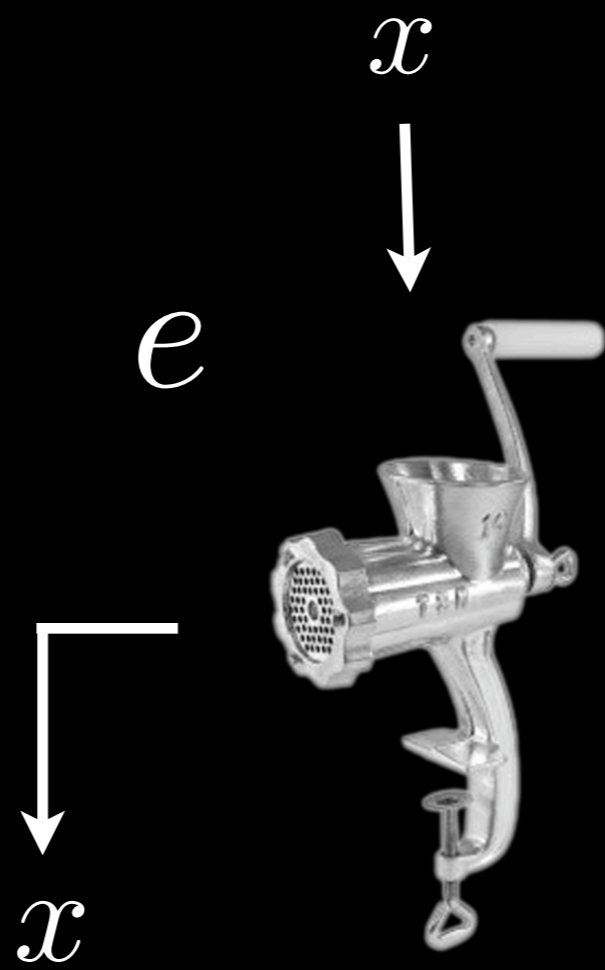
$$g: \mathcal{X} \rightarrow \mathcal{X}$$

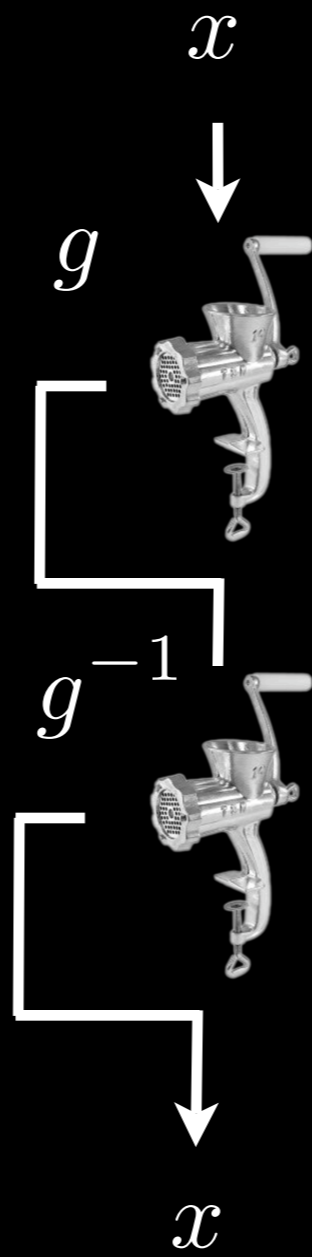












The **cyclic group** $\mathbb{Z}_n = \{0, 1, 2, \dots, n - 1\}$

$$xy = x + y \pmod n$$

Klein's Viergruppe $V = \{1, i, j, k\}$

	1	i	j	k
1	1	i	j	k
i	i	1	k	j
j	j	k	1	i
k	k	j	i	1

$$V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

The **quaternion** group $Q = \{1, i, j, k, -1, -i, -j, -k\}$

	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

$$-1^2 = 1$$

$$(-1)a = a(-1) = -a$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = k$$

The icosahedron group $I_h \cong A_5$



The **symmetric groups** S_n

group of bijections

$$\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$$

i.e., permutations of n objects

The integers \mathbb{Z}

$$xy = x + y$$

The reals \mathbb{R} and the Euclidean vector spaces \mathbb{R}^n

$$xy = x + y$$

The **rotation groups** $SO(n)$

group of $n \times n$ orthogonal matrices of det 1

The Euclidean group $ISO(n)$ and group of rigid body motions $ISO^+(n)$



Erlangen program (1872):

