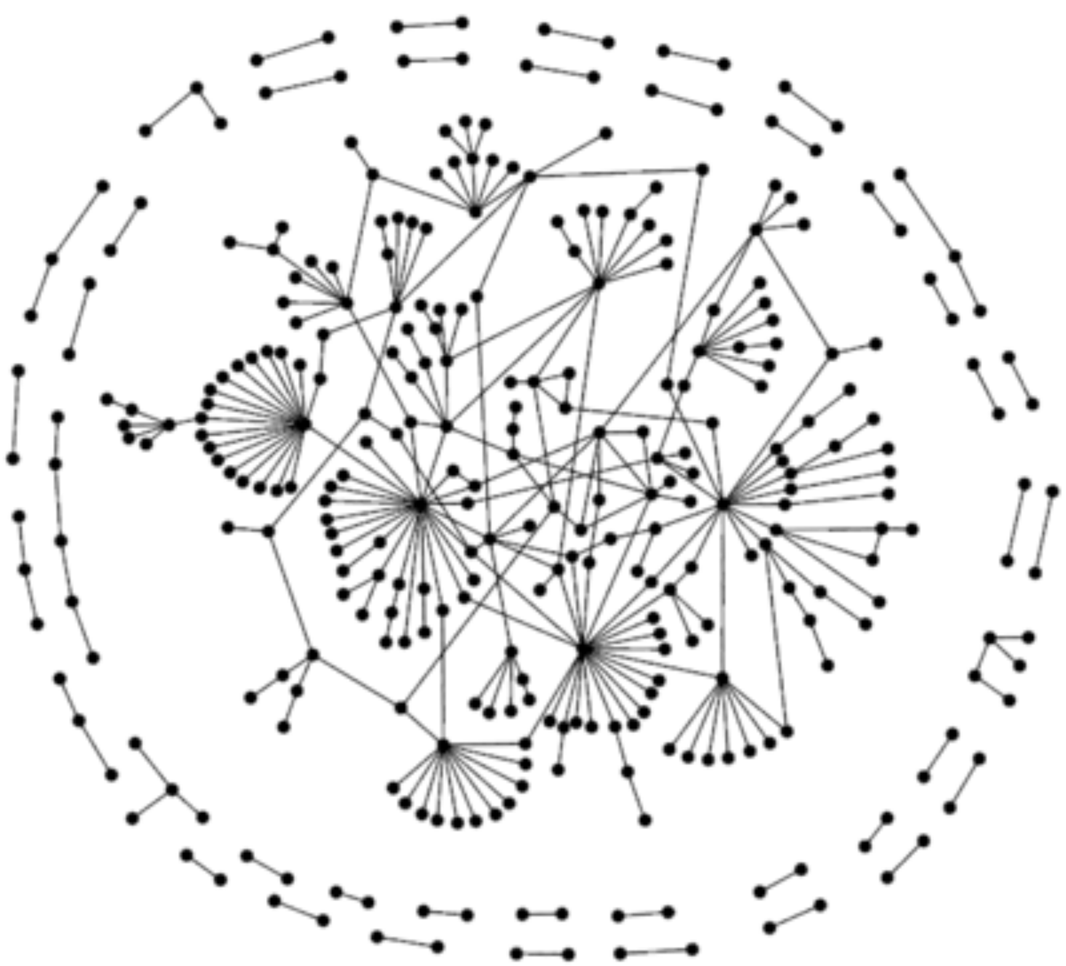
Composite Self-concordant Minimization

Volkan Cevher

*Laboratory for Information and Inference Systems-**LIONS***

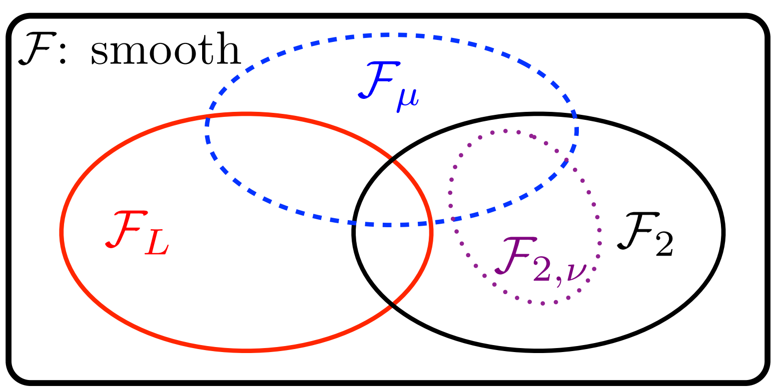
Ecole Polytechnique Federale de Lausanne (EPFL)

volkan.cevher@epfl.ch



UCL-Gatsby

June 12, 2014



joint work with

Quoc Tran Dinh

Anastasios Kyrillidis

Yen-Huan Li

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

lions@epfl

MARIE CURIE ACTIONS

FNSNF
Fonds national suisse
Schweizerischer Nationalfonds
Fondo nazionale svizzero
Swiss National Science Foundation

erc

Composite minimization

$$\text{(P)} \qquad \min_{\mathbf{x} \in \mathbb{R}^n} \{\phi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\}$$

f
convex and **smooth**

g
convex and possibly **nonsmooth**

Motivation

Problem (P) *covers many practical problems*:

- Unconstrained basic LASSO / logistic regression
- Graphical model selection / latent variable graphical model selection
- Poisson imaging reconstruction / LASSO problem with unknown variance
- Low-rank recovery / clustering
- Atomic norm regularization / off-the-grid array processing

g: ℓ_1-norm, nuclear norm or indicator functions

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- Unconstrained basic LASSO / logistic regression
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need scalable algorithms

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Composite minimization: modus operandi

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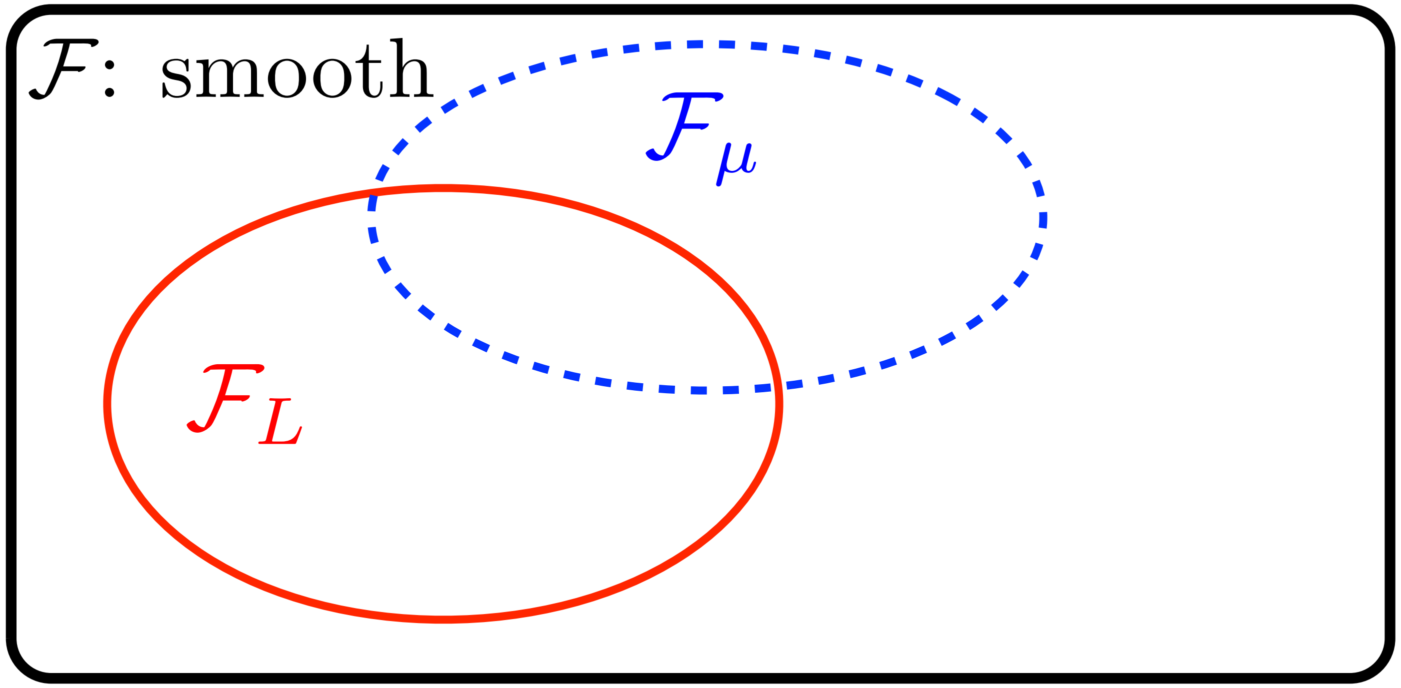
Classes of smooth functions (f)



- $\mathcal{F}_L$ - L-Lipschitz gradient
- $\mathcal{F}_\mu$ - μ-strongly convex

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|y - x\|$$

$$\mu \mathbb{I} \preceq \nabla^2 f(x) \preceq L \mathbb{I}$$

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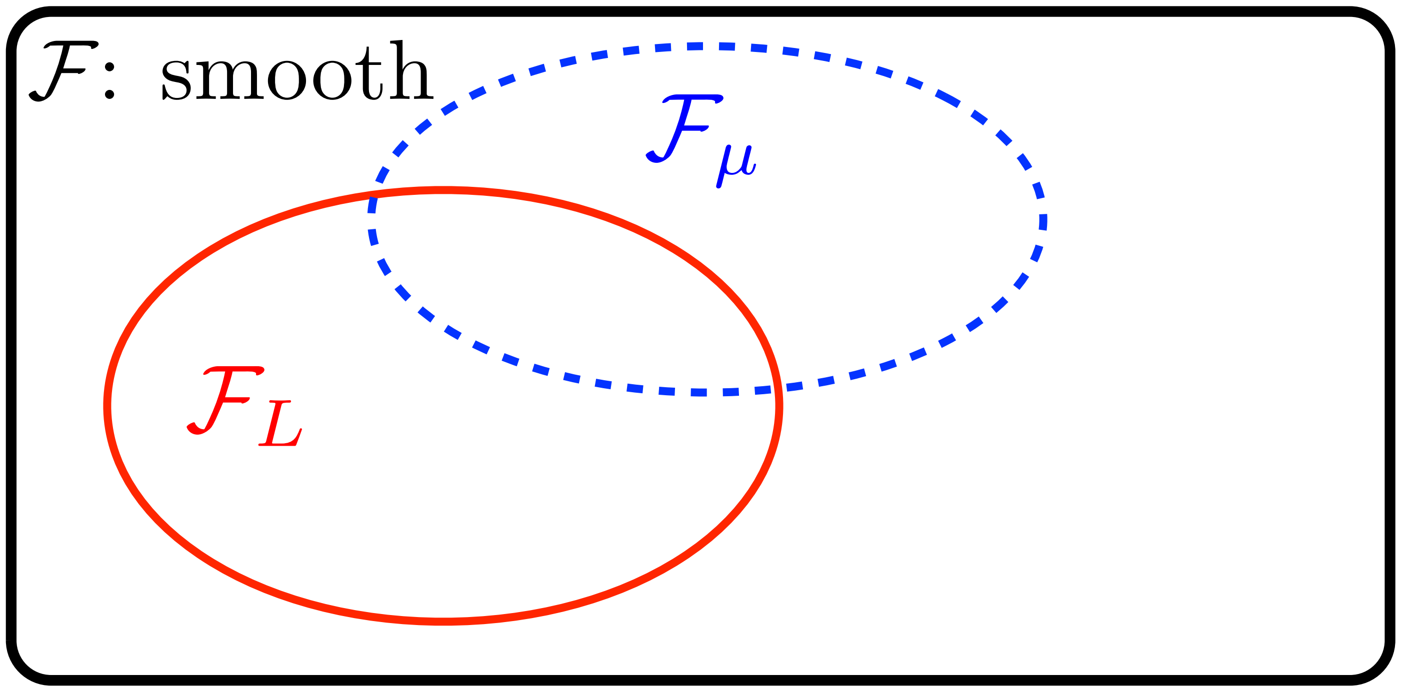

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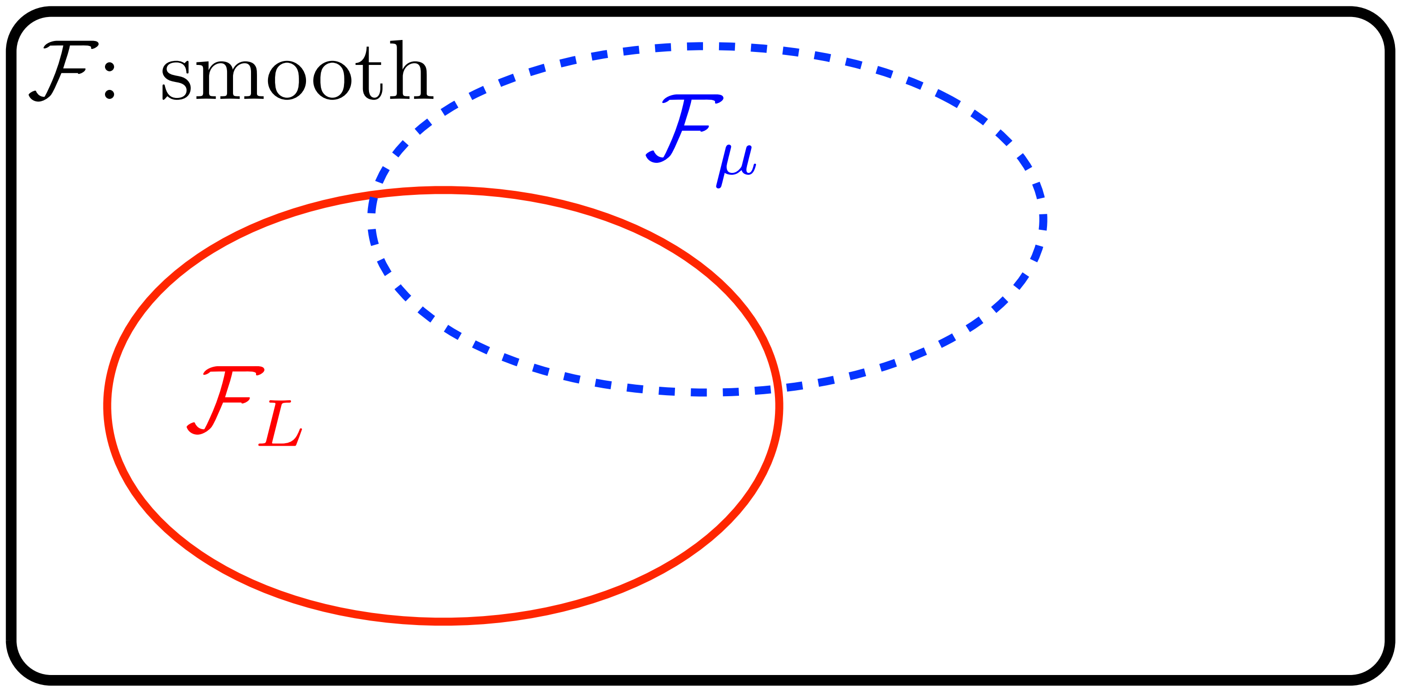

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Following **prox** computation is ***tractable:***

$$\mathrm{prox}_{\gamma g}(\mathbf{s}) := \arg\min_{\mathbf{x}} \left\{ g(\mathbf{x}) + \frac{1}{2\gamma}\|\mathbf{x} - \mathbf{s}\|_2^2 \right\}$$

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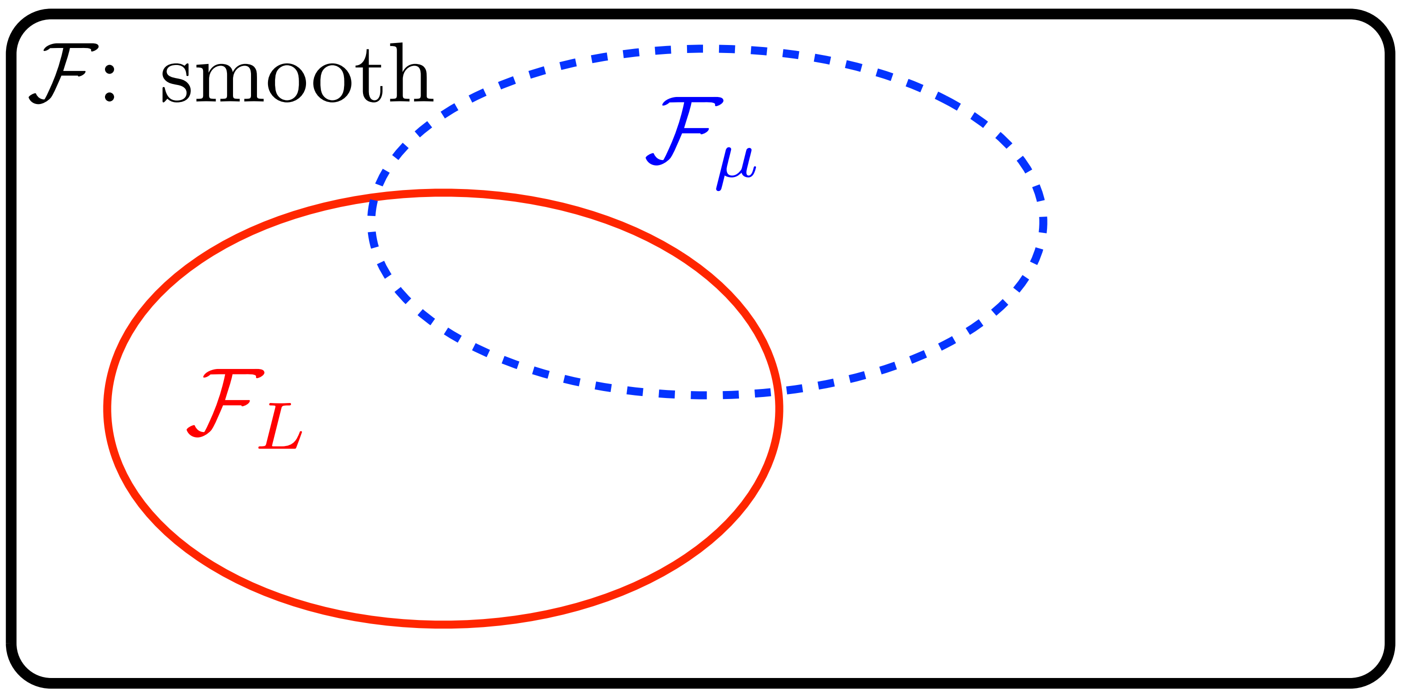

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Example:

if $g(\mathbf{x}) = \|\mathbf{x}\|_1$, then
$\mathrm{prox}_{\gamma g}(\mathbf{s}) = \mathrm{SoftThresh}(\mathbf{s}, \gamma)$

Composite minimization: modus operandi

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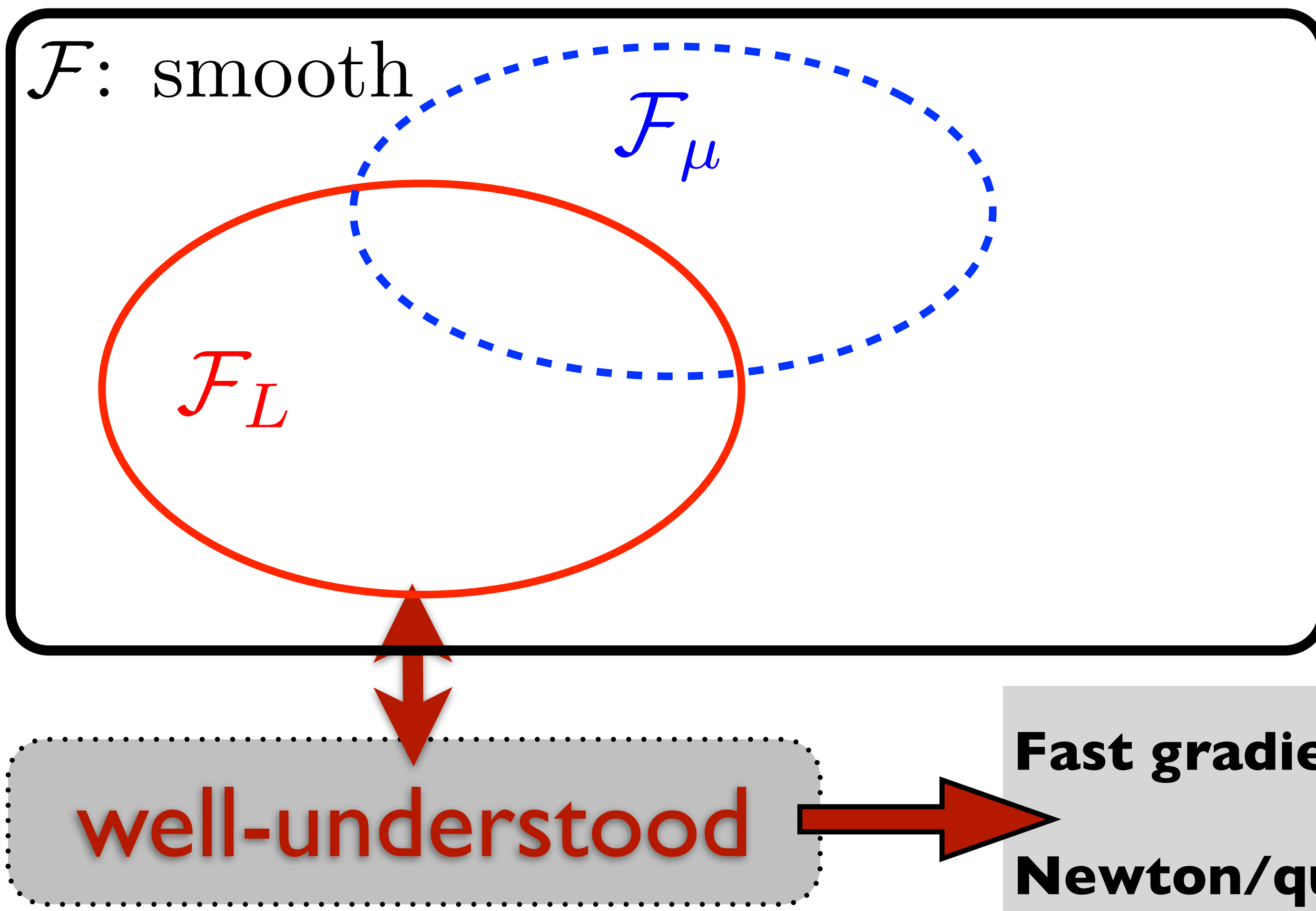

