Mini-course on representation theoretical methods in ML

2: Harmonic Analysis

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So far, we concentrated on G.

Now shift to G acting on $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{C}$.

1. G acts on $\mathcal{X}$ by $x \mapsto g(x)$

2. If $f\colon \mathcal{X} \to \mathbb{C}$ then the induced action is $f \mapsto f^g$, where

$$f^g(g(x)) = f(x) \qquad\qquad f^g(x) = f(g^{-1}(x))$$

3. If $L(\mathcal{X}) = \{f\colon G \to \mathbb{C}\}$, then $T_g\colon f \mapsto f^g$ is a linear map

$$T_g\colon L(\mathcal{X}) \to L(\mathcal{X})$$

4. Is T_g decomposable?

$$L(\mathcal{X}) = W_1 \oplus W_2 \oplus \ldots \oplus W_k$$

$$T_g(f) = [T_g]_1\left([f]_1\right) \oplus \ldots \oplus [T_g]_k\left([f]_k\right) \qquad\qquad [f]_k \in W_k$$

5. What are the elementary $[T_g]_i$ systems obeying

$$[T_{g_2}]_i\,[T_{g_1}]_i = [T_{g_2 g_1}]_i \ ?$$

1. Representation: $\rho\colon G \to \mathbb{C}^{d\times d}$

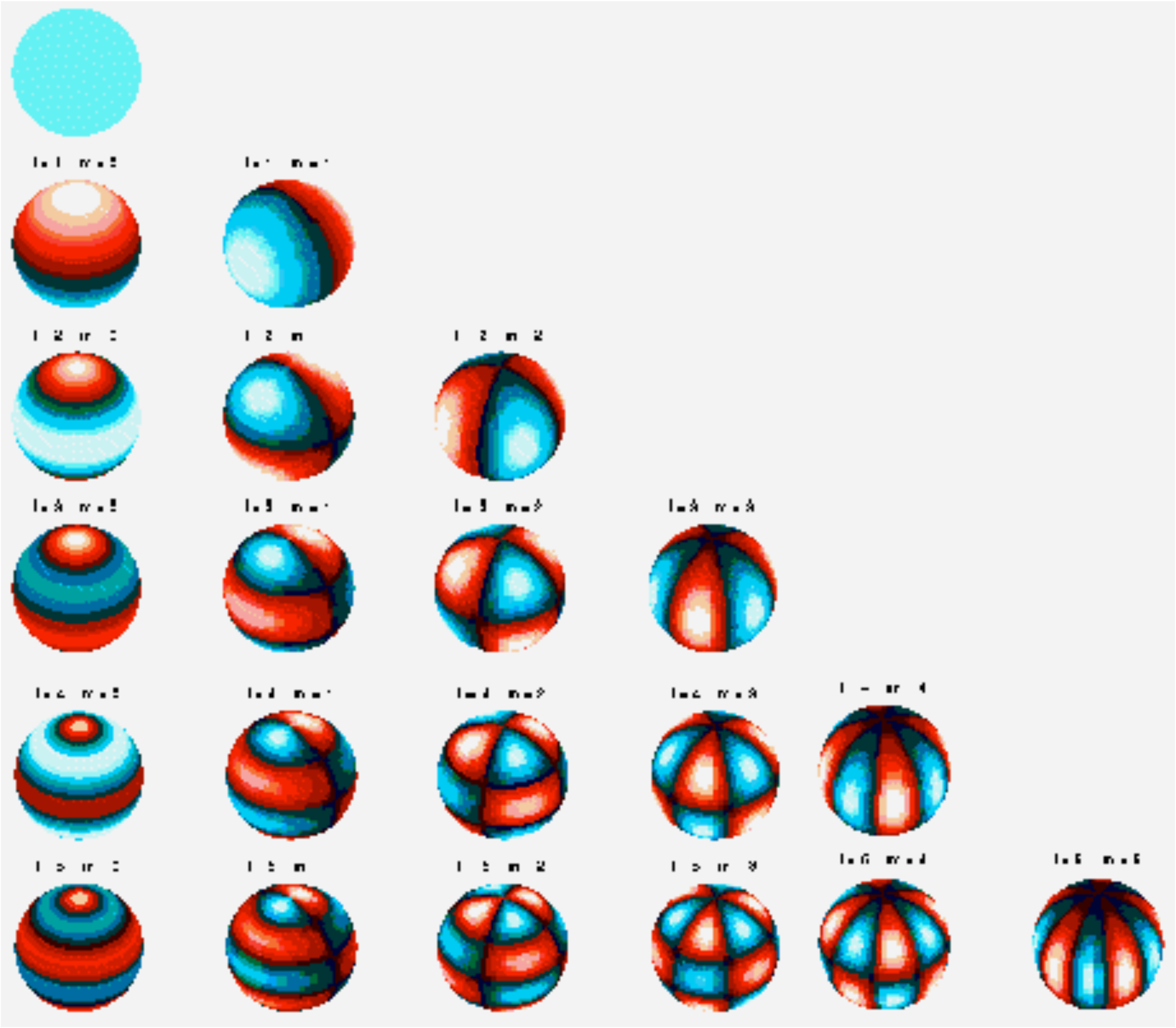
$$\rho(x)\cdot\rho(y) = \rho(xy)$$

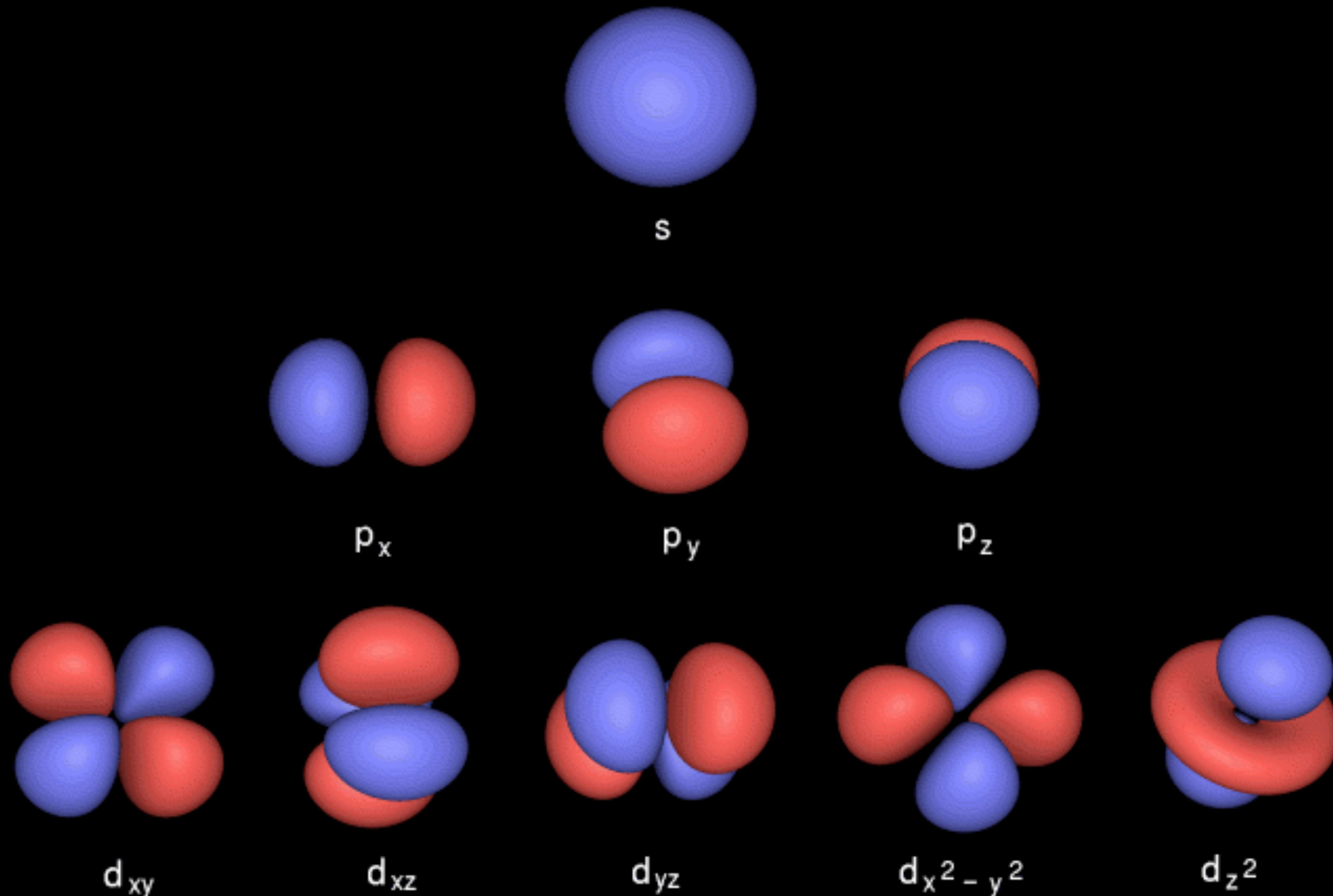
2. Equivalence: $\rho_1(x) = T^{-1}\rho_2(x)\,T$