“geometry is the study of properties invariant under a group”

The **special unitary groups** $SU(n)$

group of $n \times n$ unitary matrices of determinant 1

The **general linear group** $\text{GL}(n)$

group of $n \times n$ invertible matrices



$$G = (\mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}) \rtimes ((A_8 \times A_{12}) \rtimes \mathbb{Z}_2)$$

The **Monster** group M

$$|M| = 8080174247945128758864599049617 \dots$$

$$\dots 107570057543680000000000$$

h	#	Abelian	#	non-Abelian	total
1	1	$\langle e \rangle$	0	-	1
2	1	C_2	0	-	1
3	1	C_3	0	-	1
4	2	$C_4, C_2 \times C_2$	0	-	2
5	1	C_5	0	-	1
6	1	C_6	1	D_3	2
7	1	C_7	0	-	1
8	3	$C_8, C_2 \times C_4, C_2 \times C_2 \times C_2$	2	D_4, Q_8	5
9	2	$C_9, C_3 \times C_3$	0	-	2
10	1	C_{10}	1	D_5	2
11	1	C_{11}	0	-	1
12	2	$C_{12}, C_2 \times C_6$	3	A_4, D_6, T	5
13	1	C_{13}	0	-	1
14	1	C_{14}	1	D_7	2
15	1	C_{15}	0	-	1
16	5	$C_{16}, C_8 \times C_2, C_4 \times C_4, C_4 \times C_2 \times C_2, C_2 \times C_2 \times C_2 \times C_2$	9	$D_8, D_4 \times C_2, Q \times C_2, G_{16}^{(4)}, G_{16}^{(5)}, G_{16}^{(6)}, G_{16}^{(7)}, G_{16}^{(8)}, G_{16}^{(9)}$	14
17	1	C_{17}	0	-	1