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Fast gradient schemes (Nesterov's methods)

Newton/quasi Newton schemes

Composite minimization: an uncharted region

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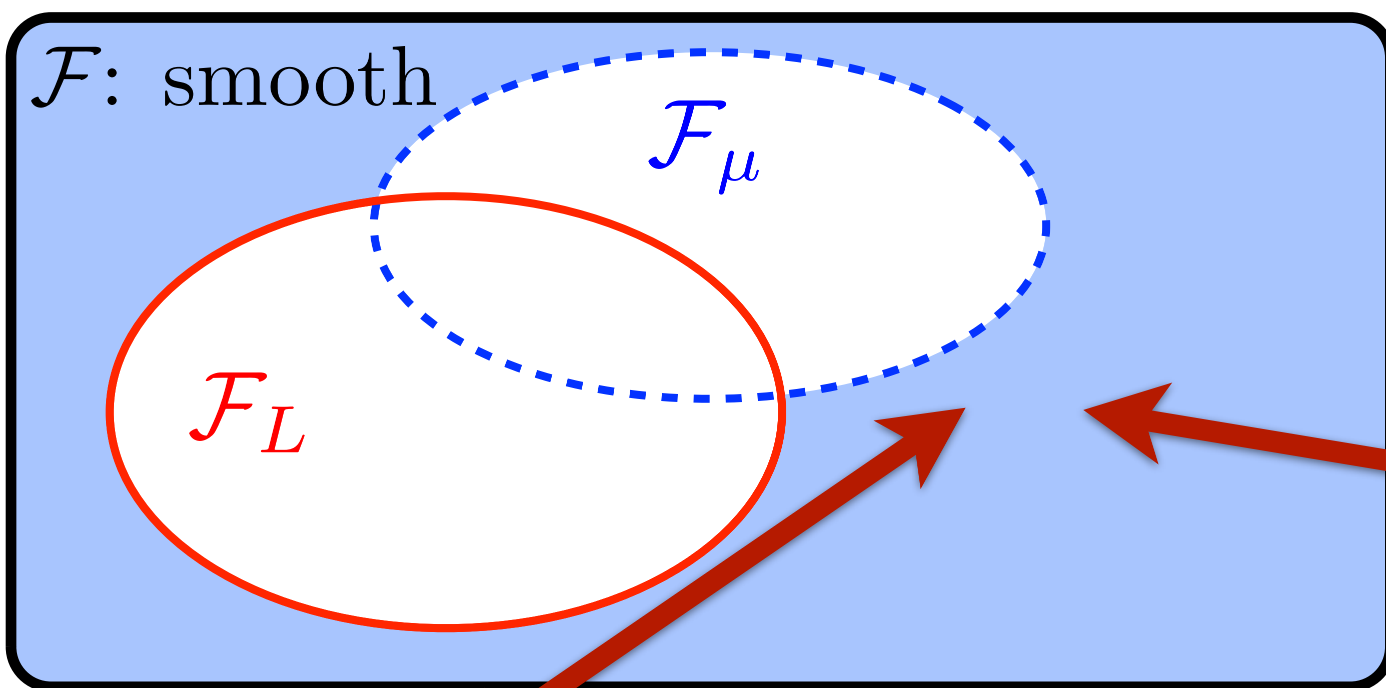

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Scalability is NOT great

~~**Fast gradient schemes (Nesterov’s methods)**~~

~~**Newton/quasi Newton schemes**~~

Composite self-concordant minimization

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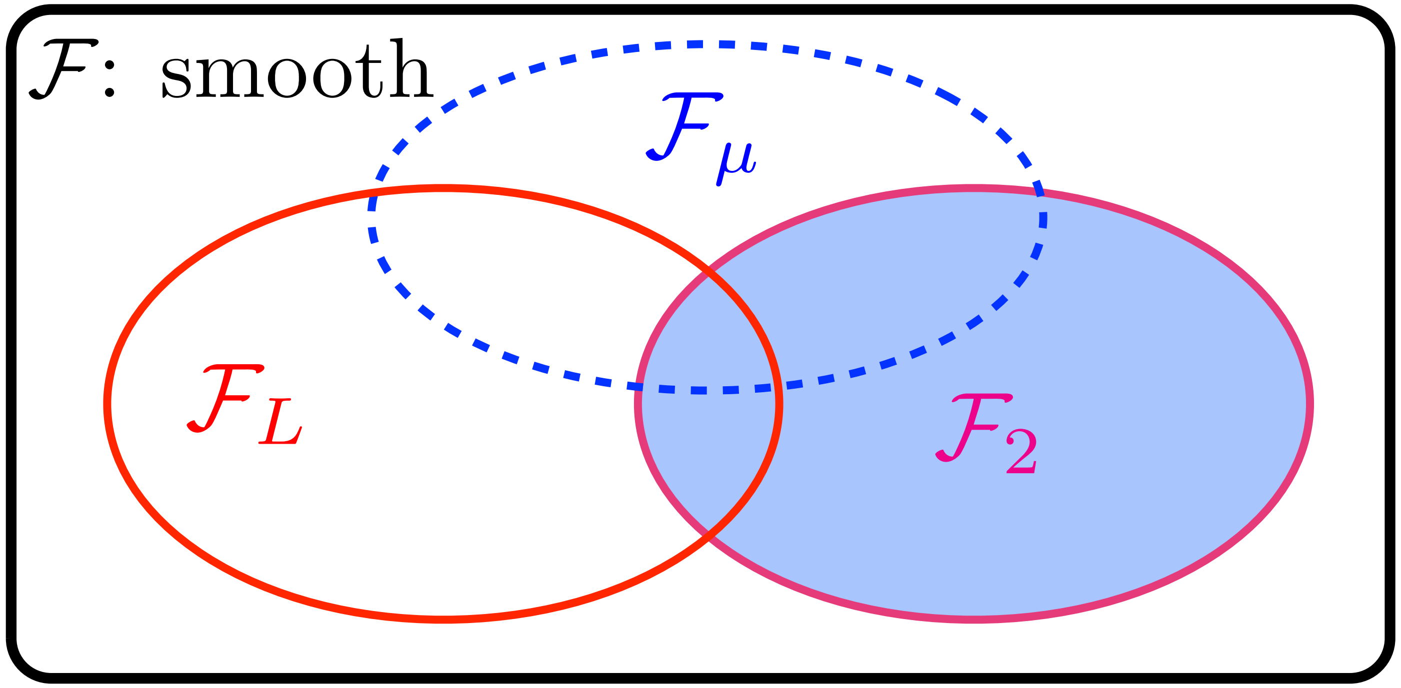

- $\mathcal{F}_L$ - L-Lipschitz gradient
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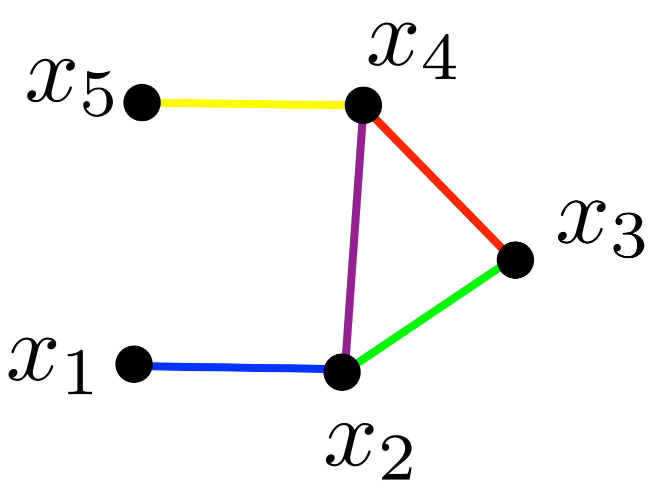
- $\mathcal{F}_2$ - self-concordant

Key structure for the interior point method

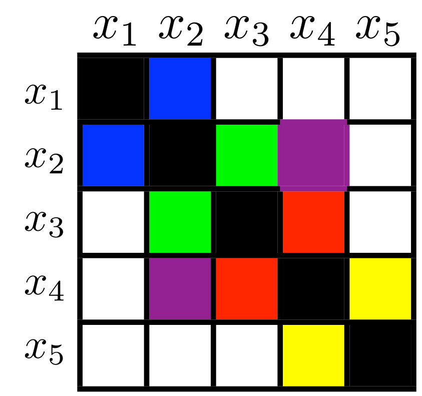
f is self-concordant if $\varphi(t) := f(\mathbf{x} + t\mathbf{d})$ satisfies $|\varphi'''(t)| \leq 2\varphi''(t)^{3/2}$

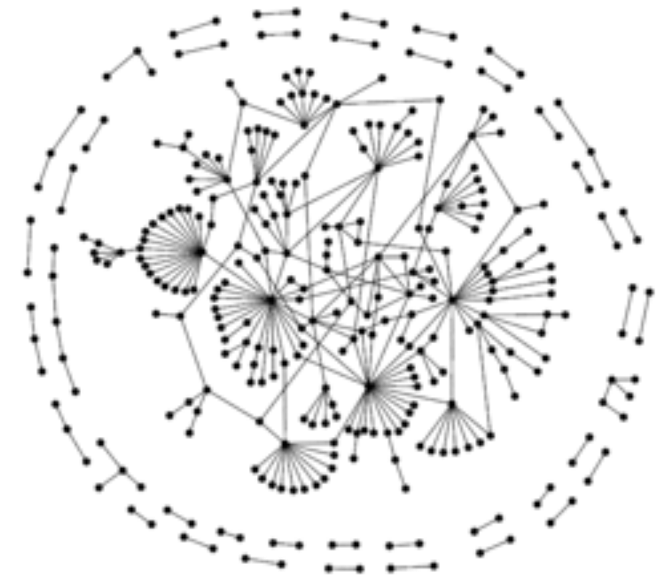
Example: Log-determinant for LMIs

- ***Application***: Graphical model selection



$\Theta =$





Given a data set $\mathcal{D} := \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, where $\mathbf{x}_i$ is a Gaussian random variable. Let $\boldsymbol{\Sigma}$ be the **covariance matrix** corresponding to the **graphical model** of the Gausian Markov random field. The aim is to learn a **sparse matrix** Θ that approximates the inverse $\boldsymbol{\Sigma}^{-1}$.

Optimization problem
$$\min_{\boldsymbol{\Theta} \succ 0} \left\{ \underbrace{-\log\det(\boldsymbol{\Theta}) + \mathrm{trace}(\boldsymbol{\Sigma}\boldsymbol{\Theta})}_{f(\mathbf{x})} + \underbrace{\rho\|\mathrm{vec}(\boldsymbol{\Theta})\|_1}_{g(\mathbf{x})} \right\}$$

Log-barrier for linear/quadratic inequalities

- Poisson imaging reconstruction via TV regularization

$$x^* \in \operatorname*{argmin}_x \left\{ \underbrace{\sum_{i=1}^m a_i^T x - \sum_{i=1}^m y_i \log(a_i^T x + b_i)}_{f(x)} + g(x) \right\}$$

- Basic pursuit denoising problem (BPDP): Barrier formulation

$$\mathbf{x}_t^* = \operatorname*{argmin}_{\mathbf{x}} \left\{ \underbrace{-t\log\left(\sigma^2 - \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2\right)}_{=: f(\mathbf{x})} + g(\mathbf{x}) \right\}$$

- LASSO problem with unknown variance

$$\mathbf{x}^* \equiv (\phi^*, \gamma^*) = \operatorname*{argmin}_{\phi,\gamma} \left\{ \underbrace{-\log(\gamma) + \frac{1}{2n}\|\gamma\mathbf{y} - \mathbf{X}\phi\|_2^2}_{=: f(\mathbf{x})} + \underbrace{\lambda\|\phi\|_1}_{=: g(\mathbf{x})} \right\}$$

- Quantum tomography ML estimator (*another presentation!*)

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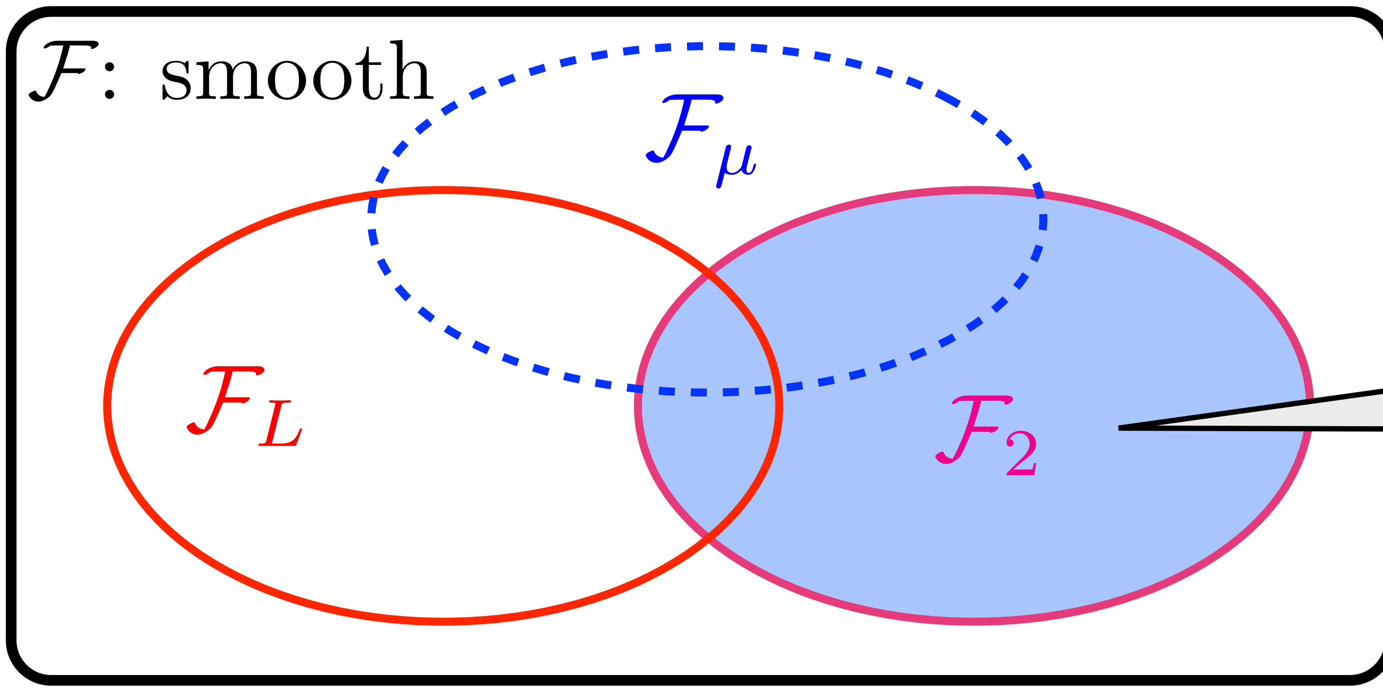

Our contributions:

i) a ***variable metric*** (path following) forward-backward framework

ii) convergence theory ***without the Lipschitz gradient assumption***

iii) novel variants and extensions for several applications & SCOPT

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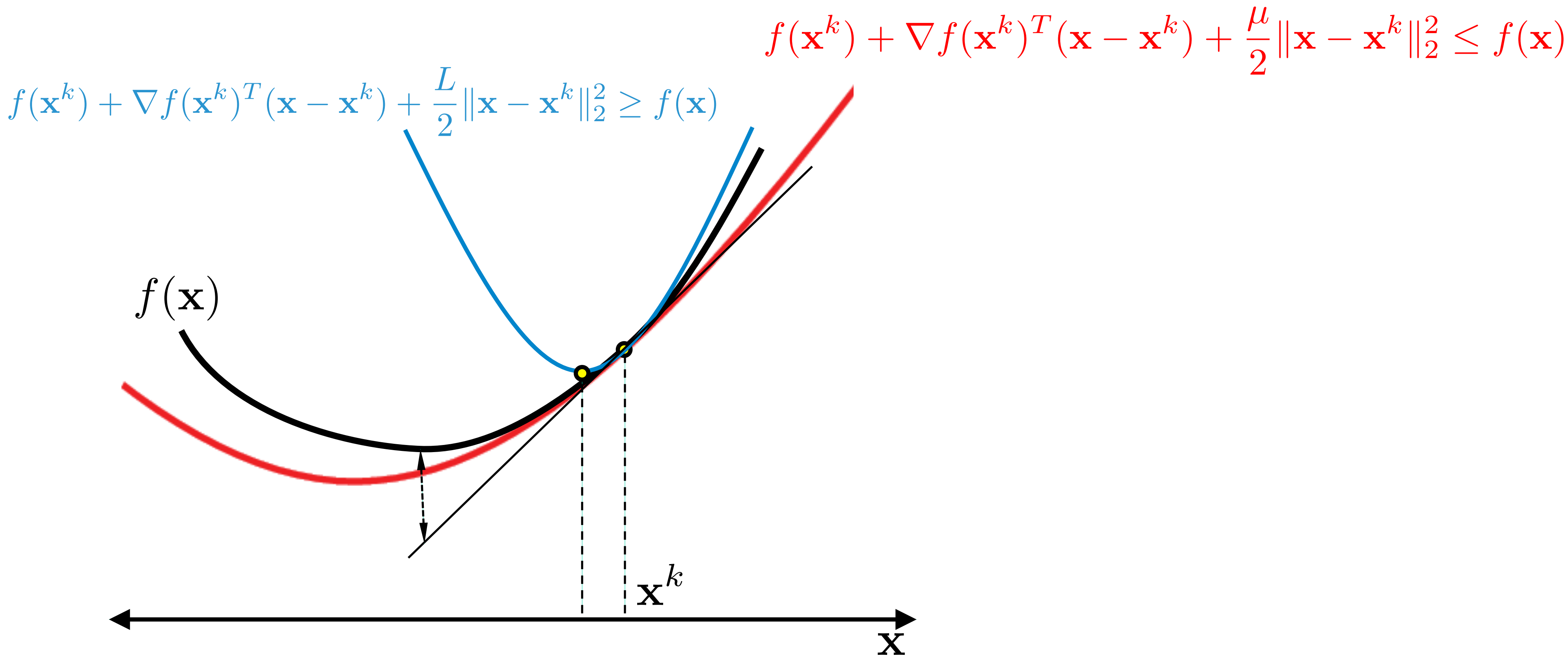
Basic algorithmic framework

