Mini-course on representation theoretical methods in ML

4. Representations of the symmetric group and the FFT

Risi Kondor (Gatsby Unit)

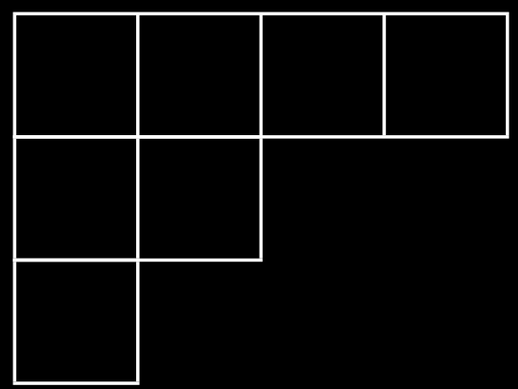
1. The trivial representation $\rho_{\mathrm{triv}}(\sigma) = (1)$

2. The alternating representation $\rho_{\mathrm{triv}}(\sigma) = \mathrm{sgn}(\sigma)$

3. The defining representation $[\rho_{\mathrm{def}}(\sigma)]_{i,j} = \delta_{\sigma(i),j}$

reducible!!!

Young diagrams



$\longleftrightarrow$ integer partitions $\lambda \vdash n$

$\longleftrightarrow$ irreducible representations $\rho \in \mathcal{R}$

Young Tableaux

1	3	6	7
2	5		
4			

standard tableau if numbers increase left to right and top to bottom

standard tableaux of shape λ $\longleftrightarrow$ dimensions of ρ_λ

Young's Orthogonal Representation (YOR)

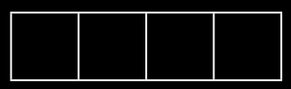
$$[\rho_\lambda(\tau_k)]_{t,t} = 1/d_t(k,k+1)$$

$$[\rho_\lambda(\tau_k)]_{\tau_k(t),t} = \sqrt{1-1/d_t(k,k+1)^2}$$

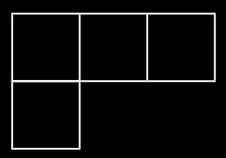
if $\tau_k(t)$ standard

All $\rho_\lambda(\sigma)$ are real!

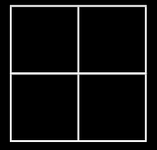
$\mathbb{S}_4$



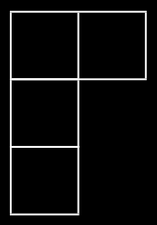
$d = 1$ $\rho_{(4)}(\sigma) = (1)$



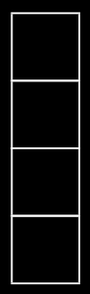
$d = 3$



$d = 2$



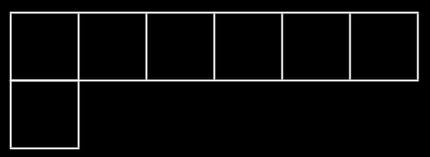
$d = 3$



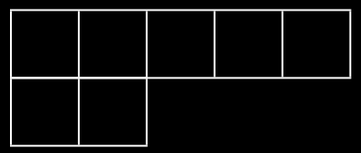
$d = 1$ $\rho_{(1,1,1,1)}(\sigma) = (\mathrm{sgn}(\sigma))$