Finite groups \mathbb{Z}_n V Q M S_n

Infinite groups

Countable groups \mathbb{Z}

Continuous groups

Lie groups

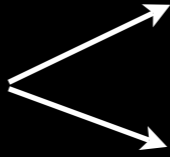
compact $SO(n)$ $SU(n)$

non-compact \mathbb{R}^n $ISO^+(n)$

Closed fields

Fields

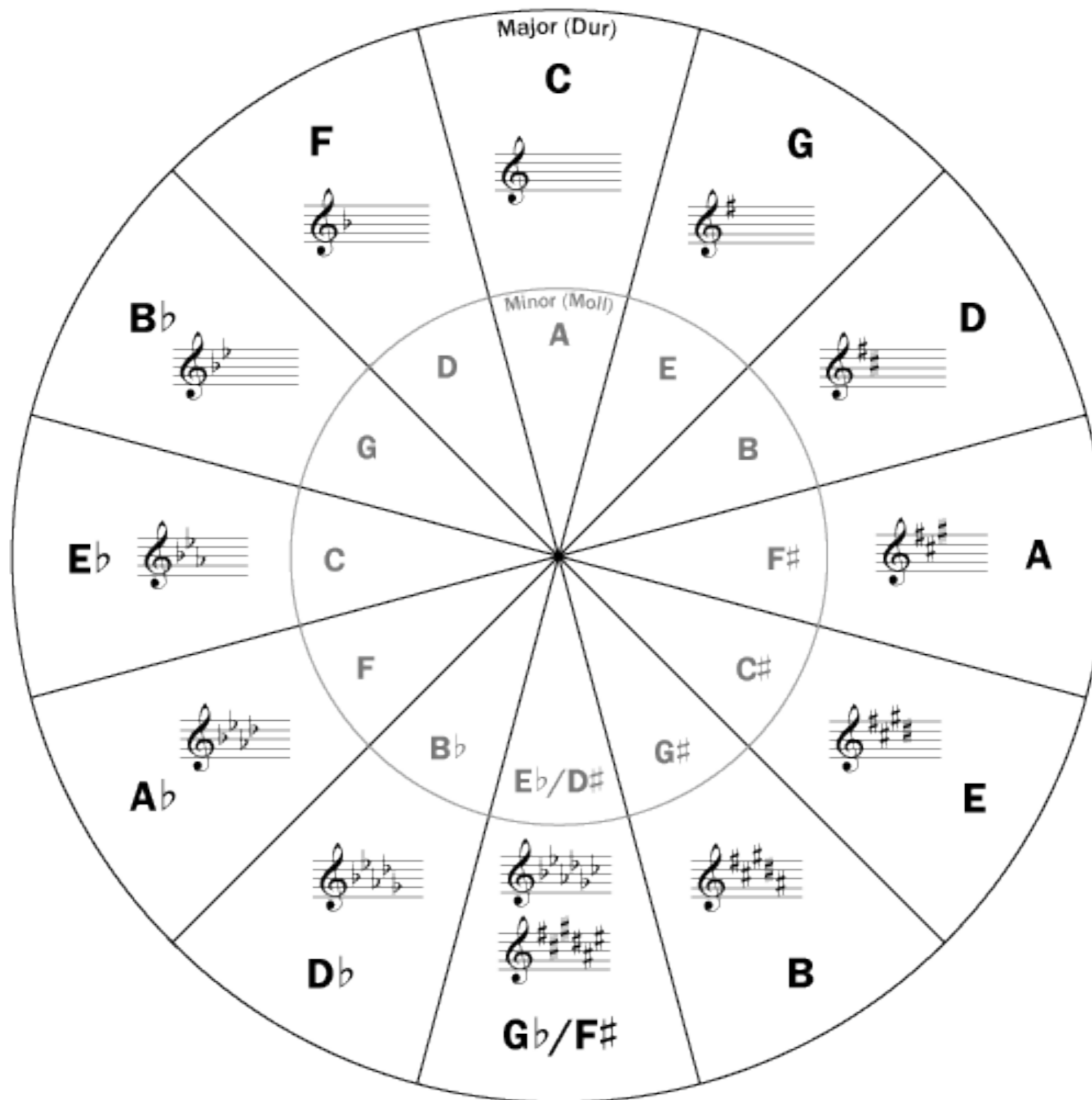
Rings

Groups  commutative
non-commutative

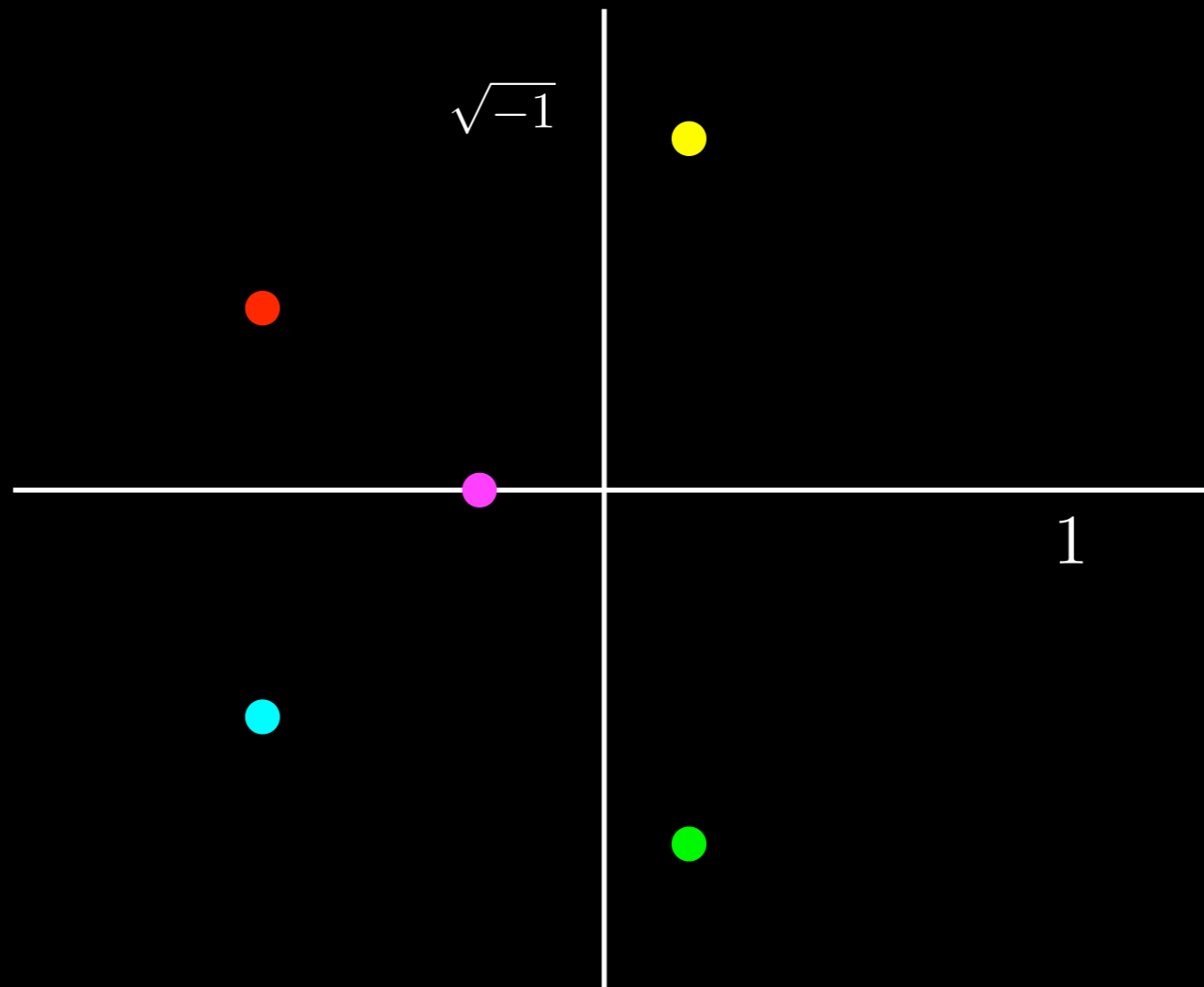
Semigroups

Applications

\mathbb{Z}_{12}

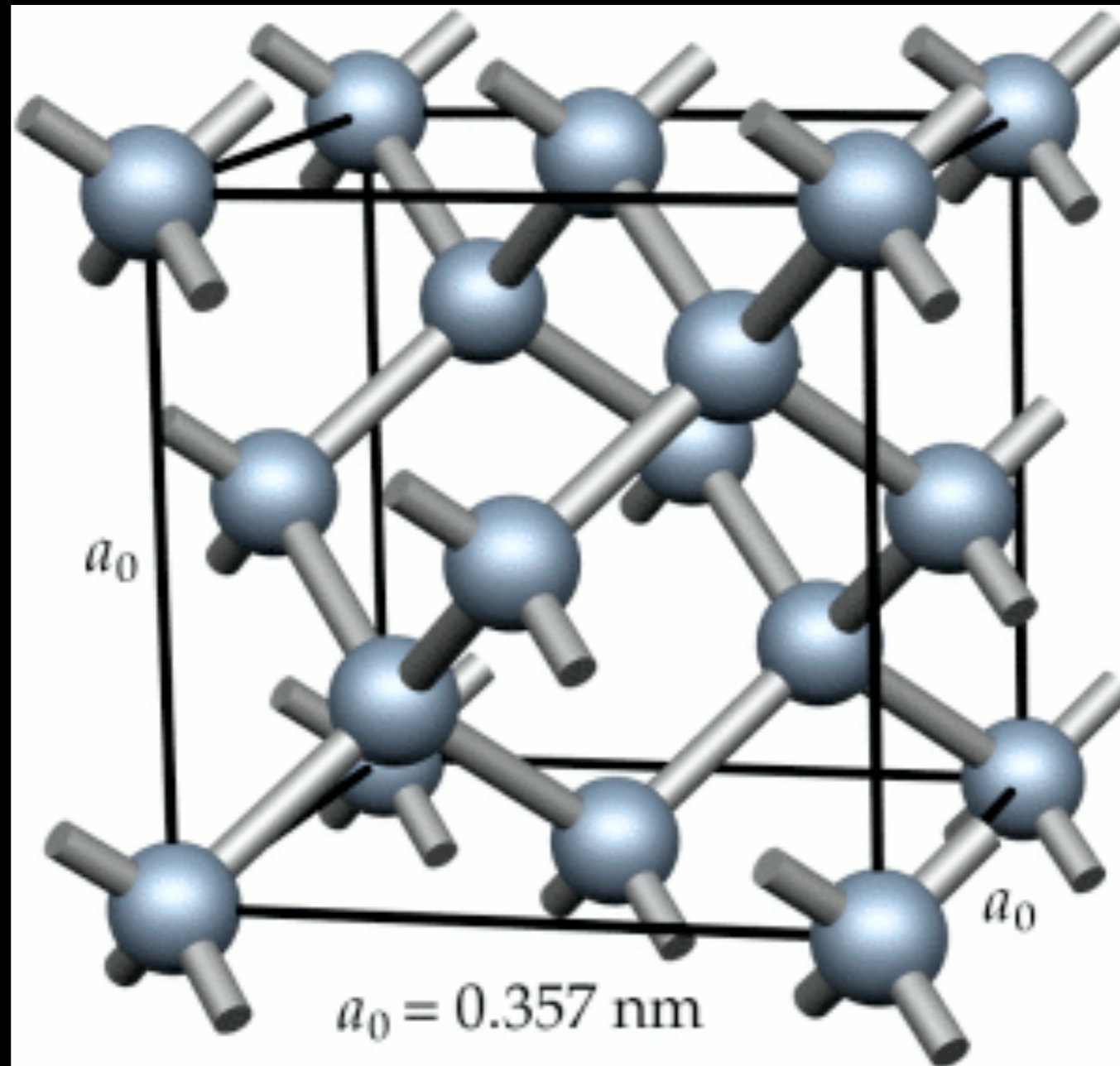


Galois Theory



$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

Crystallography



Classical Physics



Galileo Galilei
(1564-1642)

The invariance group of
classical Physics is

$$\text{ISO}(3) \times \mathbb{R}$$

Relativity

To preserve

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t^2)$$

relativity adopted the **Lorentz group** $SO^+(1, 3)$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh(\beta) & -\sinh(\beta) & 0 & 0 \\ -\sinh(\beta) & \cosh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\beta = \log \left(\frac{1 + v/c}{\sqrt{1 - v^2/c^2}} \right)$$



Albert Einstein
(1879-1955)

Noether's Theorem

Symmetry implies conservation:

$$\frac{dJ}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta}$$

(roughly)

time \rightarrow energy

space \rightarrow momentum

rotation \rightarrow angular mom.



Emmy Noether
(1882-1935)

Standard model and beyond



Eugene Wigner
(1902-1995)

symmetries \rightarrow unitary op.