A basic composite minimization framework



- **Main properties of $\mathcal{F}_{\mu,L}$**

Lower surrogate	$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \frac{\mu}{2}\|\mathbf{y} - \mathbf{x}\|_2^2$	$\mathbf{x}, \mathbf{y} \in \mathrm{dom}(f)$
Upper surrogate	$f(\mathbf{y}) \leq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \frac{L}{2}\|\mathbf{y} - \mathbf{x}\|_2^2$	$\mathbf{x}, \mathbf{y} \in \mathrm{dom}(f)$
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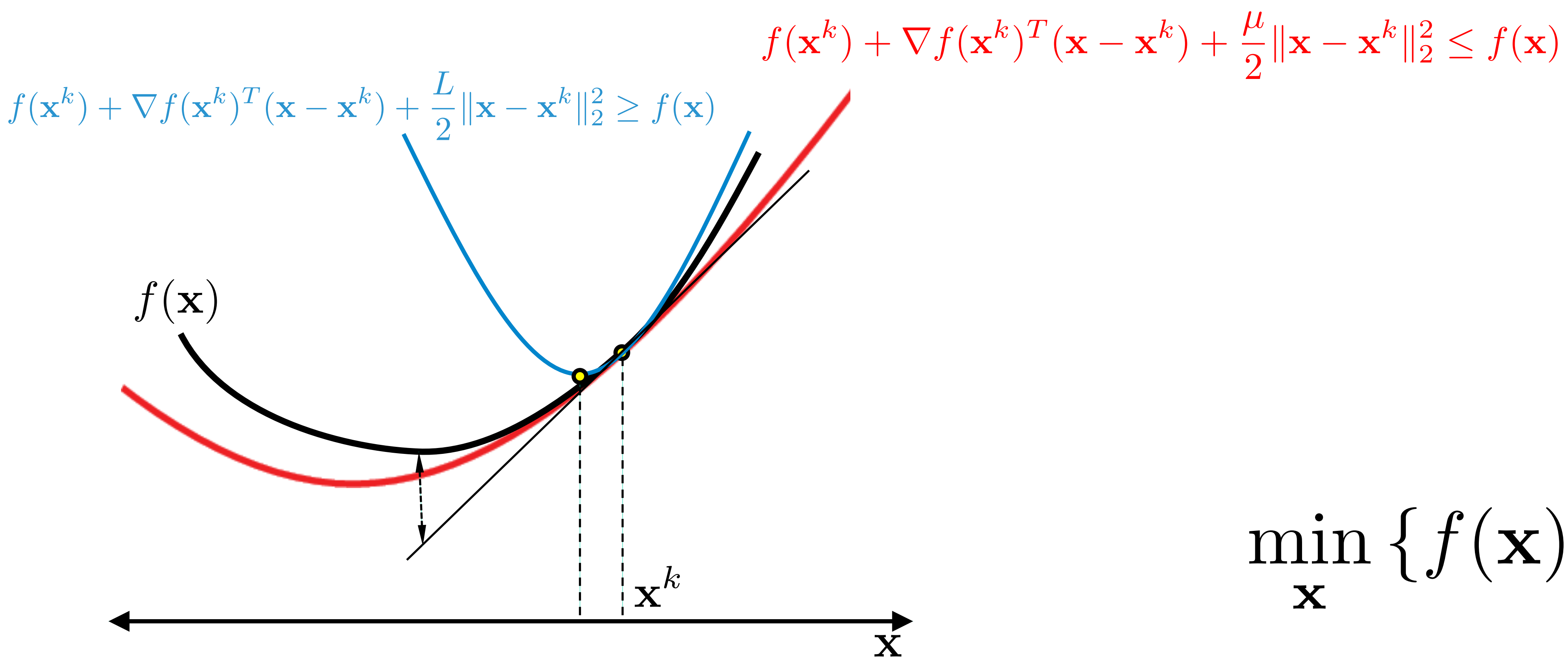

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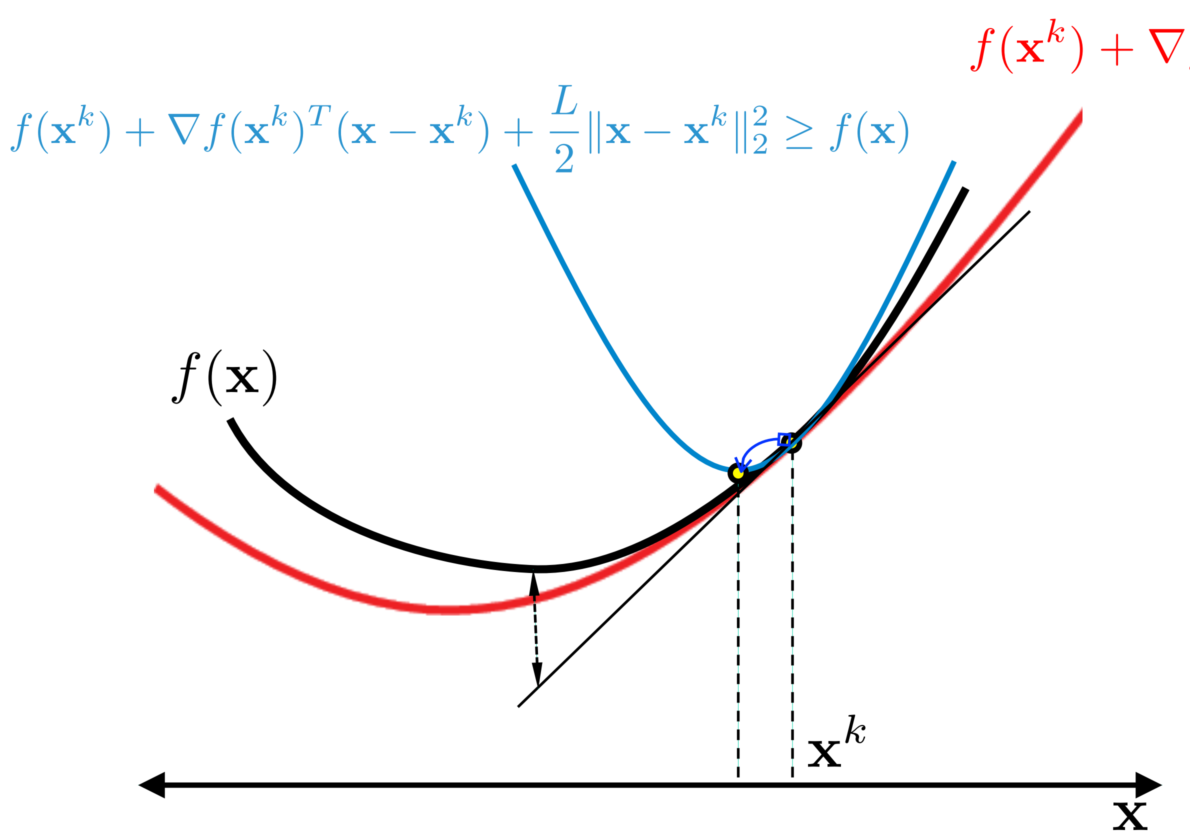

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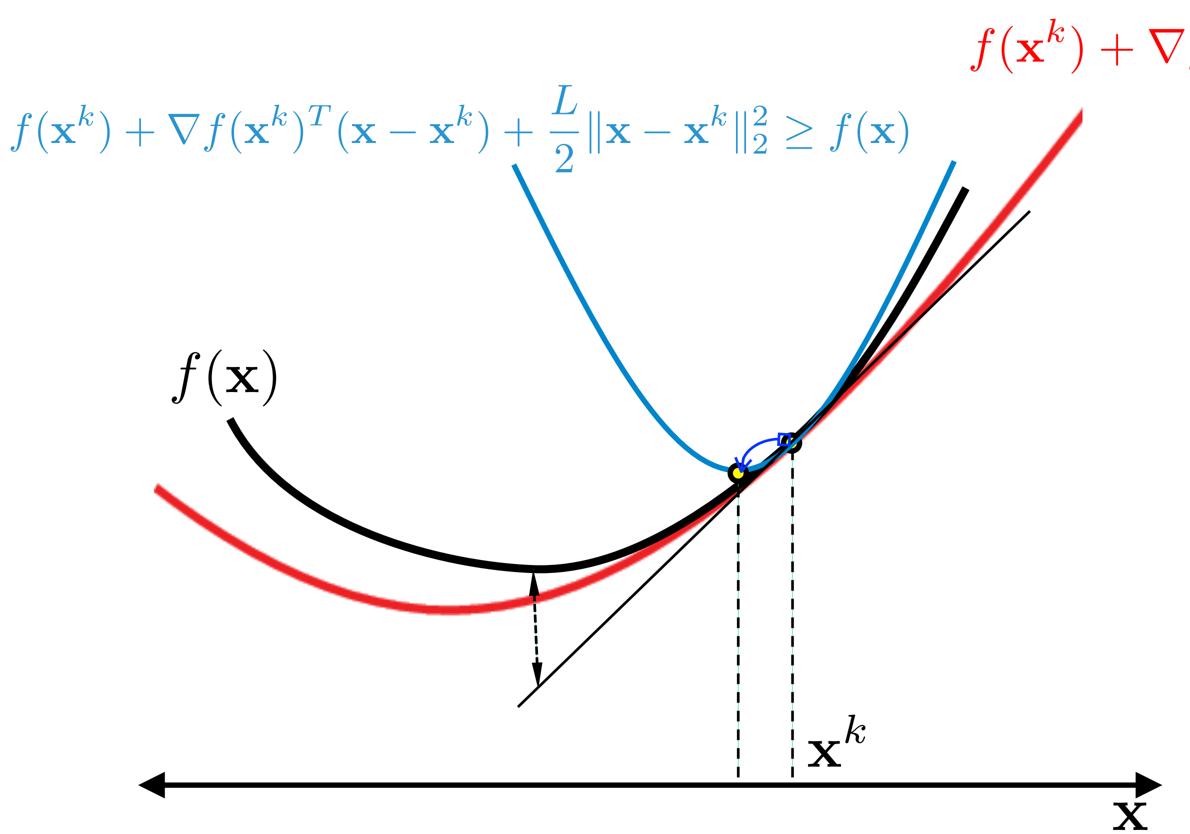

ISTA

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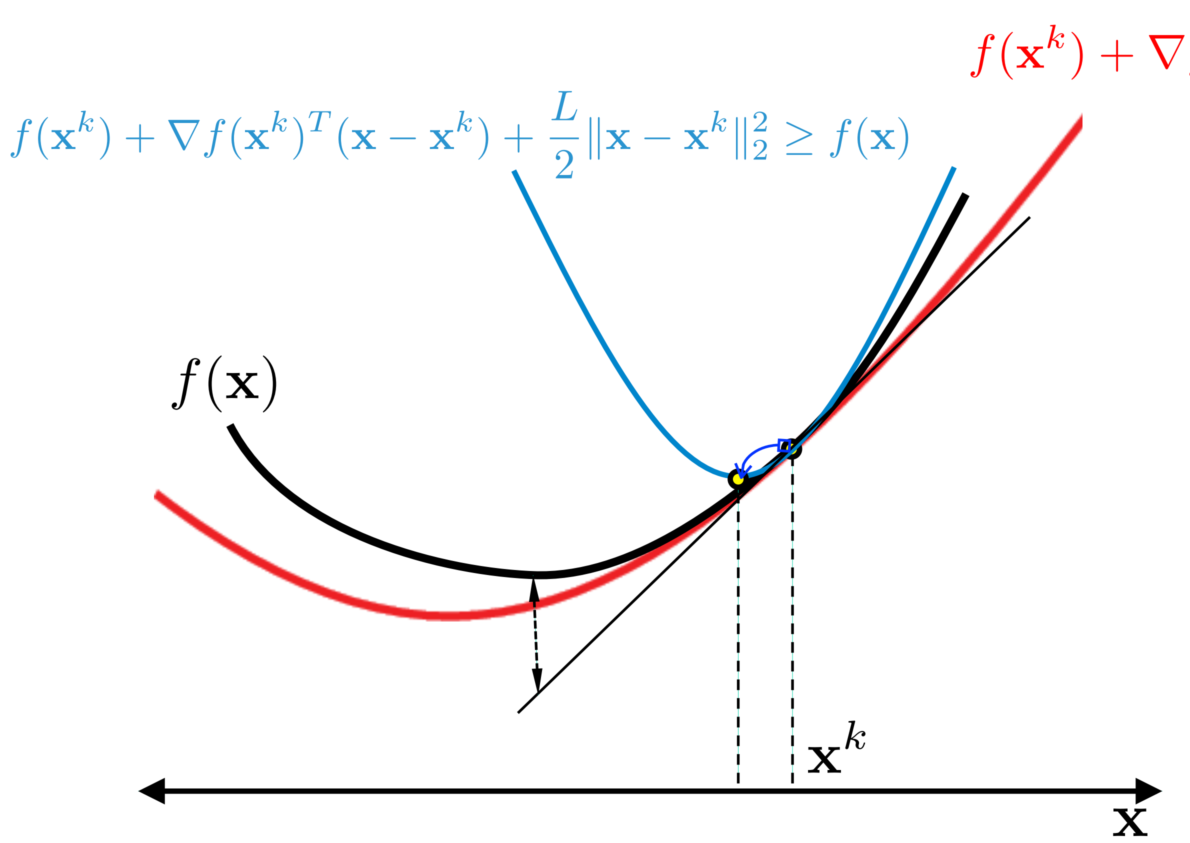

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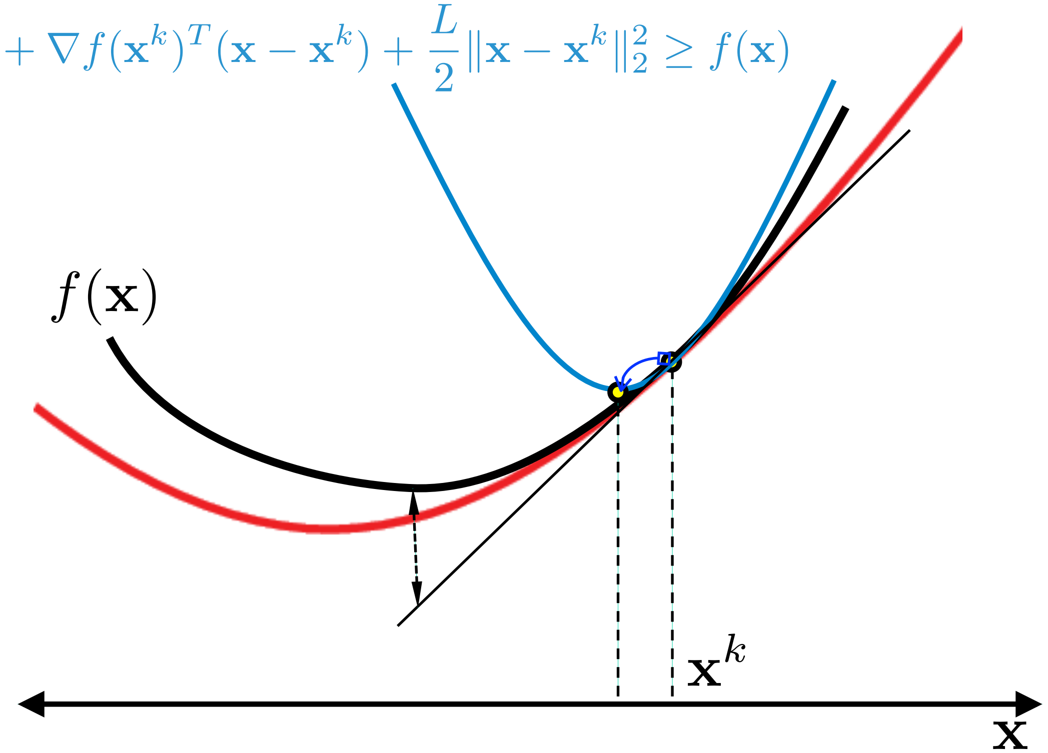

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acceleration is possible

FISTA

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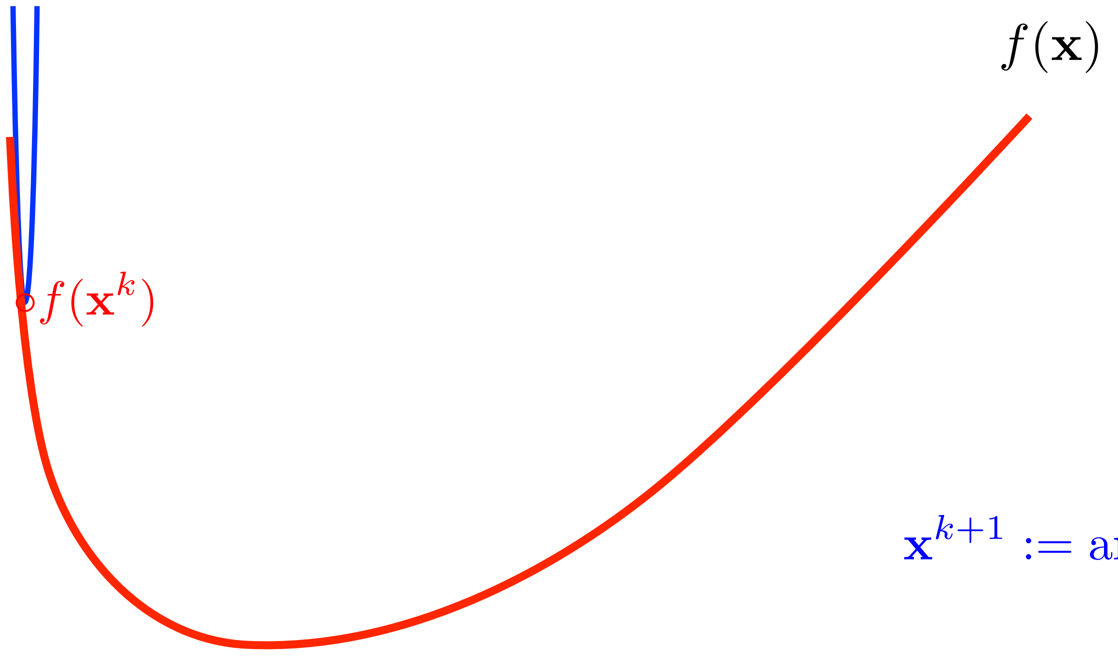
$\mathcal{F}_L$

To adapt or not to adapt?



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L is a global worst-case constant

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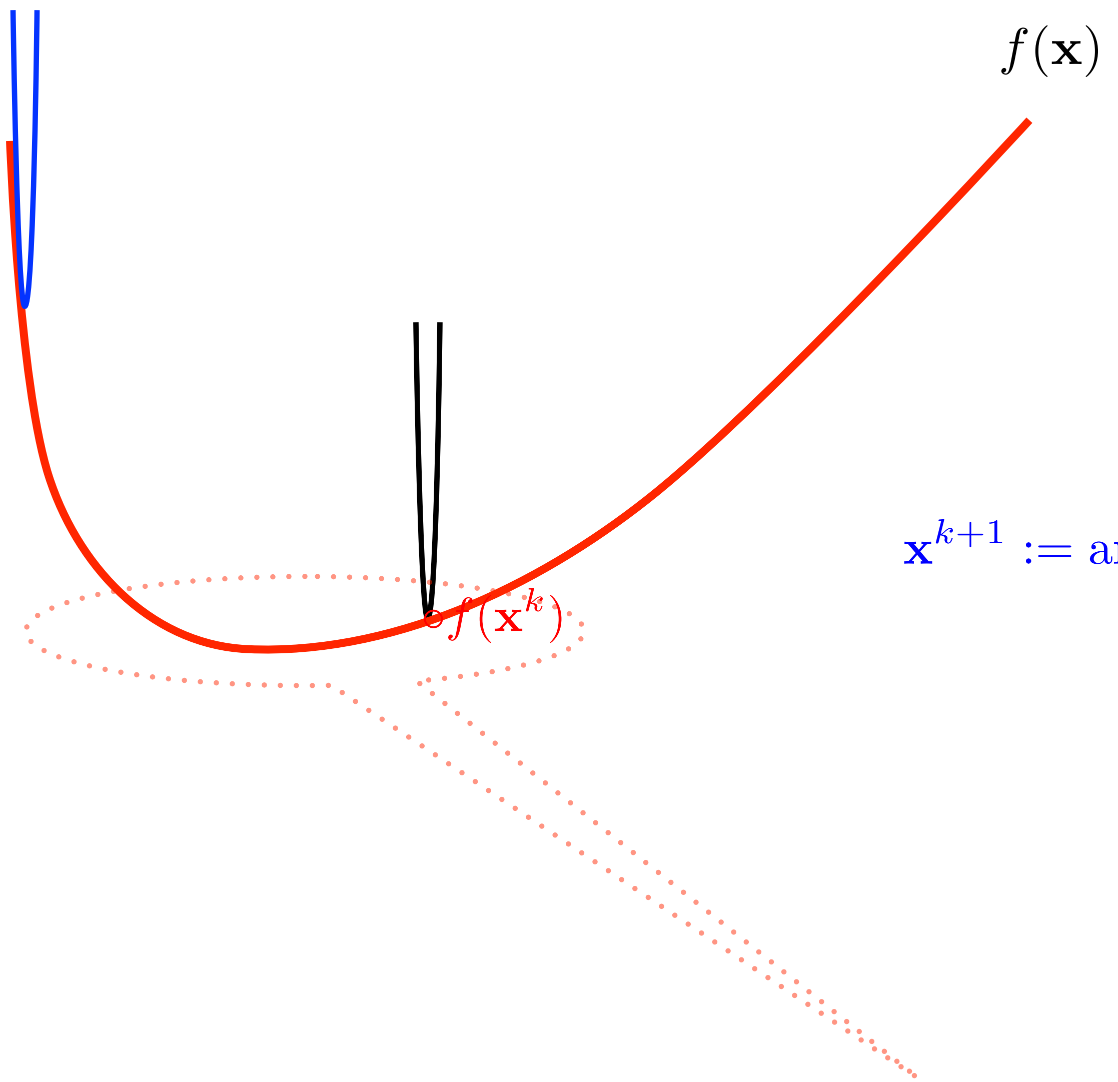