$\mathbb{S}_n$

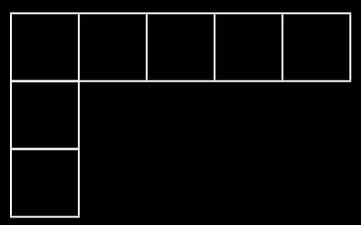
$d = 1$



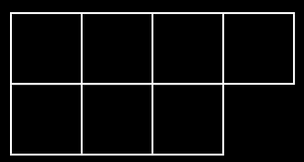
$d = n - 1$



$d = n(n-3)/2$



$d = (n-1)(n-2)/2$



$d = n(n-1)(n-5)/6$

Hook rule

$$d_\lambda = \frac{n!}{\prod_{i=1}^{n} l_i}$$

$$l_i = \text{length of hook } i$$

Restricted representations

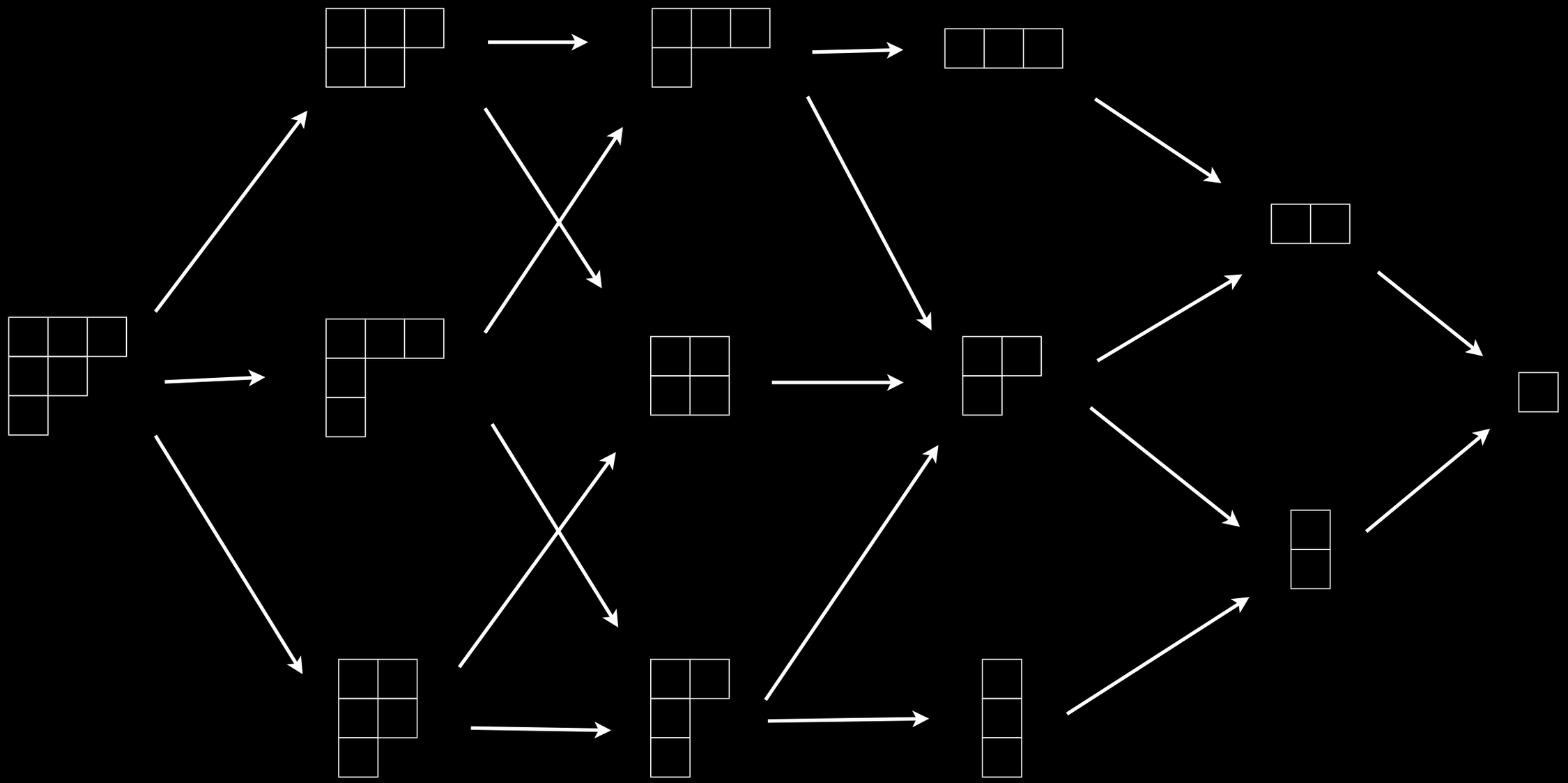
If ρ is a representation of $\mathbb{S}_n$
$\rho\downarrow_{\mathbb{S}_k}$ is a representation of $\mathbb{S}_k$ given by