3. Reducibility:

$$\rho(x) = T^{-1}\begin{pmatrix}\rho_1(x) & 0 \\ 0 & \rho_2(x)\end{pmatrix} T$$

4. Complete set of inequivalent irreducible representations: $\mathcal{R}$




s
p_x
p_y
p_z
d_{xy}
d_{xz}
d_{yz}
$d_{x^2-y^2}$
d_{z^2}

Joseph Fourier
1768-1830

$$f(x) = \sum_{k=-\infty}^{\infty} f_k\, e^{ikx} \qquad f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$$

$$f(x) = \int e^{2\pi ikx}\, \widehat{f}(k)\ dk \qquad \widehat{f}(k) = \int e^{-i2\pi kx} f(x)\, dx$$

$$f(x) = \frac{1}{n} \sum_{x=0}^{n-1} e^{2\pi ikx/n}\ \widehat{f}(k) \qquad \widehat{f}(k) = \sum_{x=0}^{n-1} e^{-2\pi ikx/n}\ f(x)$$

1. Linearity $\qquad \widehat{f+g} = \widehat{f} + \widehat{g}$

2. Unitarity $\qquad \langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle$

3. Translation $\qquad \widehat{f}'(k) = e^{-itk}\widehat{f}(k) \qquad f'(x) = f(x-t)$

4. Convolution $\qquad \widehat{f * g}(k) = \widehat{f}(k)\,\widehat{g}(k)$

$$f * g(x) = \int f(x-y)\,g(y)\,dy$$

5. Differentiation $\qquad \widehat{\partial f}(k) = 2\pi i k \widehat{f}(k) \qquad \partial f = df/dx$

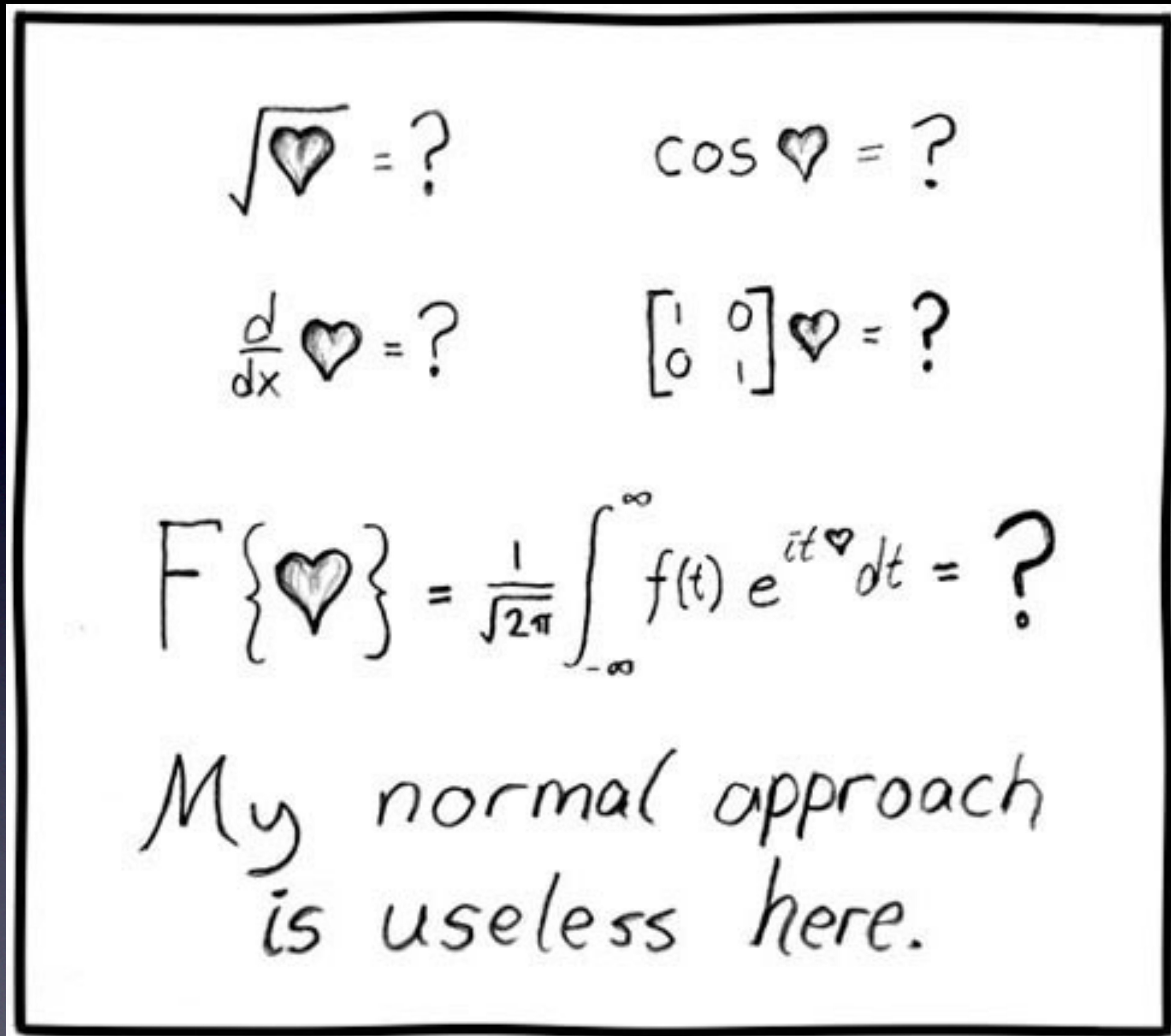
$\mathcal{F}\colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is a unitary operation that decomposes $L_2(\mathbb{R})$ into a sum of orthogonal subspaces that transform independently under translation/convolution.