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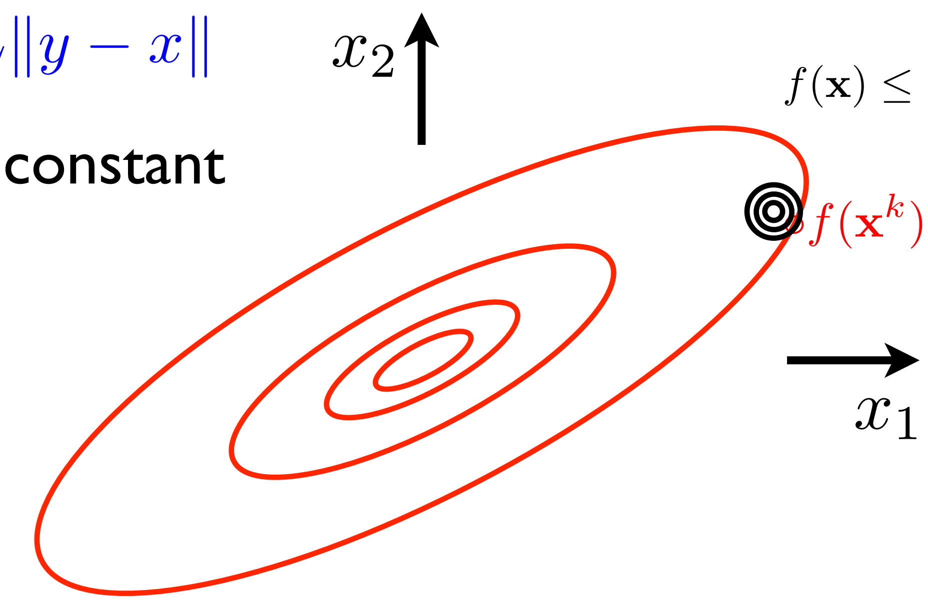

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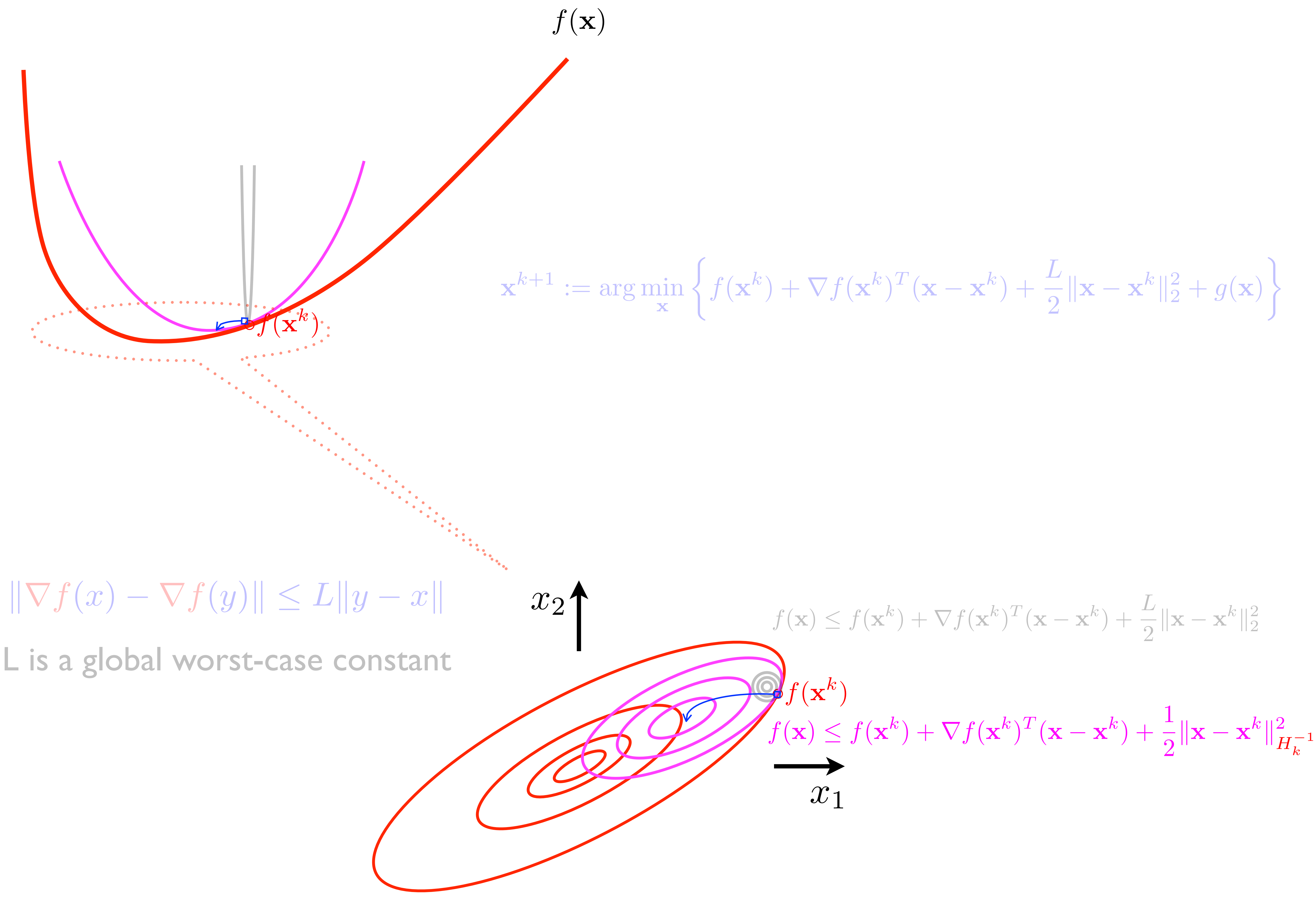

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Variable metric proximal point operator

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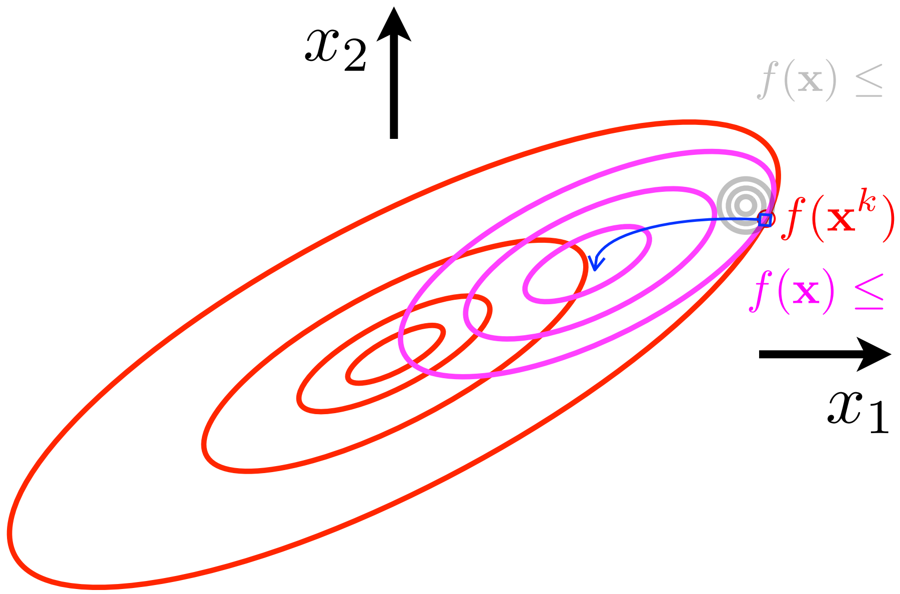

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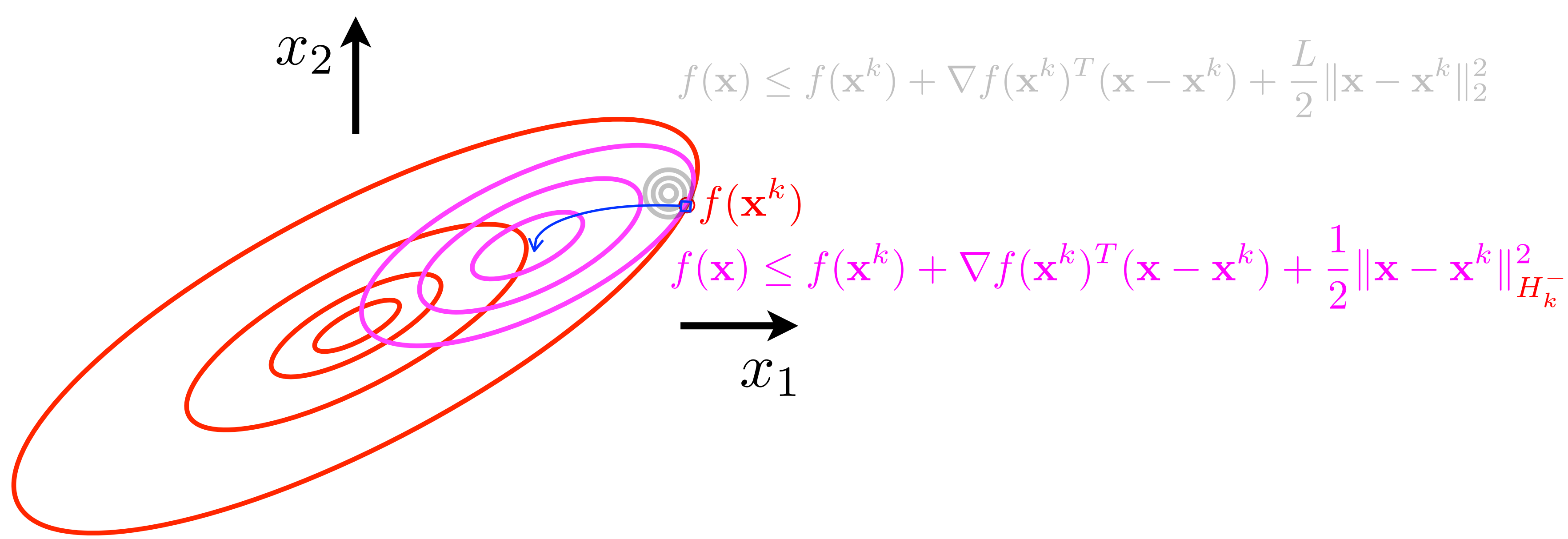

if $g(\mathbf{x}) = \|\mathbf{x}\|_1$, then
$\mathrm{prox}_{\gamma g}(\mathbf{s}) = \mathrm{SoftThresh}(\mathbf{s}, \gamma)$
$\mathrm{prox}_{H^{-1}g}(s) = \mathrm{LASSO}$

A basic variable metric minimization framework

- Proximal point scheme with variable metric [Bonnans, 1993]

Proximal point scheme with variable metric

Given x^0, generate a sequence $\{x^k\}_{k\geq 0}$ such that

$$x^{k+1} = \mathrm{prox}_{H_k^{-1}}\left(x^k - H_k{}^{-1}\nabla f(x^k)\right)$$

where H_k is symmetric positive definite

Variable metric proximal point operator

$$\mathrm{prox}_{H^{-1}g}(s) := \arg\min_x \left\{ g(x) + \frac{1}{2}\|x - s\|^2_{H^{-1}} \right\}$$

- Additional accuracy vs. computation trade-offs

Order	Example	Components	k
1-st	Accelerated gradient	$\nabla f, \mathrm{prox}_{1/L\mathbf{I}_n}$	$\mathcal{O}(\epsilon^{-1/2})$
1^+-th	BFGS	$H_k, \nabla f, \mathrm{prox}_{H_k^{-1}}$	$\mathcal{O}(\log \epsilon^{-1})$ or faster
2-nd	Proximal Newton, IPM	$\nabla^2 f, \nabla f, \mathrm{prox}_{\nabla^2 f^{-1}}$	$\mathcal{O}(\log\log \epsilon^{-1})$

Self-concordance vs. Lipschitz gradient + SC

- **Main properties of $\mathcal{F}_{\mu,L}$**

Lower surrogate	$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{\mu}{2}\lVert\mathbf{y}-\mathbf{x}\rVert_2^2$	$\mathbf{x},\mathbf{y} \in \mathrm{dom}(f)$
Upper surrogate	$f(\mathbf{y}) \leq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{L}{2}\lVert\mathbf{y}-\mathbf{x}\rVert_2^2$	$\mathbf{x},\mathbf{y} \in \mathrm{dom}(f)$
Hessian surrogates	$\mu\mathbb{I} \preceq \nabla^2 f(\mathbf{x}) \preceq L\mathbb{I}$	$\mathbf{x} \in \mathrm{dom}(f)$

Self-concordance vs. Lipschitz gradient + SC

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Lower surrogate	$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{\mu}{2}\lVert\mathbf{y}-\mathbf{x}\rVert_2^2$	$\mathbf{x}, \mathbf{y} \in \mathrm{dom}(f)$
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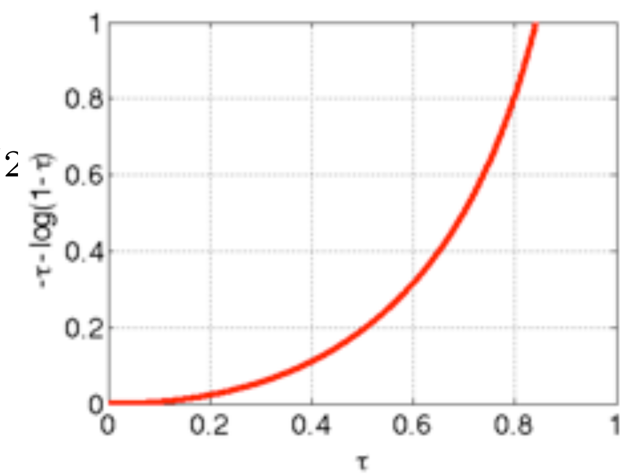
Global

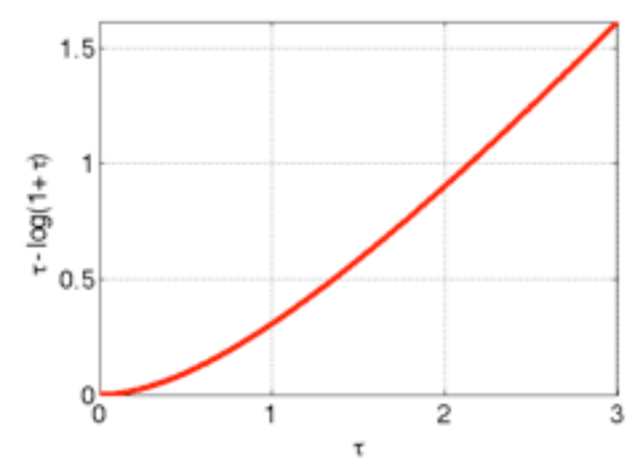
Self-concordance vs. Lipschitz gradient + SC

- **Main properties of $\mathcal{F}_2$**

Lower surrogate	$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \omega\left(\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}\right)$	$\mathbf{x},\mathbf{y} \in \mathrm{dom}(f)$
Upper surrogate	$f(\mathbf{y}) \leq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \omega_*\left(\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}\right)$	$\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}} < 1$
Hessian surrogates	$(1-\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}})^2\nabla^2 f(\mathbf{x}) \preceq \nabla^2 f(\mathbf{y}) \preceq (1-\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}})^{-2}\nabla^2 f(\mathbf{x})$	$\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}} < 1$

Utility functions: $\omega_*(\tau) = -\tau - \ln(1-\tau),\ \tau \in [0,1)$ $\qquad \omega(\tau) = \tau - \ln(1+\tau),\ \tau \geq 0$





Local norm: $\|\mathbf{u}\|_{\mathbf{x}} := \left[\mathbf{u}^T\nabla^2 f(\mathbf{x})\mathbf{u}\right]^{1/2}$

$$\frac{\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}^2}{1+\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}} \leq (\nabla f(\mathbf{y}) - \nabla f(\mathbf{x}))^T(\mathbf{y}-\mathbf{x}) \leq \underbrace{\frac{\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}^2}{1-\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}}}_{\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}<1}, \quad \forall \mathbf{x},\mathbf{y} \in \mathrm{dom}(f)$$

f is self-concordant if $\varphi(t) := f(\mathbf{x}+t\mathbf{d})$ satisfies $|\varphi'''(t)| \leq 2\varphi''(t)^{3/2}$

Self-concordance vs. Lipschitz gradient + SC

- **Main properties of $\mathcal{F}_2$**

Lower surrogate	$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \omega\left(\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}\right)$	$\mathbf{x}, \mathbf{y} \in \mathrm{dom}(f)$
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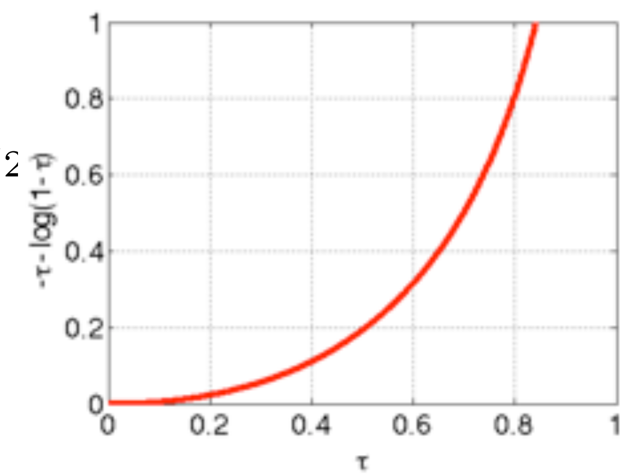

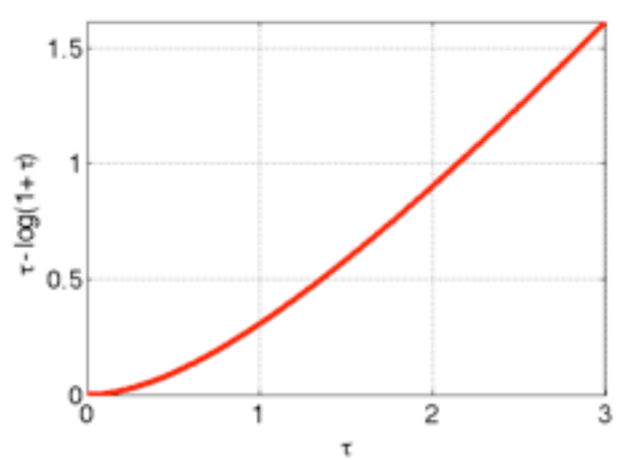

$$\frac{\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}^2}{1+\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}} \leq (\nabla f(\mathbf{y}) - \nabla f(\mathbf{x}))^T(\mathbf{y}-\mathbf{x}) \leq \underbrace{\frac{\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}^2}{1-\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}}}_{\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}<1}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathrm{dom}(f)$$

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Self-concordance: A mathematical tool

- **Main properties of $\mathcal{F}_2$**

Lower surrogate	$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \omega\left(\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}}\right)$	$\mathbf{x}, \mathbf{y} \in \mathrm{dom}(f)$
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Hessian surrogates	$(1-\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}})^2\nabla^2 f(\mathbf{x}) \preceq \nabla^2 f(\mathbf{y}) \preceq (1-\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}})^{-2}\nabla^2 f(\mathbf{x})$	$\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}} < 1$

- New variable metric framework with rigorous convergence guarantees

$$\min_{\mathbf{x}\in\mathbb{R}^n} \left\{\phi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\right\}$$

Includes several algorithms: Newton, quasi-Newton, and gradient methods...

f is self-concordant if $\varphi(t) := f(\mathbf{x}+t\mathbf{d})$ satisfies $|\varphi'''(t)| \leq 2\varphi''(t)^{3/2}$

Our composite self-concordant minimization framework

- Proximal Newton scheme $\mathbf{H}_k = \nabla^2 f(\mathbf{x}^k)$

Given $\mathbf{x}^0$, generate a sequence $\{\mathbf{x}^k\}_{k\geq 0}$ such that

$$\boxed{\mathbf{x}^{k+1} := \mathbf{x}^k + \alpha_k \mathbf{d}^k_{\mathbf{H}_k}}$$

where $\alpha_k \in (0, 1]$ is step-size, $\mathbf{d}^k_{\mathbf{H}_k}$ is a search direction