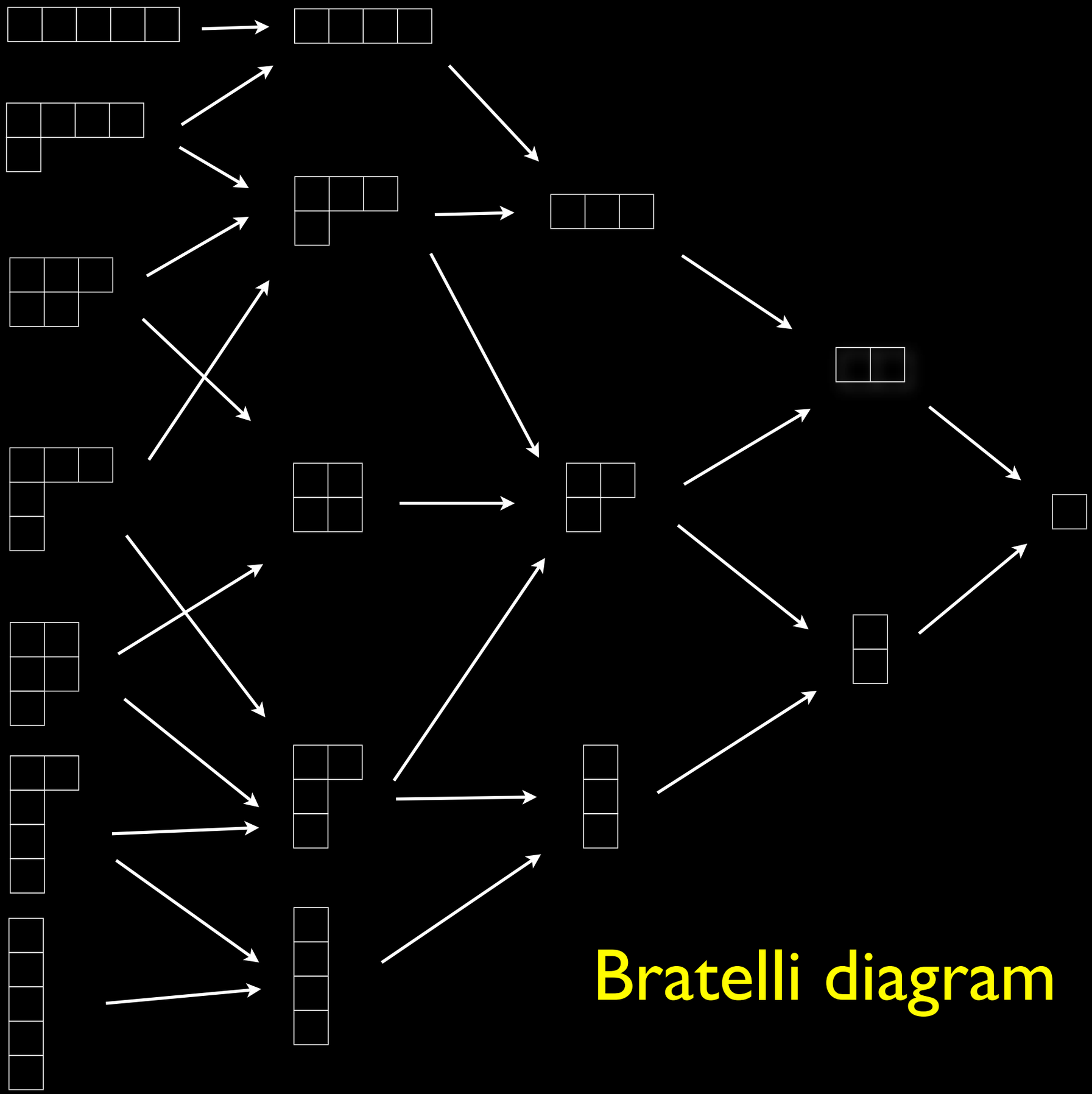
$$\rho\downarrow_{\mathbb{S}_k}(\sigma) = \rho(\sigma) \qquad\qquad \sigma \in \mathbb{S}_k.$$

Young's Rule

In YOR

$$\rho_\lambda \downarrow_{\mathbb{S}_{n-1}} (\sigma) = \bigoplus_{\substack{\lambda^- \vdash n-1 \\ \lambda^- < \lambda}} \rho_{\lambda^-}(\sigma)$$





Bratelli diagram