$$L_2(\mathbb{R}) = \bigoplus_k V_k \qquad \begin{array}{l} T_t V_k = V_k \\ f, g \in V_k \quad \Rightarrow f * g \in V_k \end{array}$$

This is purely an algebraic statement about how the translation group acts on $L_2(\mathbb{R})$.

On the other hand, $\mathcal{F}$ also picks out subspaces of different degrees of smoothness.

Can we generalize all this to groups?


√♥ = ?
cos ♥ = ?
d/dx ♥ = ?
[1 0; 0 1] ♥ = ?
F{♥} = 1/√2π ∫ from −∞ to ∞ f(t) e^(it♥) dt = ?
My normal approach is useless here.

xkcd.com

Convolution

Regard $f : G \to \mathbb{C}$ as $f = \sum_{x \in G} f(x)\, \boldsymbol{e}_x$.

Define $\boldsymbol{e}_x \cdot \boldsymbol{e}_y = \boldsymbol{e}_{xy}$, $\quad f g = \sum_{x \in G} \sum_{y \in G} f(x)\, f(y)\, \boldsymbol{e}_x\, \boldsymbol{e}_y$.

isotypics: irreducible subalgebras

$$\mathbb{C}G = V_1 \oplus V_2 \oplus \ldots \oplus V_k$$

group algebra

For compact groups this decomposition is unique.

1. $v \in V_i$ is really a linear operator $v \colon V_i \to V_i$

$$u \mapsto vu = v * u$$

2. In particular, $\{ [\boldsymbol{e}_x]_i \mid x \in G \}$ is a set of operators obeying

$$[\boldsymbol{e}_x]_i \, [\boldsymbol{e}_y]_i = [\boldsymbol{e}_{xy}]_i$$

3. These are the irreducible representations

$$\rho_i(x) = [\boldsymbol{e}_x]_i$$

4. On non-commutative groups $\boldsymbol{e}_{xy} \neq \boldsymbol{e}_{yx}$, so some V_i must be multidimensional.

5. Any f can be written as $f = \sum_{x \in g} f(x)\, \boldsymbol{e}_x$, so

$$[f]_i = \sum_{x \in g} f(x) \; [\boldsymbol{e}_x]_i = \sum_{x \in G} f(x)\, \rho_i(x)$$

6. This is the Fourier transform!

The Fourier transform on a group is

$$\widehat{f}(\rho) = \sum_{x \in G} f(x)\, \rho(x) \qquad\qquad \rho \in \mathcal{R}$$

The inverse transform is

$$f(x) = \frac{1}{|\,G\,|} \sum_{\rho \in \mathcal{R}} d_\rho \, \mathrm{tr}\left[\widehat{f}(\rho)\, \rho(x^{-1})\right]$$

$$\widehat{f}(\rho) = \sum_{x \in G} f(x)\, \rho(x) \qquad f(x) = \frac{1}{|G|} \sum_{\rho \in \mathcal{R}} d_\rho \operatorname{tr}\left[\widehat{f}(\rho)\, \rho(x^{-1})\right]$$

1. Linearity: $\widehat{f+g} = \widehat{f} + \widehat{g}$
2. Unitarity: $\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle$
3. Left-translation: $\widehat{f^z(\rho)} = \rho(z)\, \widehat{f}(\rho)$
4. Convolution: $\widehat{f * g}(\rho) = \widehat{f}(\rho)\, \widehat{g}(\rho)$
5. The individual components correspond to different levels of smoothness.

The Fourier transform $\mathcal{F}\colon f \to \widehat{f}$ is an isomorphism

$$\mathcal{F}\colon \mathbb{C}G \to \bigoplus_{\rho \in \mathcal{R}} \mathbb{C}^{d_\rho \times d_\rho}.$$