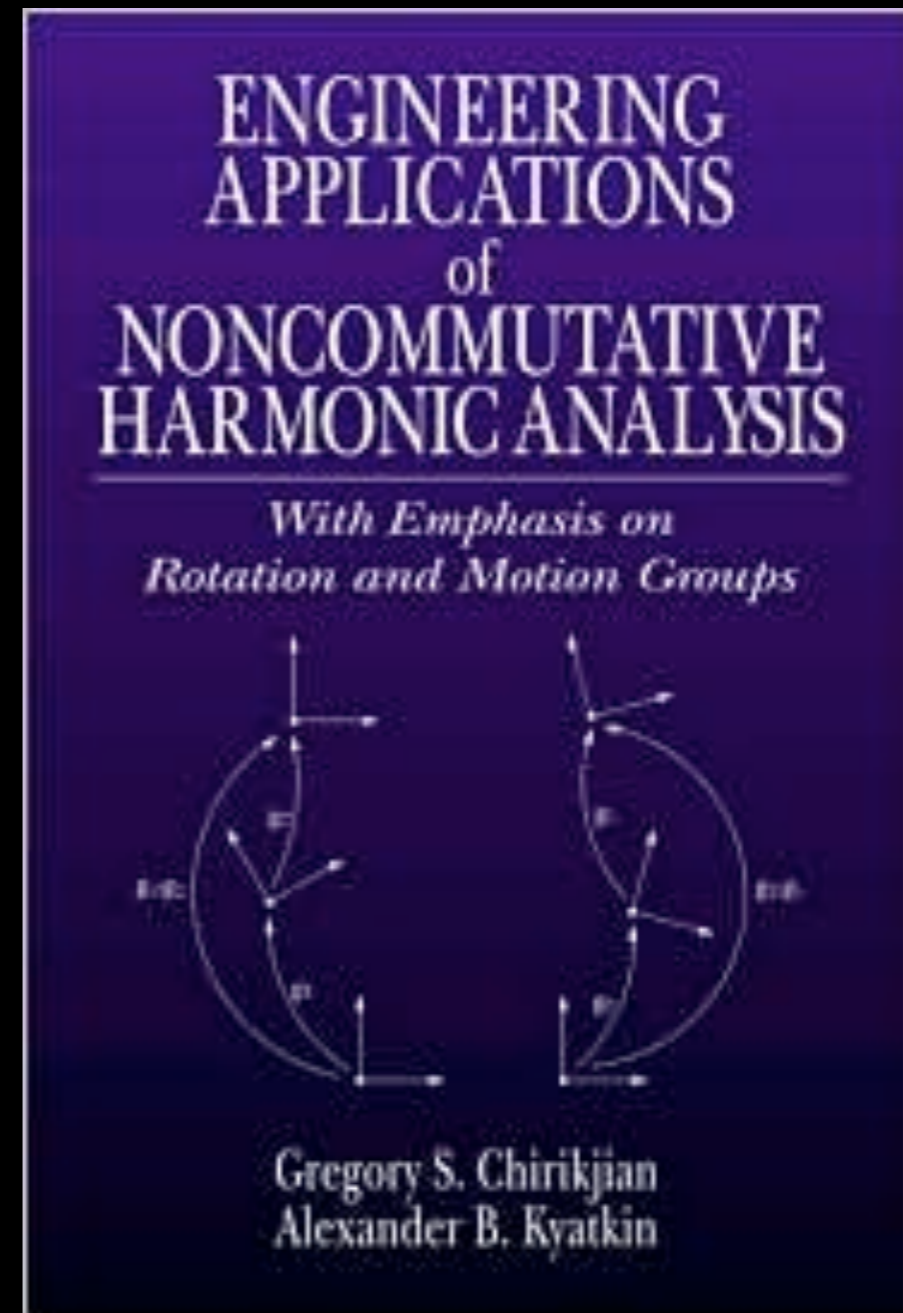
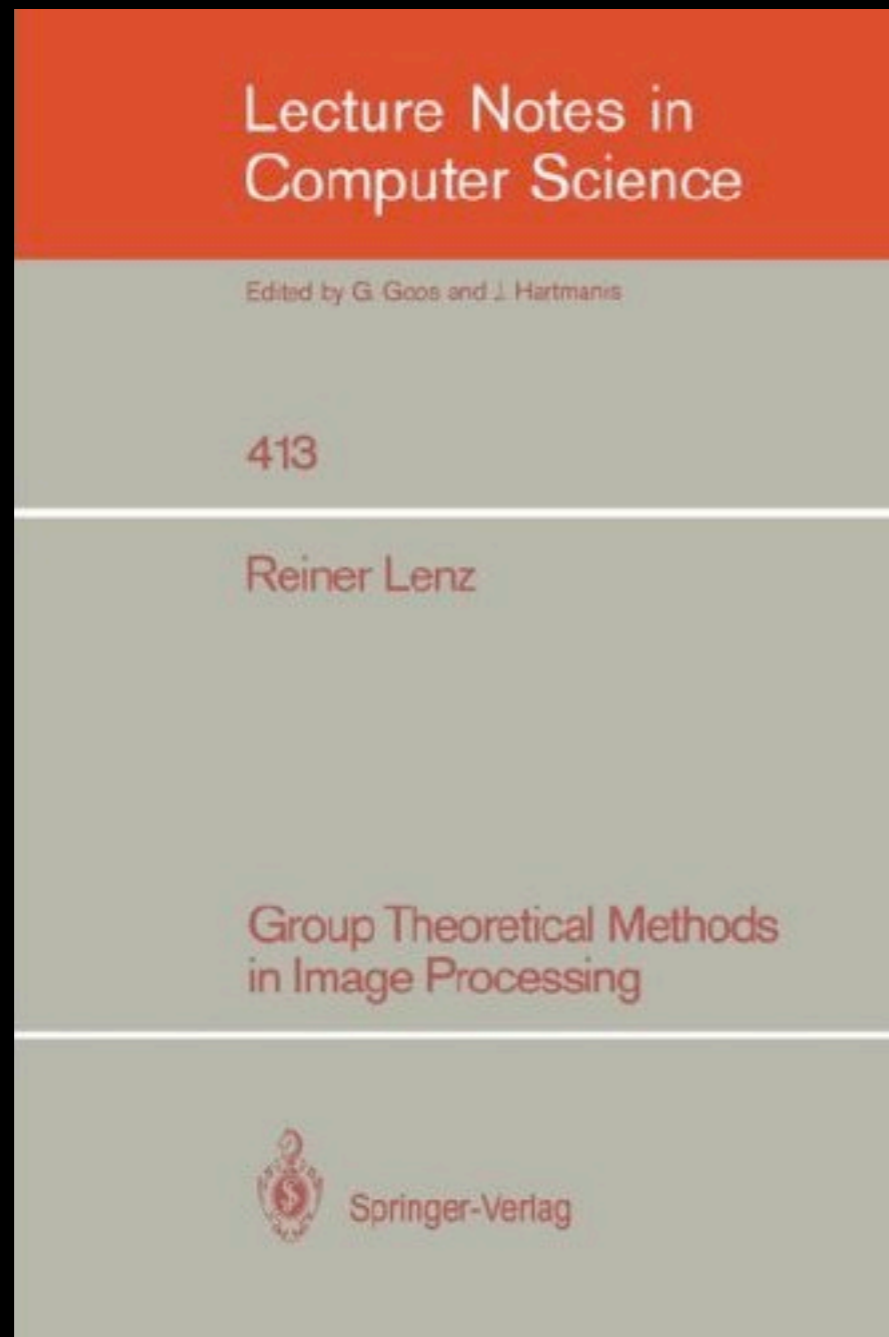
observables \rightarrow generators