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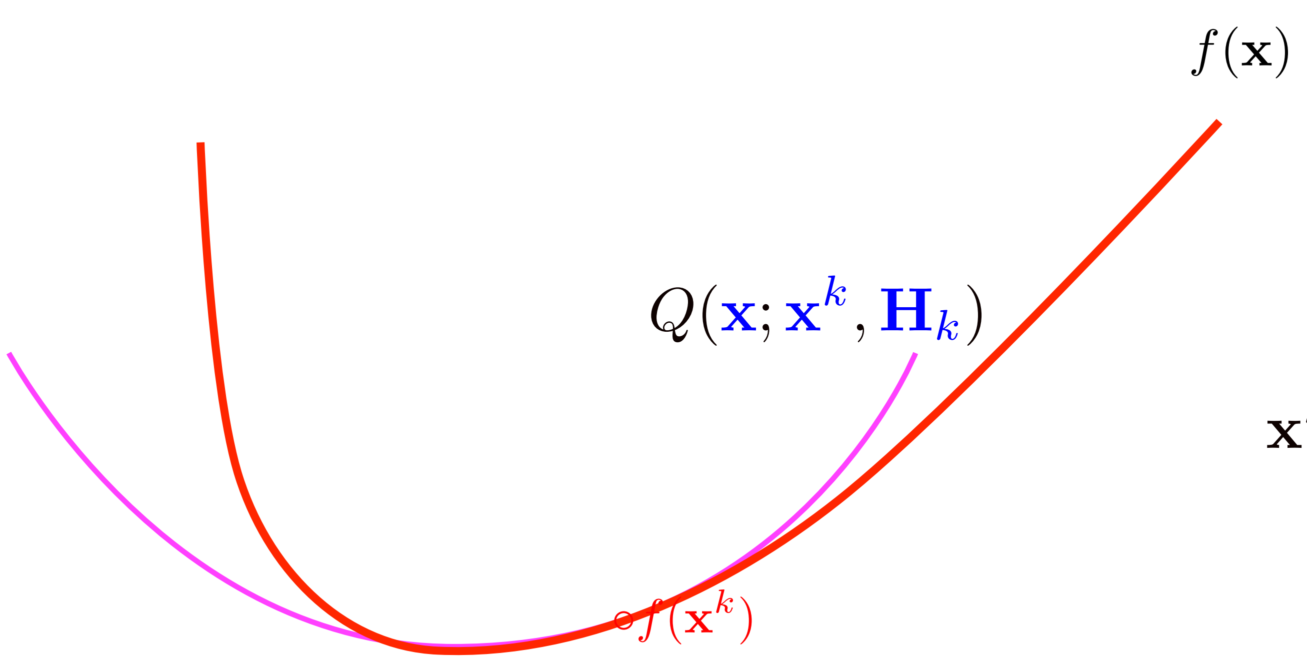
- How to compute the Proximal Newton direction?

$$\mathbf{d}^k_{\mathbf{H}_k} := \arg\min_{\mathbf{d}} \left\{ f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}_k \mathbf{d} + g(\mathbf{x}^k + \mathbf{d}) \right\}, \quad \mathbf{H}_k = \nabla^2 f(\mathbf{x}^k)$$



$$\mathbf{s}^k_g := \arg\min_{\mathbf{x}\in\mathrm{dom}(F)} \left\{ Q(\mathbf{x}; \mathbf{x}^k, \mathbf{H}_k) + g(\mathbf{x}) \right\}$$

$$\mathbf{x}^{k+1} := \mathbf{x}^k + \alpha_k \left(\mathbf{s}^k_g - \mathbf{x}^k \right)$$

Our composite self-concordant minimization framework

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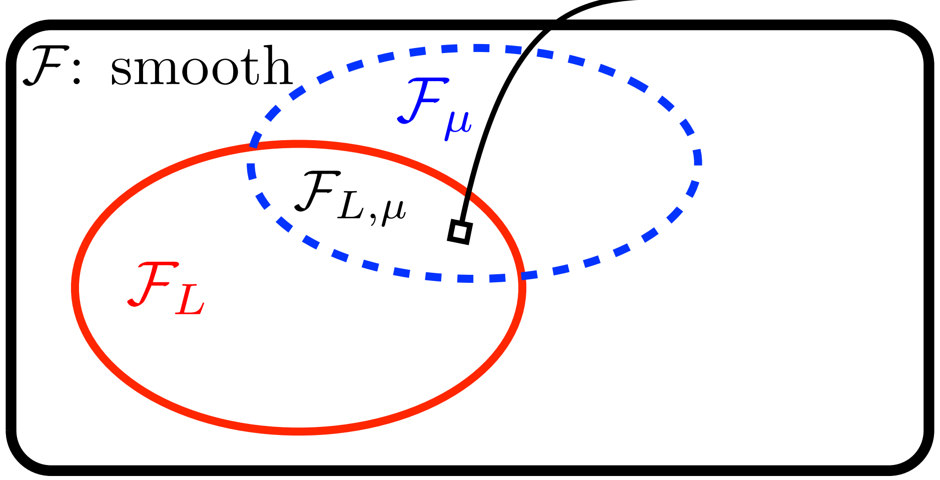

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|y - x\|$$

$$\mu\mathbb{I} \preceq \nabla^2 f(x) \preceq L\mathbb{I}$$

Fast gradient schemes (Nesterov's methods)

Newton/quasi Newton schemes

Our composite self-concordant minimization framework

- Proximal Newton scheme

Key contribution:
step size selection procedure

$\mathbf{H}_k = \nabla^2 f(\mathbf{x}^k)$

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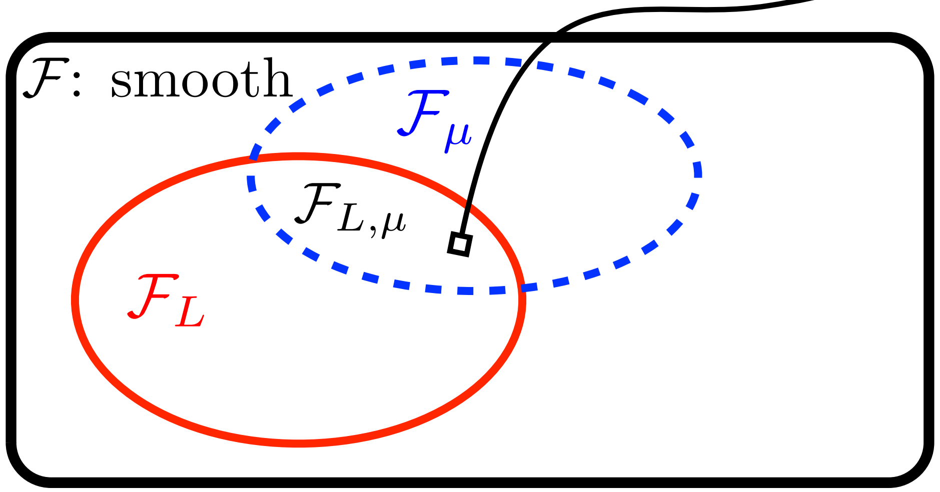

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|y - x\|$$

$$\mu\mathbb{I} \preceq \nabla^2 f(x) \preceq L\mathbb{I}$$

Fast gradient schemes (Nesterov's methods)

Newton/quasi Newton schemes

How do we compute the step-size?

- Upper surrogate of f

$$f(\mathbf{x}^{k+1}) \leq f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)^T(\mathbf{x}^{k+1} - \mathbf{x}^k) + \omega^*(\|\mathbf{x}^{k+1} - \mathbf{x}^k\|_{\mathbf{x}^k}), \quad \|\mathbf{x}^{k+1} - \mathbf{x}^k\|_{\mathbf{x}^k} < 1$$

- Convexity of g and optimality condition of the subproblem

$$g(\mathbf{x}^{k+1}) - g(\mathbf{x}^k) \leq -\alpha_k \nabla f(\mathbf{x}^k)^T \mathbf{d}^k_{\mathbf{H_k}} - \alpha_k \|\mathbf{d}^k_{\mathbf{H_k}}\|^2_{\mathbf{H}_k}.$$

How do we compute the step-size?

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$$\boxed{\phi(\mathbf{x}^{k+1}) \leq \phi(\mathbf{x}^k) - \alpha_k \|\mathbf{d}^k_{\mathbf{H}_k}\|^2_{\mathbf{H}_k} + \omega_*\left(\alpha_k \|\mathbf{d}^k_{\mathbf{H}_k}\|_{\mathbf{x}^k}\right)}$$

How do we compute the step-size?

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- When $\mathbf{H}_k \equiv \nabla^2 f(\mathbf{x}^k)$, $\lambda_k := \|\mathbf{d}^k_{\mathbf{H}_k}\|_{\mathbf{x}^k}$

$$\boxed{\phi(\mathbf{x}^{k+1}) \leq \phi(\mathbf{x}^k) - \underbrace{[\alpha_k \lambda_k - \omega_*(\alpha_k \lambda_k)]}_{\psi(\alpha_k)}}$$

maximize $\psi(\alpha_k)$ to get optimal α_k^*

$$\boxed{\alpha_k^* = \frac{1}{\lambda_k + 1} \in (0, 1]}$$

Analytic complexity

- Worst-case complexity to obtain an ε-approximate solution

$$\#\text{iterations} = \left\lfloor \frac{\phi(\mathbf{x}^0) - \phi(\mathbf{x}^*)}{0.021} \right\rfloor + O\left(\ln\ln\left(\frac{4.56}{\varepsilon}\right)\right)$$

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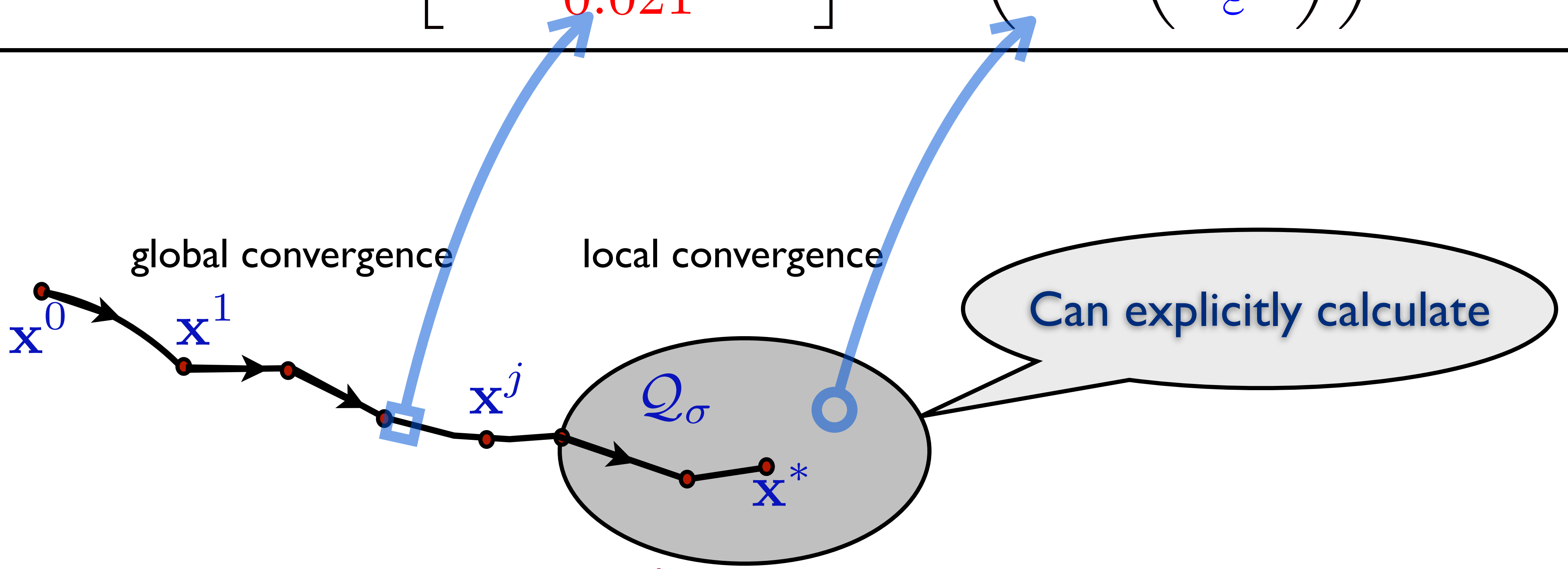
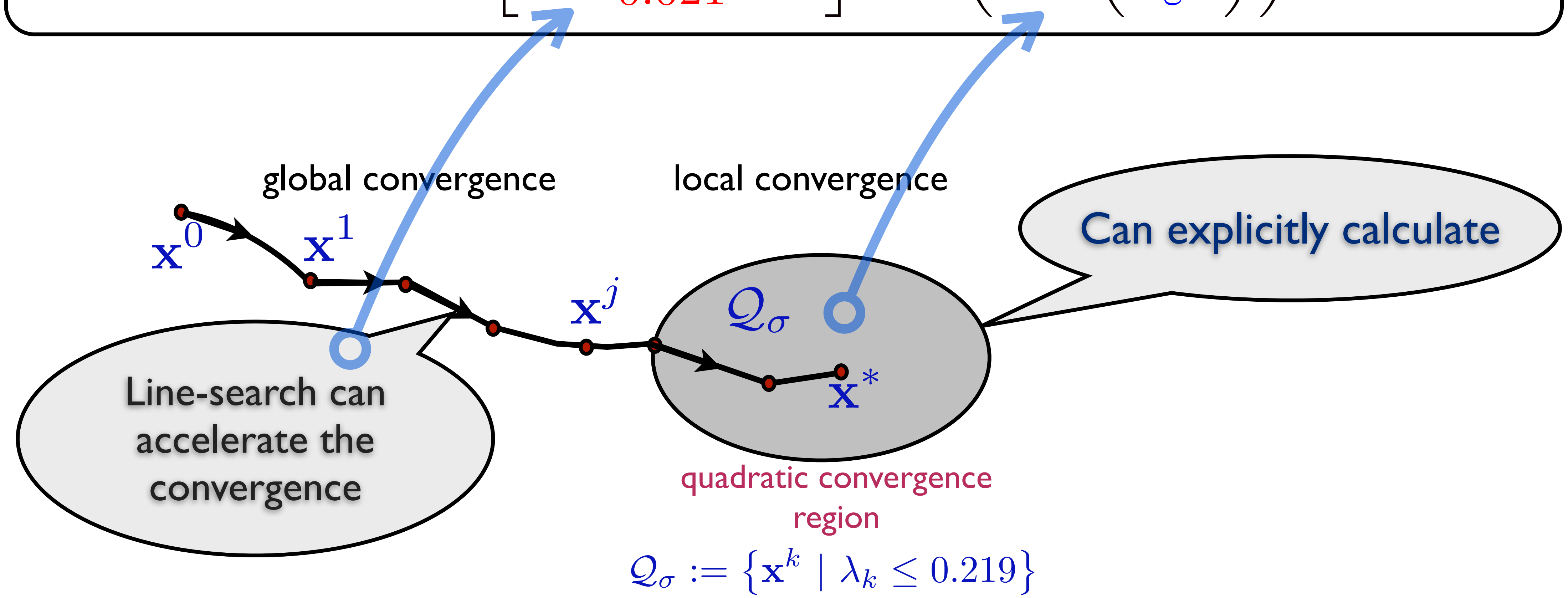


$$\mathcal{Q}_\sigma := \{\mathbf{x}^k \mid \lambda_k \le 0.219\}$$

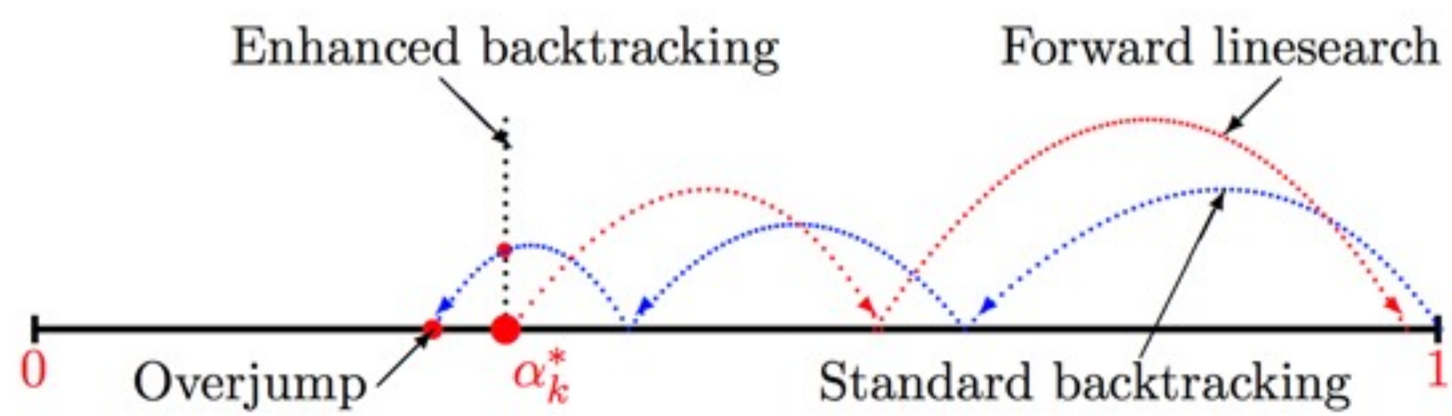
Analytic complexity

- Worst-case complexity to obtain an ε-approximate solution

$$\#\text{iterations} = \left\lfloor \frac{\phi(\mathbf{x}^0) - \phi(\mathbf{x}^*)}{0.021} \right\rfloor + O\left(\ln\ln\left(\frac{4.56}{\varepsilon}\right)\right)$$



- Line-search enhancement



Graphical model selection

- Objective:

$$\min_{\boldsymbol{\Theta} \succ 0} \left\{ \underbrace{-\log\det(\boldsymbol{\Theta}) + \mathrm{trace}(\boldsymbol{\Sigma\Theta})}_{f(\mathbf{x})} + \underbrace{\rho \|\mathrm{vec}(\boldsymbol{\Theta})\|_1}_{g(\mathbf{x})} \right\}$$

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- Gradient and Hessian (large-scale, special structure)
  - Gradient of f: $\nabla f(\mathbf{x}) = \mathbf{vec}(\boldsymbol{\Sigma} - \boldsymbol{\Theta}^{-1})$.
  - Hessian of f: $\nabla^2 f(\mathbf{x}) = \boldsymbol{\Theta}^{-1} \otimes \boldsymbol{\Theta}^{-1}$

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  - Gradient of f: $\nabla f(\mathbf{x}) = \mathbf{vec}(\Sigma - \Theta^{-1})$.
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- Dual approach for solving subproblem (SP)

Primal subproblem	Dual subproblem (SPGL)
$\min_{\Delta} \left\{ \frac{1}{2}\mathrm{trace}((\Theta_i^{-1}\Delta)^2) + \mathrm{trace}(\mathbf{R}_i\Delta) + \rho\|\mathrm{vec}(\Delta)\|_1 \right\}$	$\min_{\|\mathrm{vec}(\mathbf{U})\|_\infty \leq 1} \left\{ \frac{1}{2}\mathrm{trace}((\Theta_i\mathbf{U})^2) + \mathrm{trace}(\mathbf{Q}_i\mathbf{U}) \right\}$
$\mathbf{R}_i := \Sigma - 2\Theta_i^{-1}$	$\mathbf{Q}_i := \rho^{-1}[\Theta_i\Sigma\Theta_i - 2\Theta_i]$

Unconstrained LASSO problem

Graphical model selection

- Objective:

$$\min_{\Theta \succ 0} \left\{ \underbrace{-\log\det(\Theta) + \mathrm{trace}(\Sigma\Theta)}_{f(\mathbf{x})} + \underbrace{\rho\|\mathrm{vec}(\Theta)\|_1}_{g(\mathbf{x})} \right\}$$

- Gradient and Hessian (large-scale, special structure)
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No Cholesky decomposition and matrix inversion

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Primal subproblem	Dual subproblem (SPGL)
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$\mathbf{R}_i := \Sigma - 2\Theta_i^{-1}$	$\mathbf{Q}_i := \rho^{-1}[\Theta_i\Sigma\Theta_i - 2\Theta_i]$

Unconstrained LASSO problem

Graphical model selection

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No Cholesky decomposition and matrix inversion

- Dual approach for solving subproblem (SP)

Primal subproblem	Dual subproblem (SPGL)
Unconstrained LASSO problem	$\min_{\|\mathrm{vec}(\mathbf{U})\|_\infty \le 1} \left\{ \frac{1}{2}\mathrm{trace}((\boldsymbol{\Theta}_i\mathbf{U})^2) + \mathrm{trace}(\mathbf{Q}_i\mathbf{U}) \right\}$
$\mathbf{R}_i := \boldsymbol{\Sigma} - 2\boldsymbol{\Theta}_i^{-1}$	$\mathbf{Q}_i := \rho^{-1}[\boldsymbol{\Theta}_i\boldsymbol{\Sigma}\boldsymbol{\Theta}_i - 2\boldsymbol{\Theta}_i]$

- How to compute proximal Newton decrement $\lambda_i := \|\mathbf{d}^i\|_{\mathbf{x}^i}$?

$$\lambda_i := [p - 2\mathrm{trace}(\mathbf{W}_i) + \mathrm{trace}(\mathbf{W}_i^2)]^{1/2}, \quad \mathbf{W}_i = \boldsymbol{\Theta}_i(\boldsymbol{\Sigma} - \rho\mathbf{U}^*)$$