pure states \rightarrow dimensions of
irreps

$SU(3)$, $SU(6)$, ...

lead to quarks and even
stranger animals...

Engineering



Machine Learning

Invariances

images, graphs, etc.

Permutation problems

ranking

multi-object tracking

Search

optimization over combinatorial
structures

Structure

Isomorphism

$G \cong H$ if there is a **bijection** $\psi: G \rightarrow H$ s.t.

$$\psi(g_1) \psi(g_2) = \psi(g_1 g_2)$$

If ψ is only surjective, then it is a **homomorphism**.

Direct product

$$G \times H = \{ (g, h) \mid g \in G, h \in H \}$$

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

e.g., $\mathbb{Z}_2^n, \mathbb{R}^n$

Semi-direct product

If H acts on G (by automorphisms), then

$$G \rtimes H = \{ (x, \Lambda) \mid x \in G, \Lambda \in H \}$$

$$(x', \Lambda')(x, \Lambda) = (x' \Lambda'(x), \Lambda' \Lambda)$$

e.g., $\text{ISO}_3^+ \cong \mathbb{R}^3 \rtimes \text{SO}(3) \qquad x \mapsto Rx + b$

Wreath product

If H is a subgroup of S_n , then

$$G \wr H \cong G^n \rtimes H$$

$$(g'_1, g_2, \dots, g'_n; \sigma')(g_1, g_2, \dots, g_n; \sigma) = \\ (g'_1 g_{\sigma^{-1}(1)}, g'_2 g_{\sigma^{-1}(2)}, \dots, g'_n g_{\sigma^{-1}(n)}; \sigma' \sigma)$$