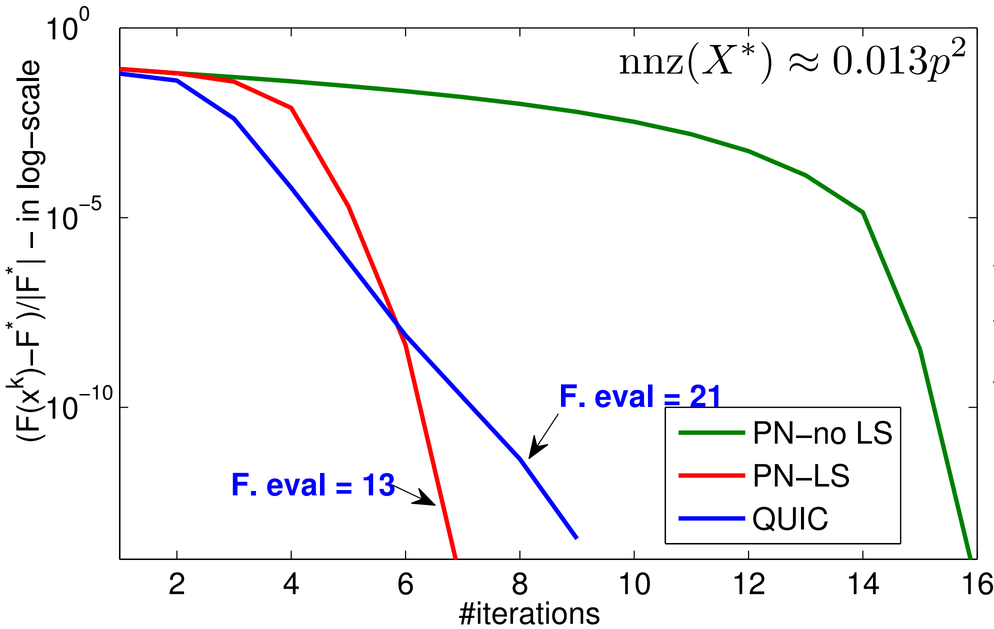
Graphical model selection: numerical examples

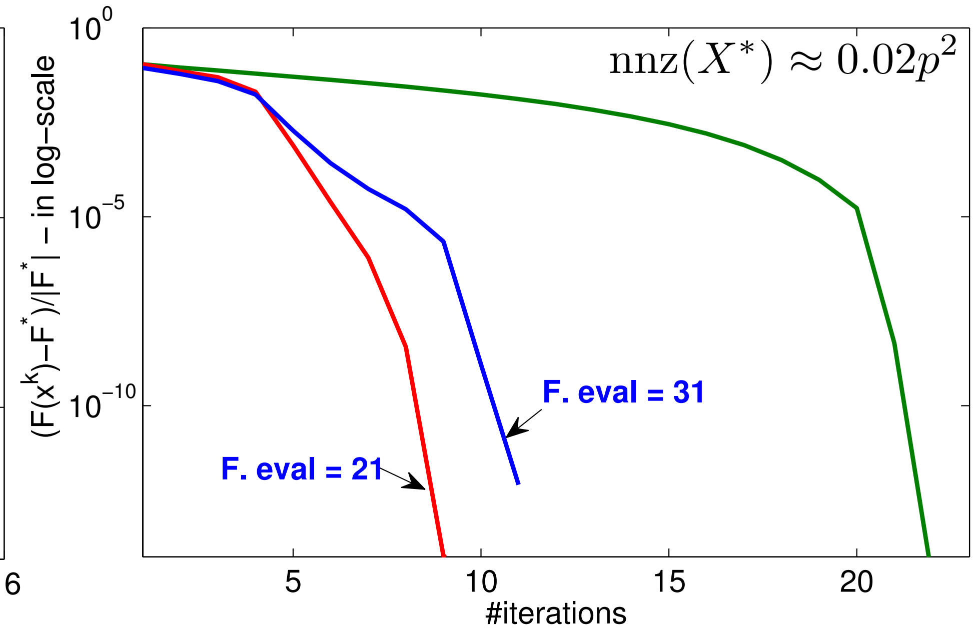
Our method vs QUIC [Hseih2011]

- QUIC subproblem solver: special block-coordinate descent

- Our subproblem solver: general proximal algorithms

Convergence behaviour [rho = 0.5]: Lymph [p = 587] (left), Leukemia [p = 1255] (right)





Step-size selection strategies: Arabidopsis [p = 834], Leukemia [p = 1255], Hereditary [p = 1869]

	Synthetic ($\rho = 0.01$)			Arabidopsis ($\rho = 0.5$)			Leukemia ($\rho = 0.1$)			Hereditary ($\rho = 0.1$)		
LS SCHEME	#iter	#chol	#Mm	#iter	#chol	#Mm	#iter	#chol	#Mm	#iter	#chol	#Mm
NoLS	25.4	-	3400	18	-	1810	44	-	9842	72	-	20960
BtkLS	25.5	37.0	2436	11	25	718	15	50	1282	19	63	2006
E-BtkLS	25.5	36.2	2436	11	24	718	15	49	1282	15	51	1282
FwLS	18.1	26.2	1632	10	17	612	12	34	844	14	44	1126

*Details: "A proximal Newton framework for composite minimization: Graph learning without Cholesky decompositions and matrix inversions," ICML'13 and lions.epfl.ch/publications.

Graphical model selection: numerical examples

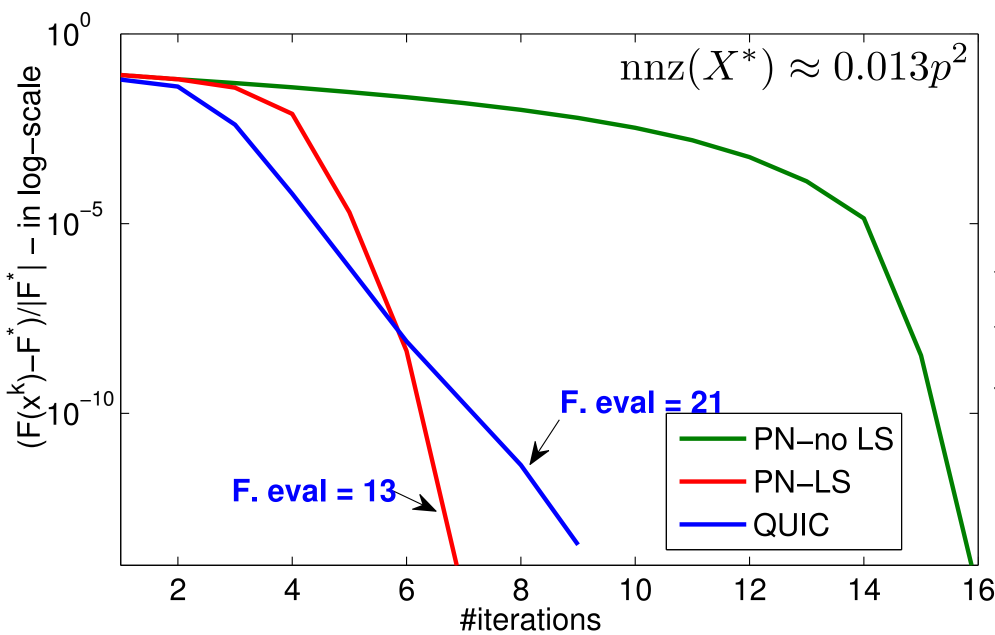
Our method vs QUIC [Hseih2011]

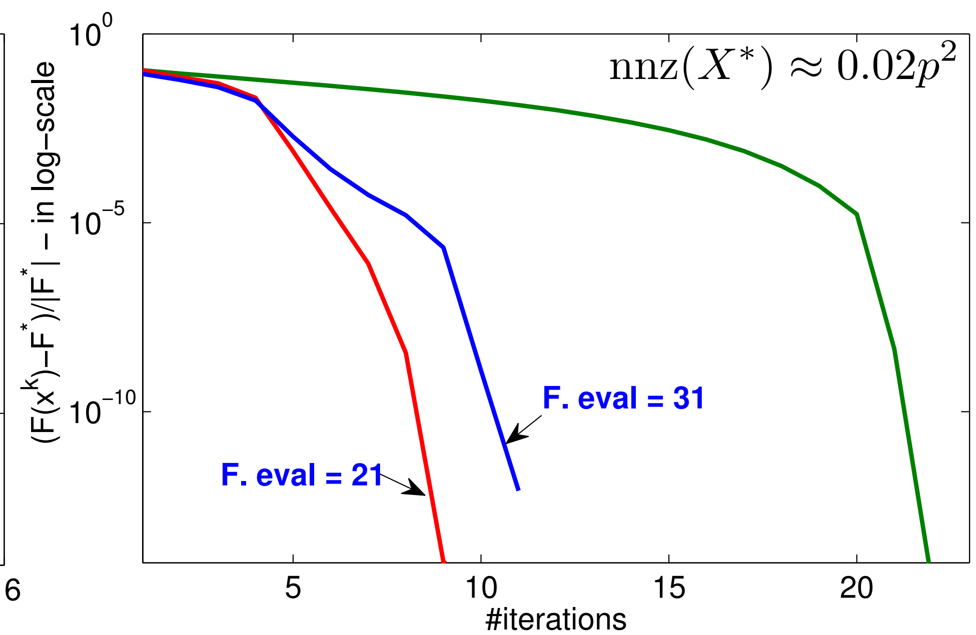
- QUIC subproblem solver: special block-coordinate descent

On the average x5 acceleration (up to x15) over Matlab QUIC

- Our subproblem solver: general proximal algorithms

Convergence behaviour [rho = 0.5]: Lymph [p = 587] (left), Leukemia [p = 1255] (right)





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*Details: "A proximal Newton framework for composite minimization: Graph learning without Cholesky decompositions and matrix inversions," ICML'13 and lions.epfl.ch/publications.

Composite minimization: alternatives?

(P) $$\min_{\mathbf{x}\in\mathbb{R}^n} \{\phi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\}$$

f
convex and **smooth**

g
convex and possibly **nonsmooth**

Existing numerical approaches

- Splitting methods

  - **Forward-backward**: *applicable if **f** has Lipschitz gradient*

  - **Douglas-Rachford** decomposition: ***f*** *and* ***g*** *have "tractable" proximity operators*

- Augmented Lagrangian methods (e.g., D-R again)

$$\min_{\mathbf{x}\in\mathbb{R}^n} \{\phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + g(\mathbf{y})\}$$
$$\text{s.t.} \quad \boxed{\mathbf{x} - \mathbf{y} = 0}$$

$-\log\det$: full eigen decomposition

$-\log$ (with a linear operator): non-linear system

Prox operator of self-concordant functions are costly!

- New theory for AL/ADMM/decomposition (*another presentation!*)

Our “cheaper” variable metric strategies

- Proximal **gradient** scheme*

Given $\mathbf{x}^0$, generate a sequence $\{\mathbf{x}^k\}_{k\geq 0}$ such that

$$\boxed{\mathbf{x}^{k+1} := \mathbf{x}^k + \alpha_k \mathbf{d}^k_{\mathbf{H}_k}}$$

where $\alpha_k \in (0,1]$ is step-size, $\mathbf{d}^k_{\mathbf{H}_k}$ is a search direction

- How to compute the search direction?

$$\mathbf{d}^k_{\mathbf{H}_k} := \arg\min_{\mathbf{d}} \left\{ f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}_k \mathbf{d} + g(\mathbf{x}^k + \mathbf{d}) \right\}, \quad \mathbf{H}_k = \mathbf{D}_k : \text{ diagonal}$$

- No Lipschitz assumption

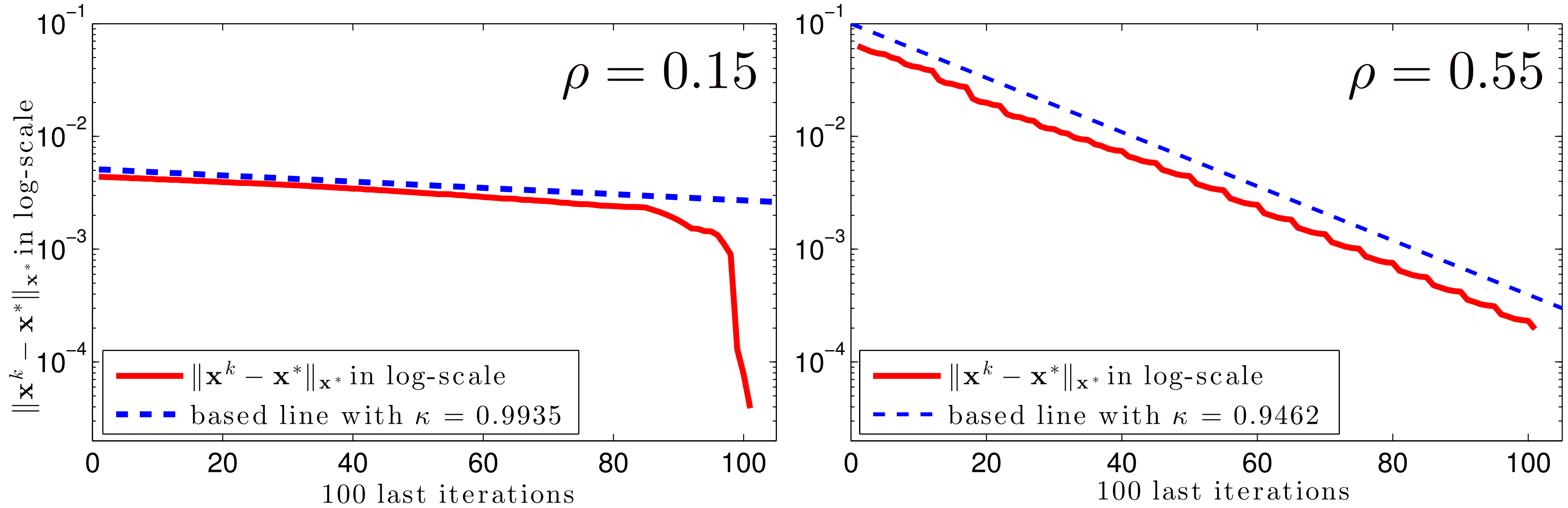
A new predictor corrector scheme (with local linear convergence*)

- Proximal **quasi**-Newton scheme (BFGS updates)*

*Details: “Composite Self-Concordant Minimization,” arxiv and lions.epfl.ch/publications.

New theory: Local linear convergence of the PG

Graph learning: Lymph [p = 587]



$$\min_{\boldsymbol{\Theta} \succ 0} \left\{ \underbrace{-\log\det(\boldsymbol{\Theta}) + \mathrm{trace}(\boldsymbol{\Sigma}\boldsymbol{\Theta})}_{f(\mathbf{x})} + \underbrace{\rho\|\mathrm{vec}(\boldsymbol{\Theta})\|_1}_{g(\mathbf{x})} \right\}$$

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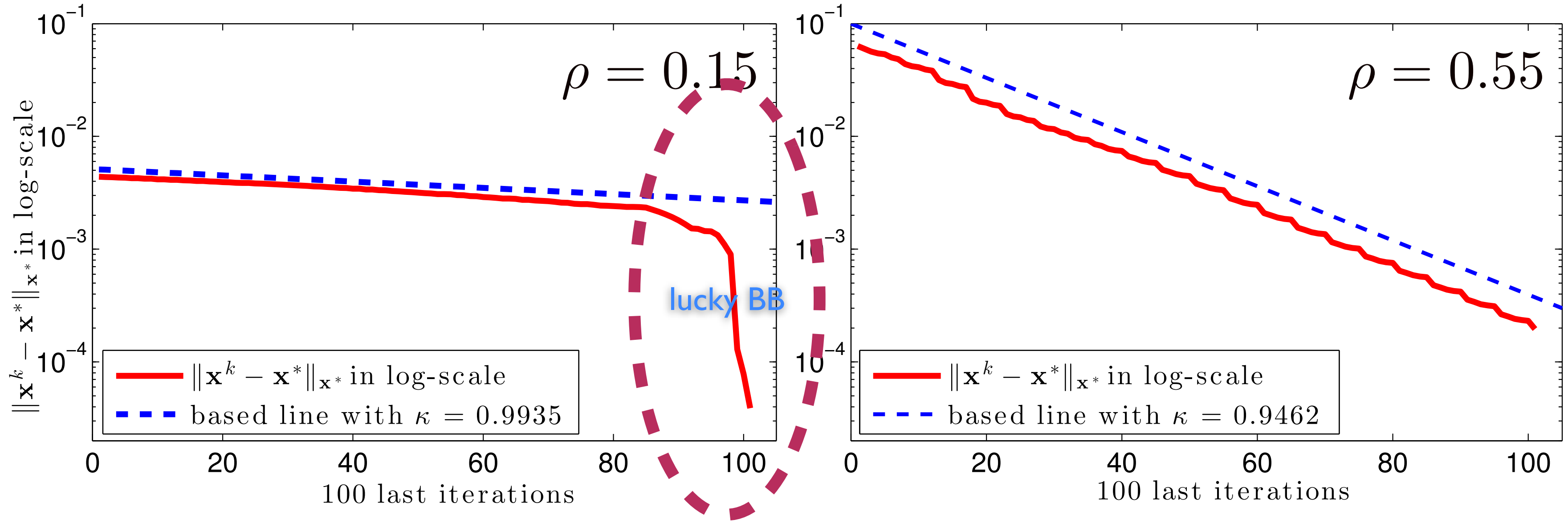

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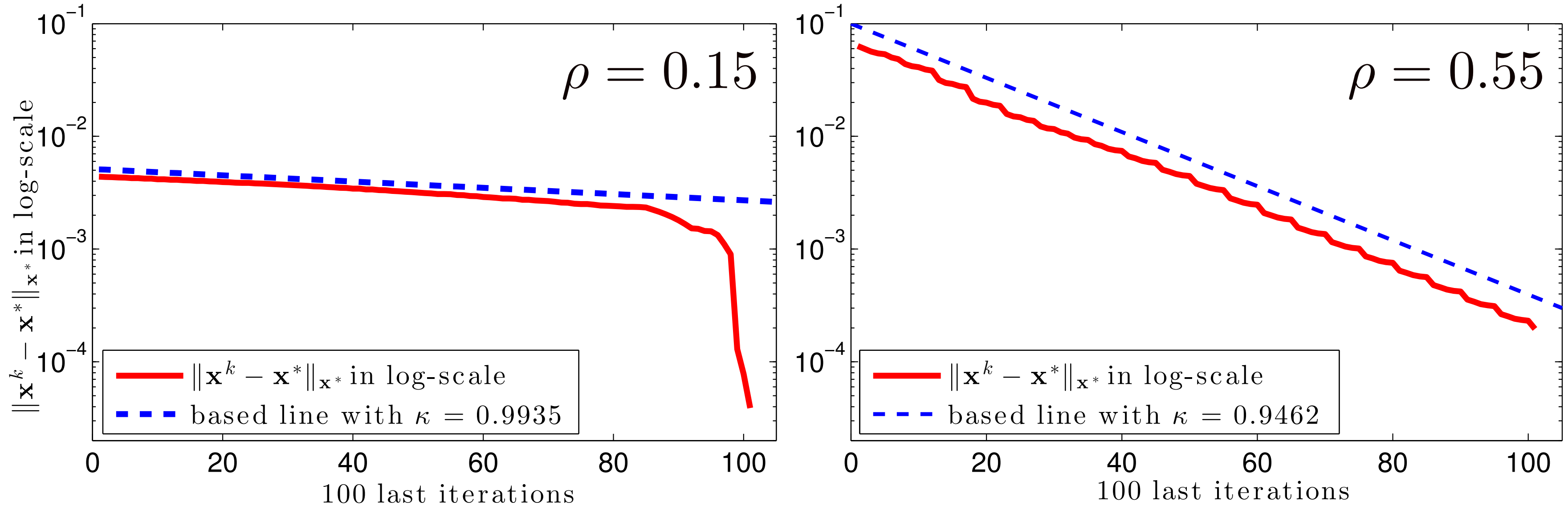

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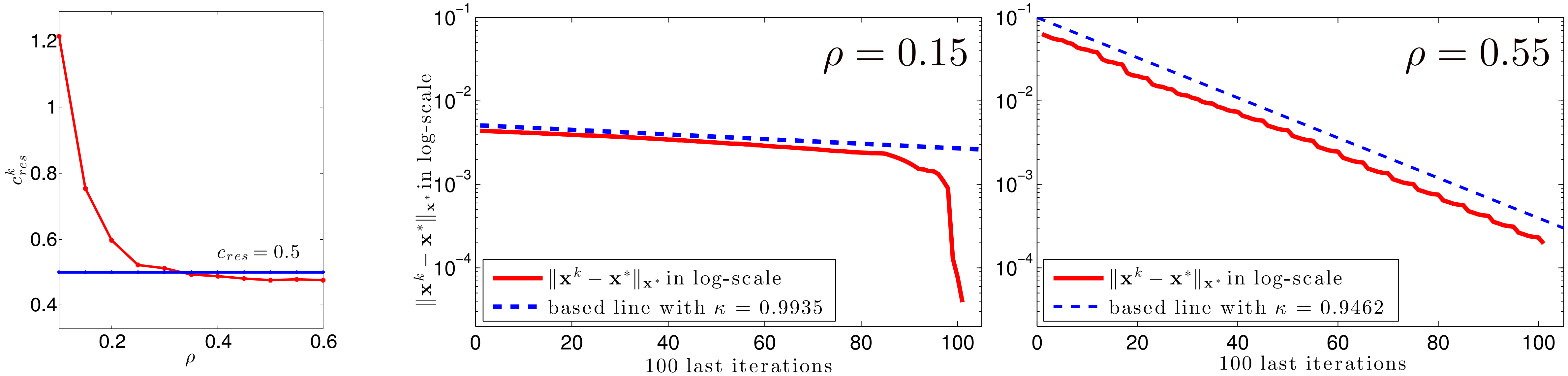