Subgroups

$H \subseteq G$ is a subgroup of G if it is closed wrt. the group operation.

left cosets: $xH = \{ xh \mid h \in H \}$

x_1H, x_2H, \dots, x_kH partition G

e.g., take $G = \mathbb{Z}$ and $H = 3\mathbb{Z}$

Normal subgroups

$$H \triangleleft G \quad \text{if} \quad yH = Hy \quad \text{for any} \quad y \in G$$

Want to define $(xH)(yH) = (xyH)$.

If $x' = xh_1$ and $y' = yh_2$ must still get the same coset.

$$xh_1yh_2 = xyh \quad \text{for some } h \in H$$

$$h_1yh_2 = yh$$

$$h_1y = yh$$

$$y^{-1}h_1y = h$$

$$y^{-1}h_1y \in H$$

$$\longrightarrow \quad yH = Hy \quad \text{for any } y \in G$$

Normal subgroups

$H \triangleleft G$ if $yH = Hy$ for any $y \in G$

$\longrightarrow G/H$ is also a group

No normal factors \longrightarrow simple group

Jordan–Hölder theorem

Up to permutation of factors the subnormal series

$$1 \triangleleft H_1 \triangleleft \cdots \triangleleft H_k = G$$

is unique for any finite group.

The Enormous Theorem (1983)

Every finite simple group belongs to one of the following classes:

- \mathbb{Z}_p ,
- A_n ($n \geq 5$),
- simple groups of Lie type,
- the 26 sporadic groups.

Group	Order (sequence A001228 in OEIS)	1SF	Factorized order
F_1 or M	8080174247945128758864599049617107570057543680000000000	$\approx 8 \times 10^{53}$	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$
F_2 or B	4154781481226426191177580544000000	$\approx 4 \times 10^{33}$	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
$F_{i_{24}}$ or F_{3+}	1255205709190661721292800	$\approx 1 \times 10^{24}$	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$F_{i_{23}}$	4089470473293004800	$\approx 4 \times 10^{18}$	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
$F_{i_{22}}$	64561751654400	$\approx 6 \times 10^{13}$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
F_3 or Th	90745943887872000	$\approx 9 \times 10^{16}$	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
Ly	51765179004000000	$\approx 5 \times 10^{16}$	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
F_5 or HN	273030912000000	$\approx 3 \times 10^{14}$	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
Co_1	4157776806543360000	$\approx 4 \times 10^{18}$	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Co_2	42305421312000	$\approx 4 \times 10^{13}$	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_3	495766656000	$\approx 5 \times 10^{11}$	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
ON	460815505920	$\approx 5 \times 10^{11}$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Suz	448345497600	$\approx 4 \times 10^{11}$	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Ru	145926144000	$\approx 1 \times 10^{11}$	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
He	4030387200	$\approx 4 \times 10^9$	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
McL	898128000	$\approx 9 \times 10^8$	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
HS	44352000	$\approx 4 \times 10^7$	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
J_4	86775571046077562880	$\approx 9 \times 10^{19}$	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
J_3 or HJM	50232960	$\approx 5 \times 10^7$	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_2 or HJ	604800	$\approx 6 \times 10^5$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
J_1	175560	$\approx 2 \times 10^5$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
M_{24}	244823040	$\approx 2 \times 10^8$	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{23}	10200960	$\approx 1 \times 10^7$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{22}	443520	$\approx 4 \times 10^5$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{12}	95040	$\approx 1 \times 10^5$	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{11}	7920	$\approx 8 \times 10^3$	$2^4 \cdot 3^2 \cdot 5 \cdot 11$

Summary

Groups are the elementary building blocks of structure in mathematics.

Symmetries and systems of transformations always have a group lurking in the background.

Three types of product, subgroups, cosets, normal subgroup, factor group, composition series.

Very powerful and general machinery.

Next time: putting groups to work

1. Acting on vector spaces
2. Representation theory
3. Harmonic analysis