Theory is based on the notion of "restricted strong convexity"

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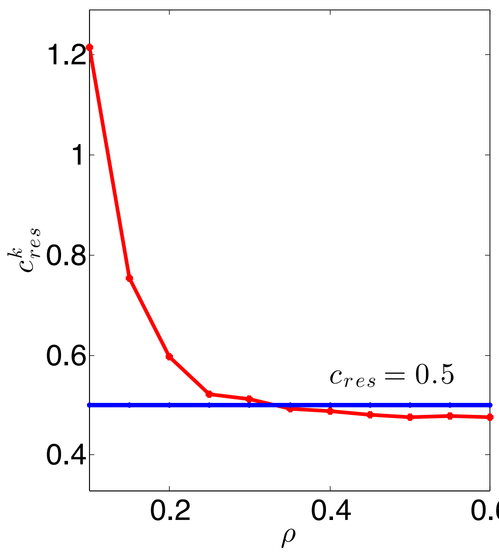

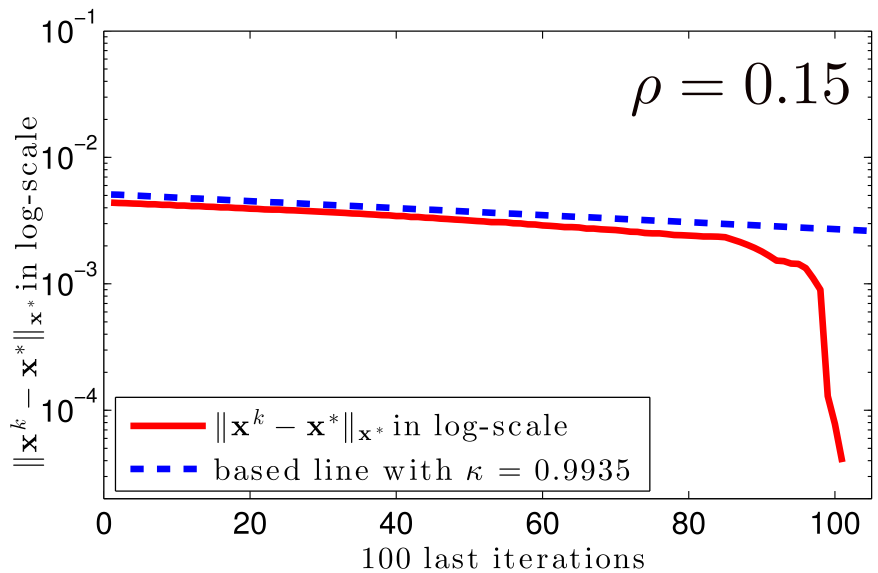

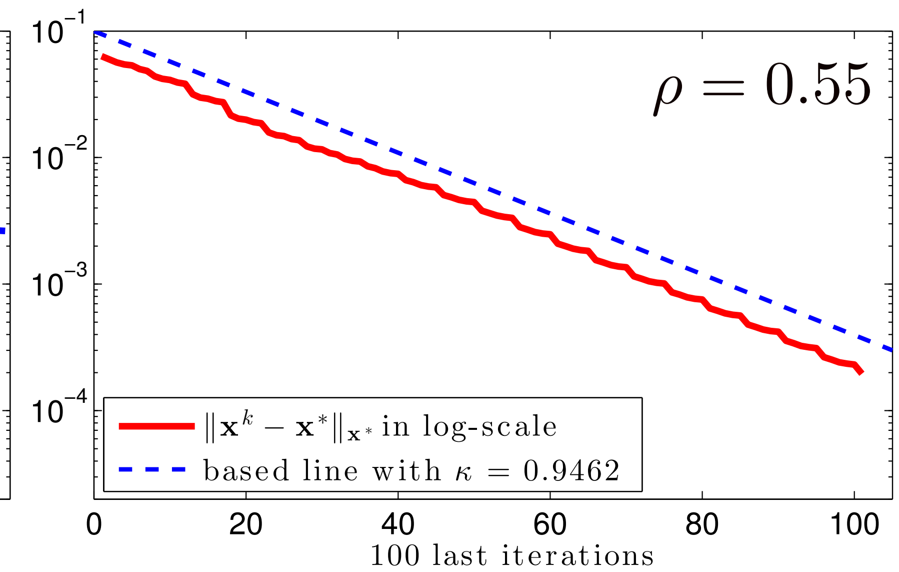

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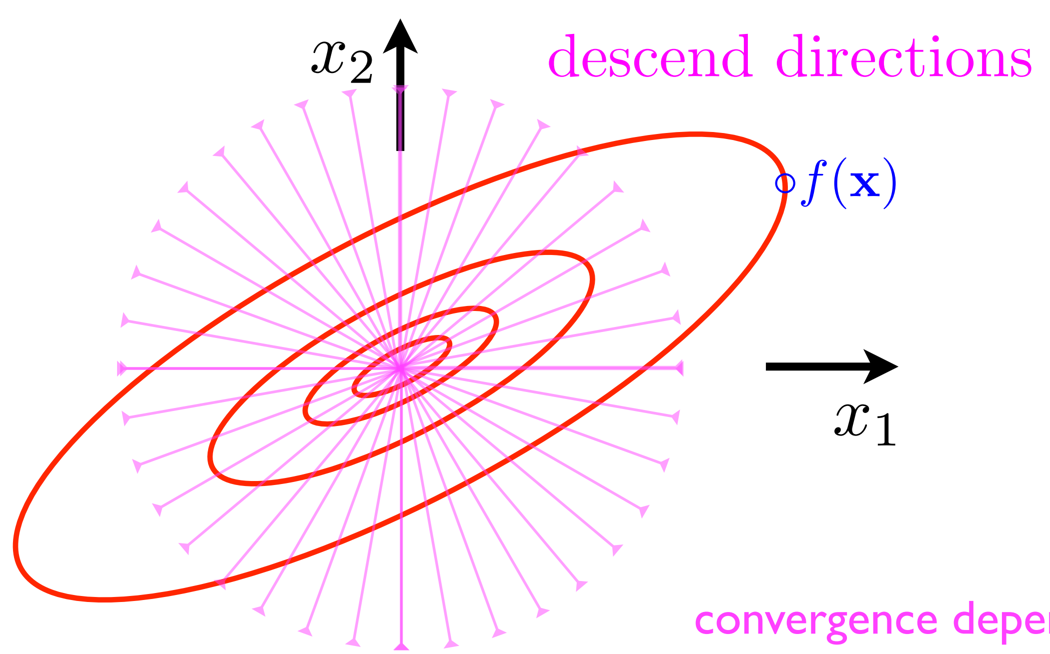
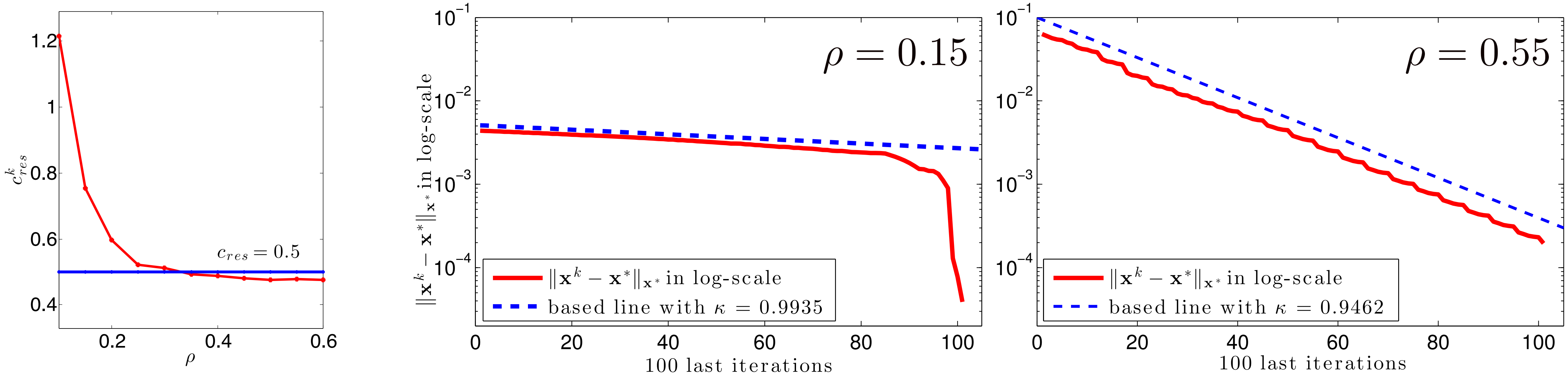


convergence depends on the full condition number

New theory: Local linear convergence of the PG

Graph learning: Lymph [p = 587]



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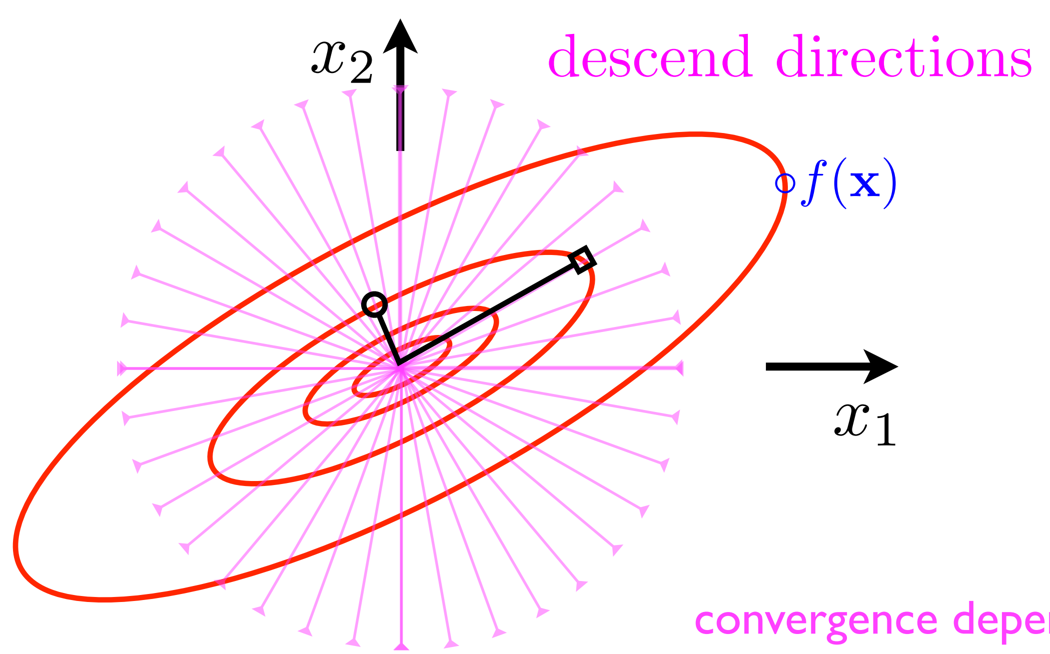

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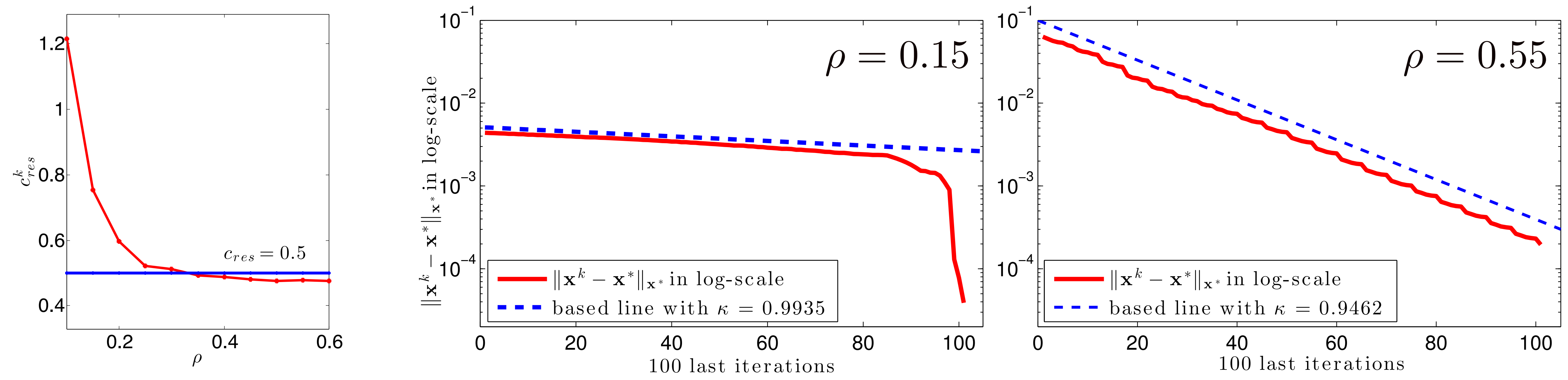

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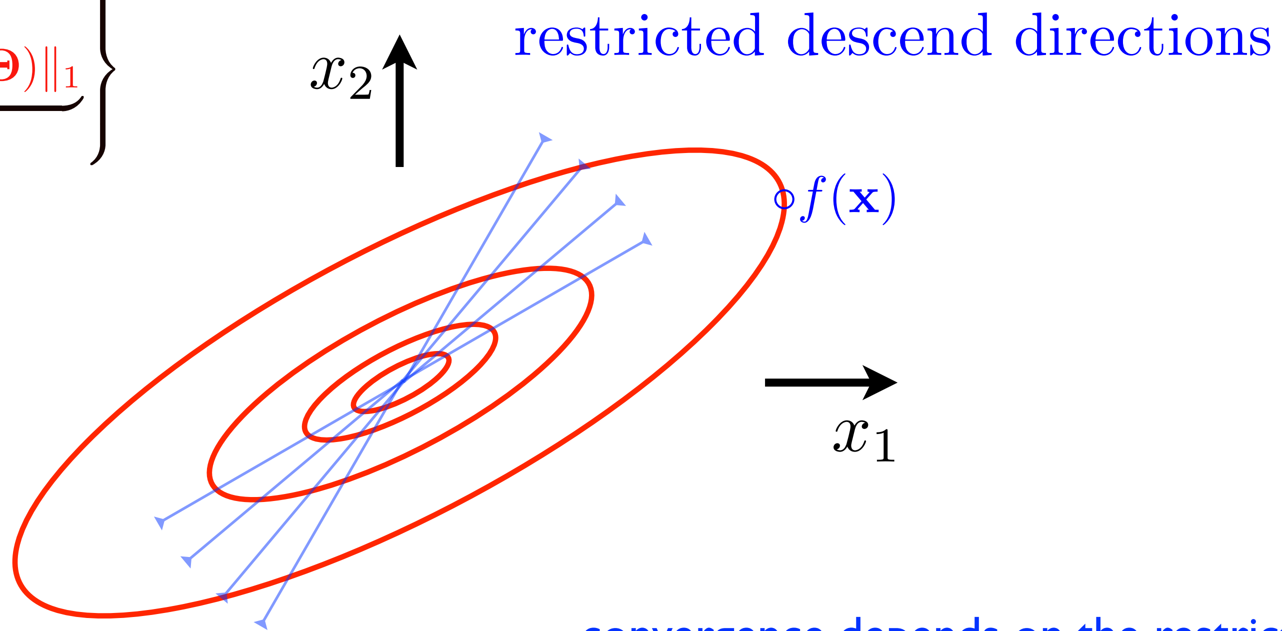

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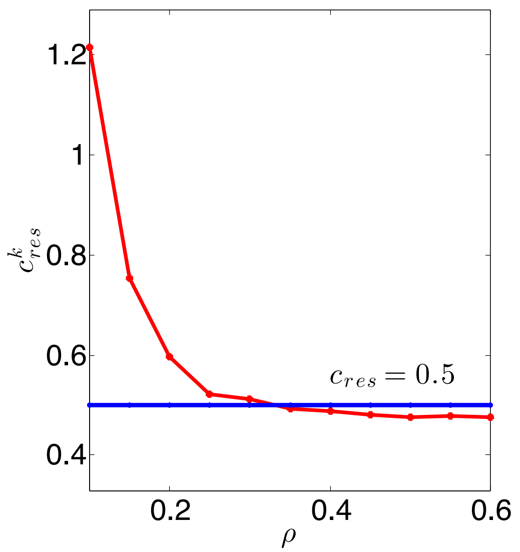
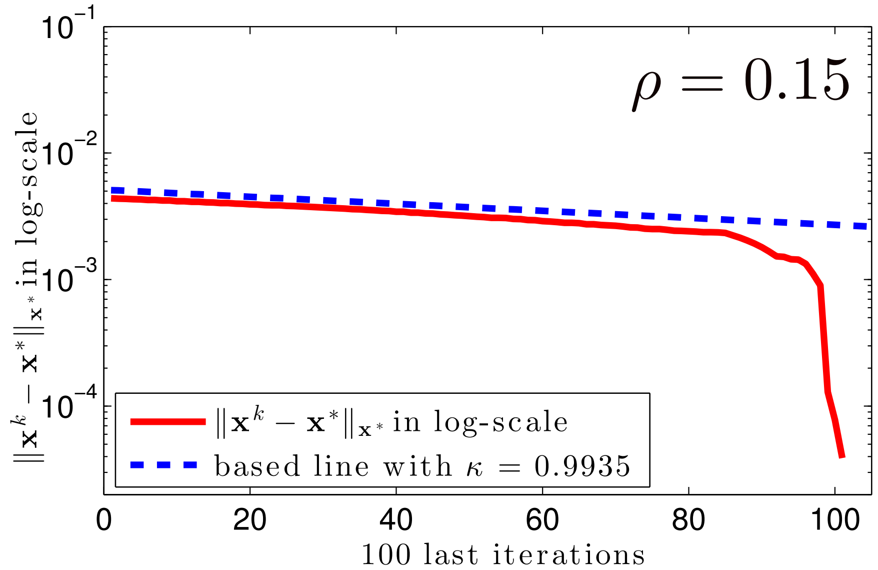
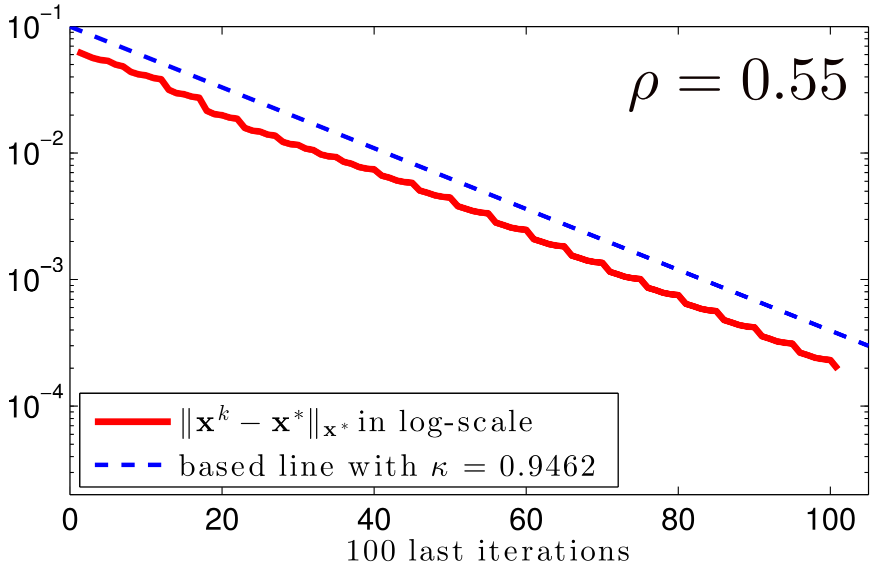
New theory: Local linear convergence of the PG

Graph learning: Lymph [p = 587]







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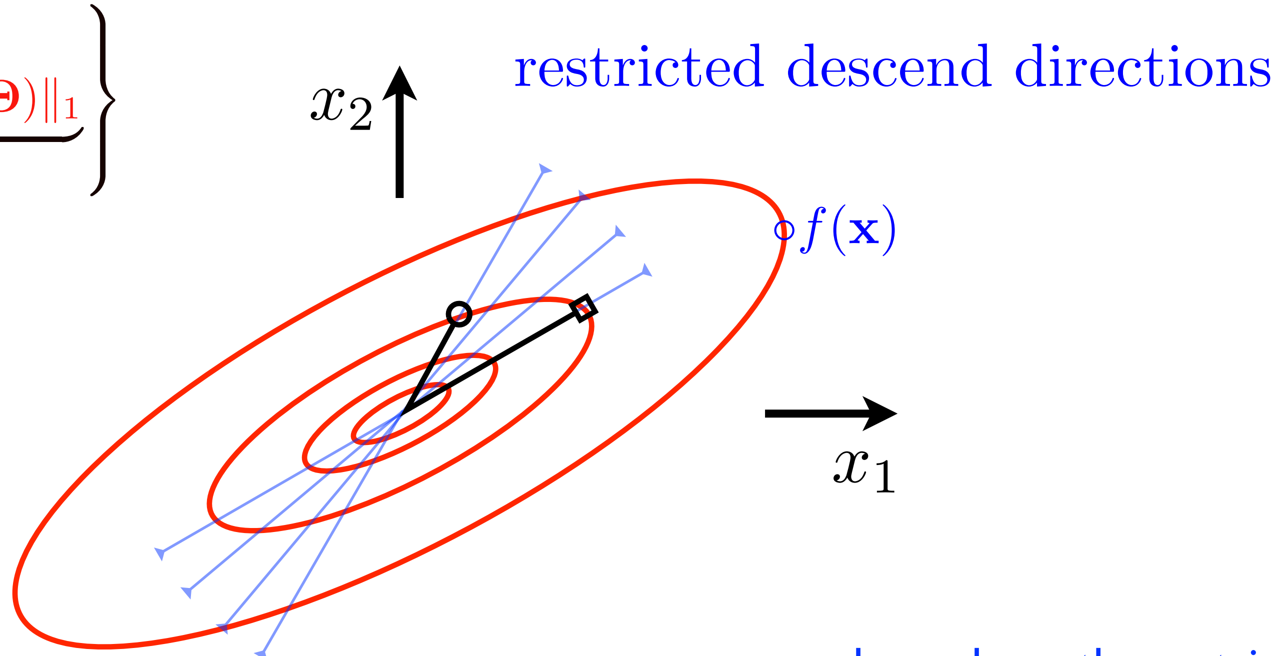

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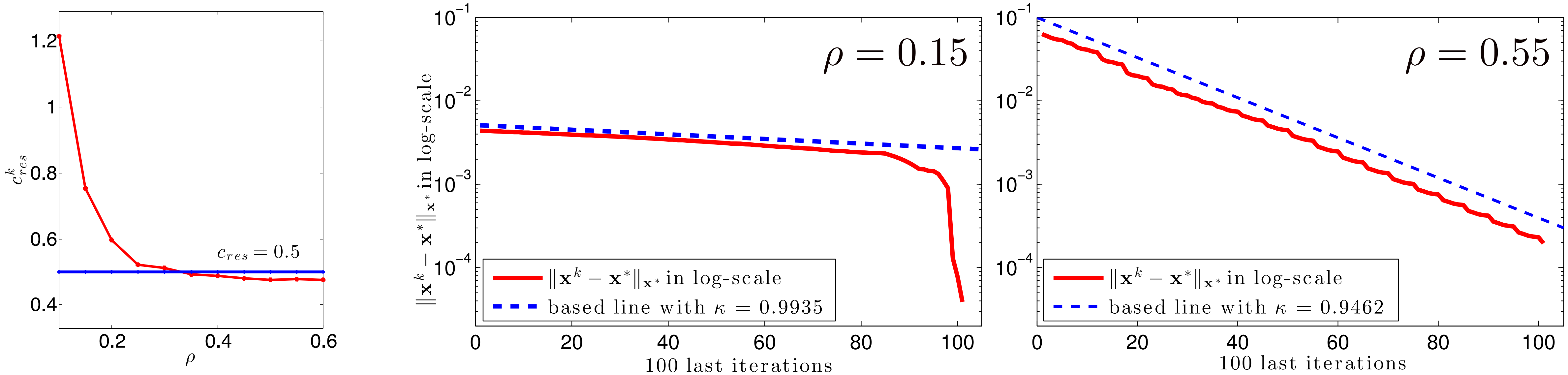
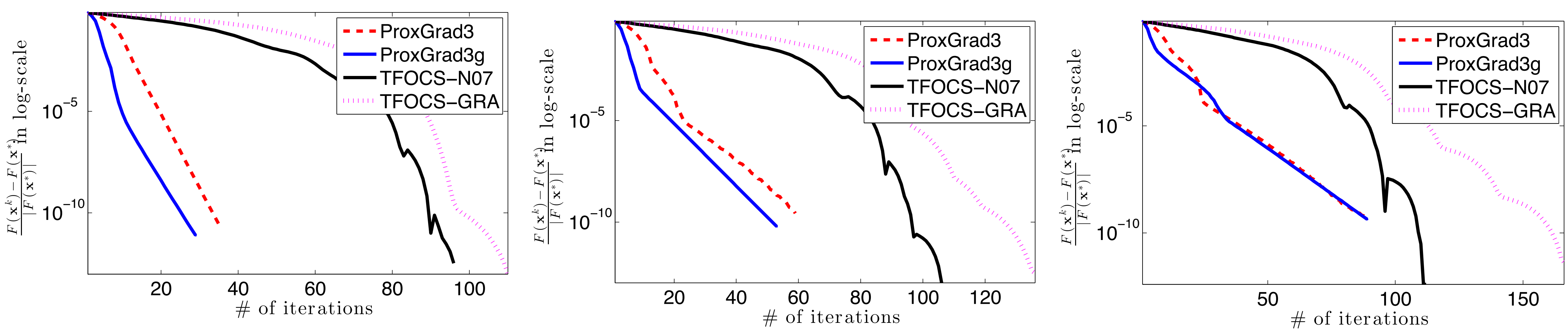


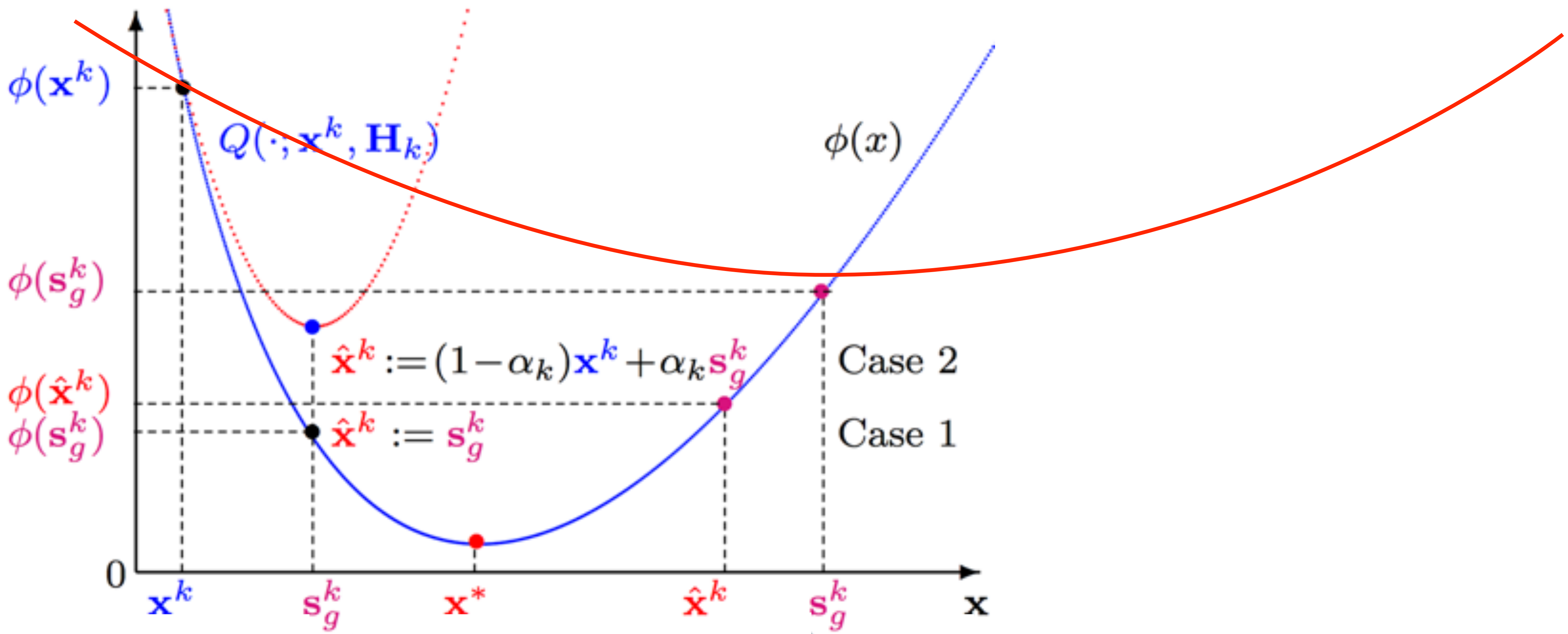
Heteroschedastic LASSO [rho decreases from left to right]



$$\mathbf{x}^* \equiv (\phi^*, \gamma^*) = \underset{\phi,\gamma}{\operatorname{argmin}} \Big\{ \underbrace{-\log(\gamma) + \frac{1}{2n}\|\gamma\mathbf{y} - \mathbf{X}\phi\|_2^2}_{=: f(\mathbf{x})} + \underbrace{\rho\|\phi\|_1}_{=: g(\mathbf{x})} \Big\}$$

Proximal gradient scheme: new engineering

- A ***greedy*** enhancement $\min_{\mathbf{x}\in\mathbb{R}^n} \{\phi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\}$



$$\hat{\mathbf{x}}^k := (1-\alpha_k)\mathbf{x}^k + \alpha_k\mathbf{s}_g^k$$

prox: $\mathbf{s}_g^k := \arg\min_{\mathbf{x}\in\mathrm{dom}(F)} \{Q(\mathbf{x};\mathbf{x}^k,\mathbf{D}_k) + g(\mathbf{x})\}$

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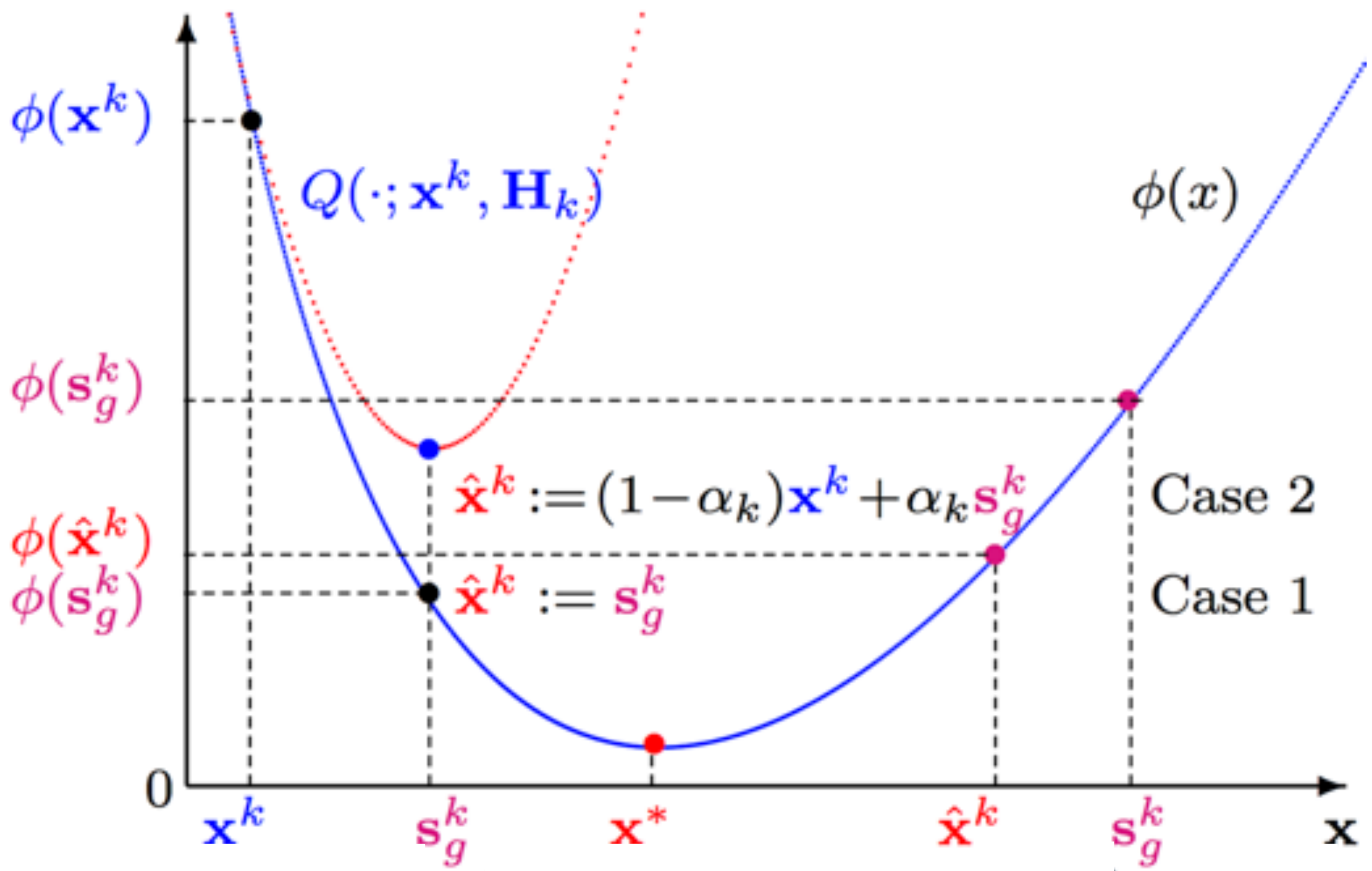

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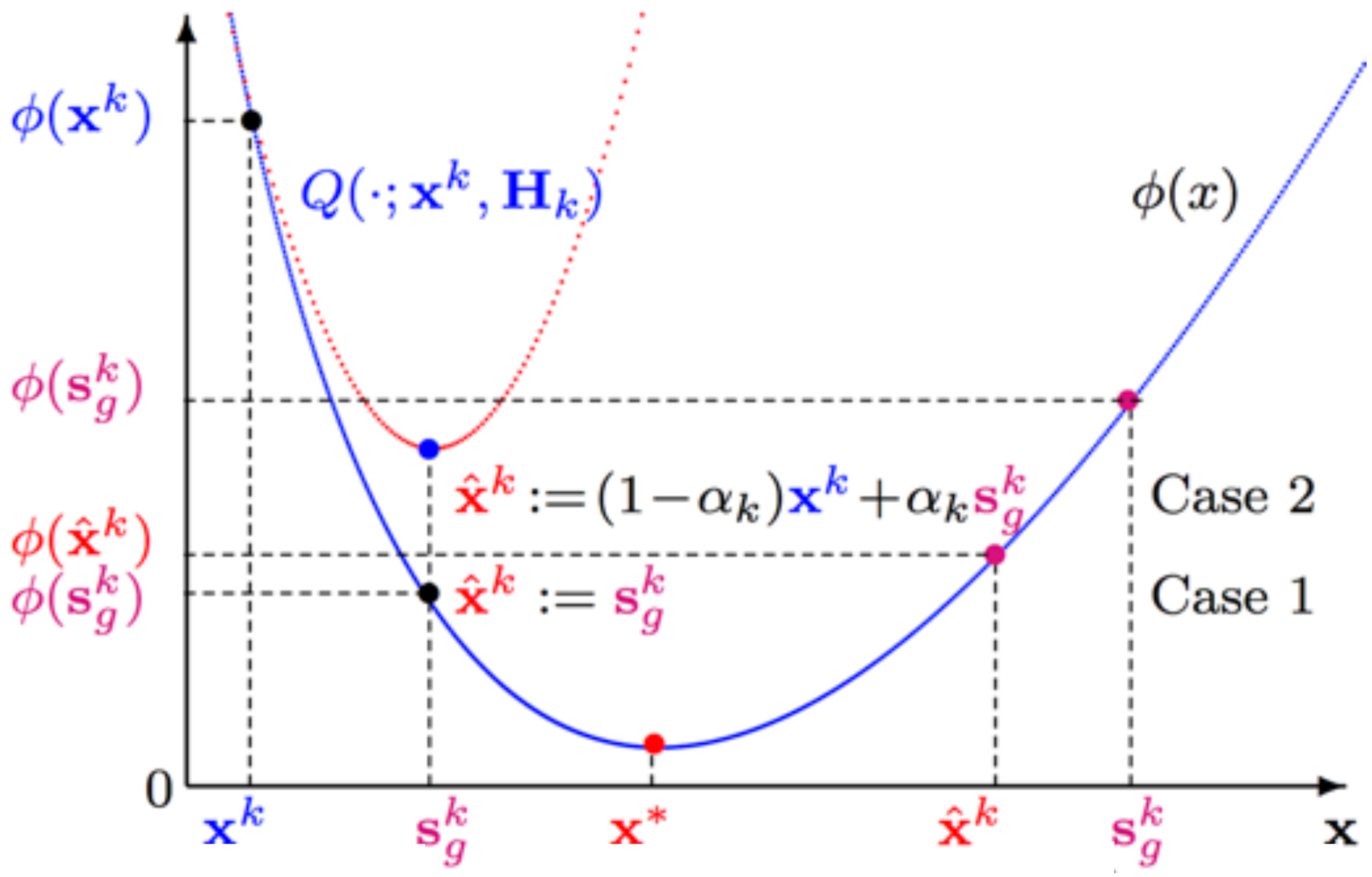

slows down convergence!

$$\hat{\mathbf{x}}^k := (1-\alpha_k)\mathbf{x}^k + \alpha_k\mathbf{s}_g^k$$

prox: $\mathbf{s}_g^k := \arg\min_{\mathbf{x}\in\mathrm{dom}(F)} \{Q(\mathbf{x};\mathbf{x}^k,\mathbf{D}_k) + g(\mathbf{x})\}$

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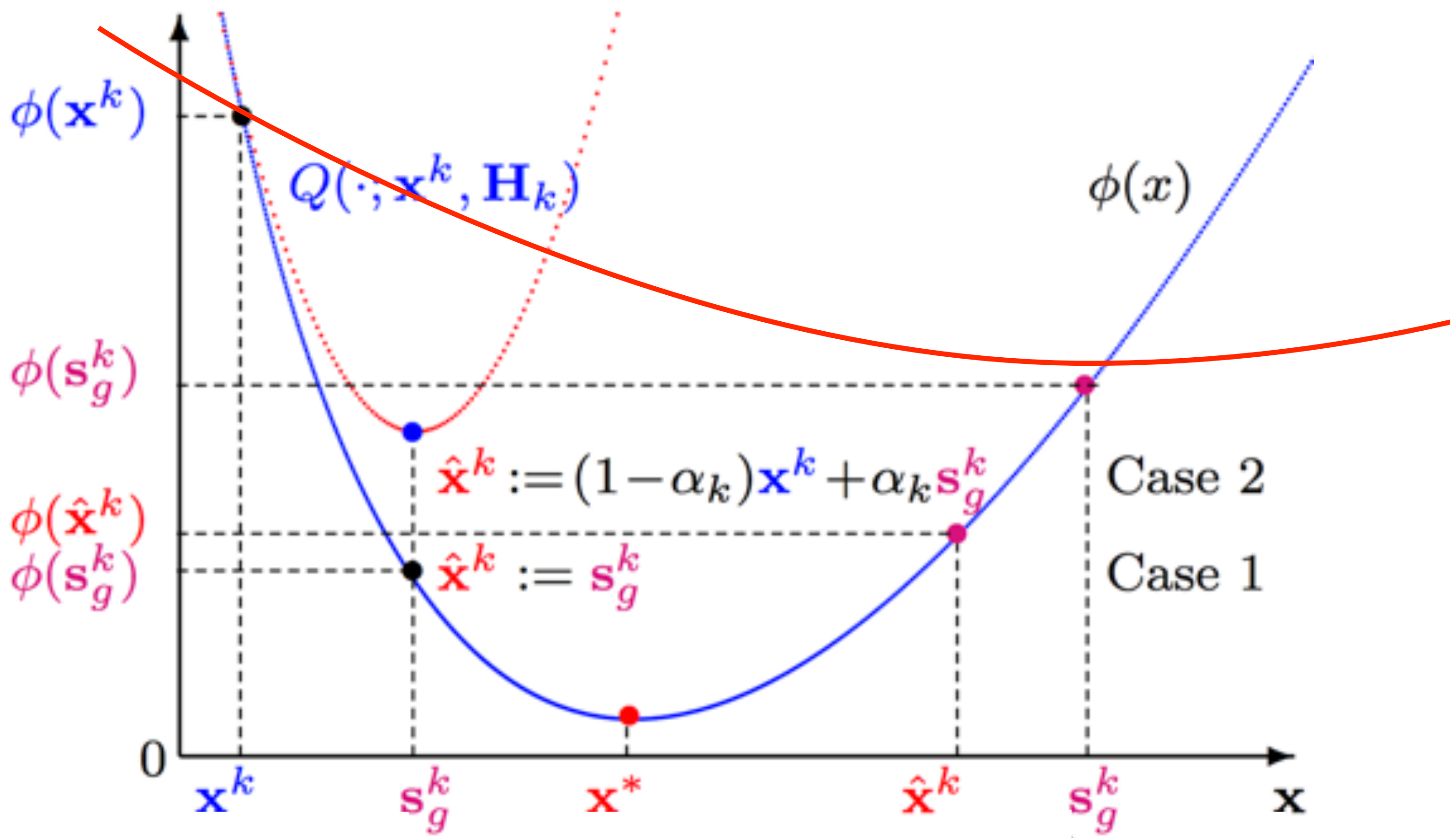

simple decision:

based on the function values

$\phi(\mathbf{s}_g^k)$, $\phi(\hat{\mathbf{x}}^k)$ and $\phi(\mathbf{x}^k)$

$$\hat{\mathbf{x}}^k := (1-\alpha_k)\mathbf{x}^k + \alpha_k\mathbf{s}_g^k$$

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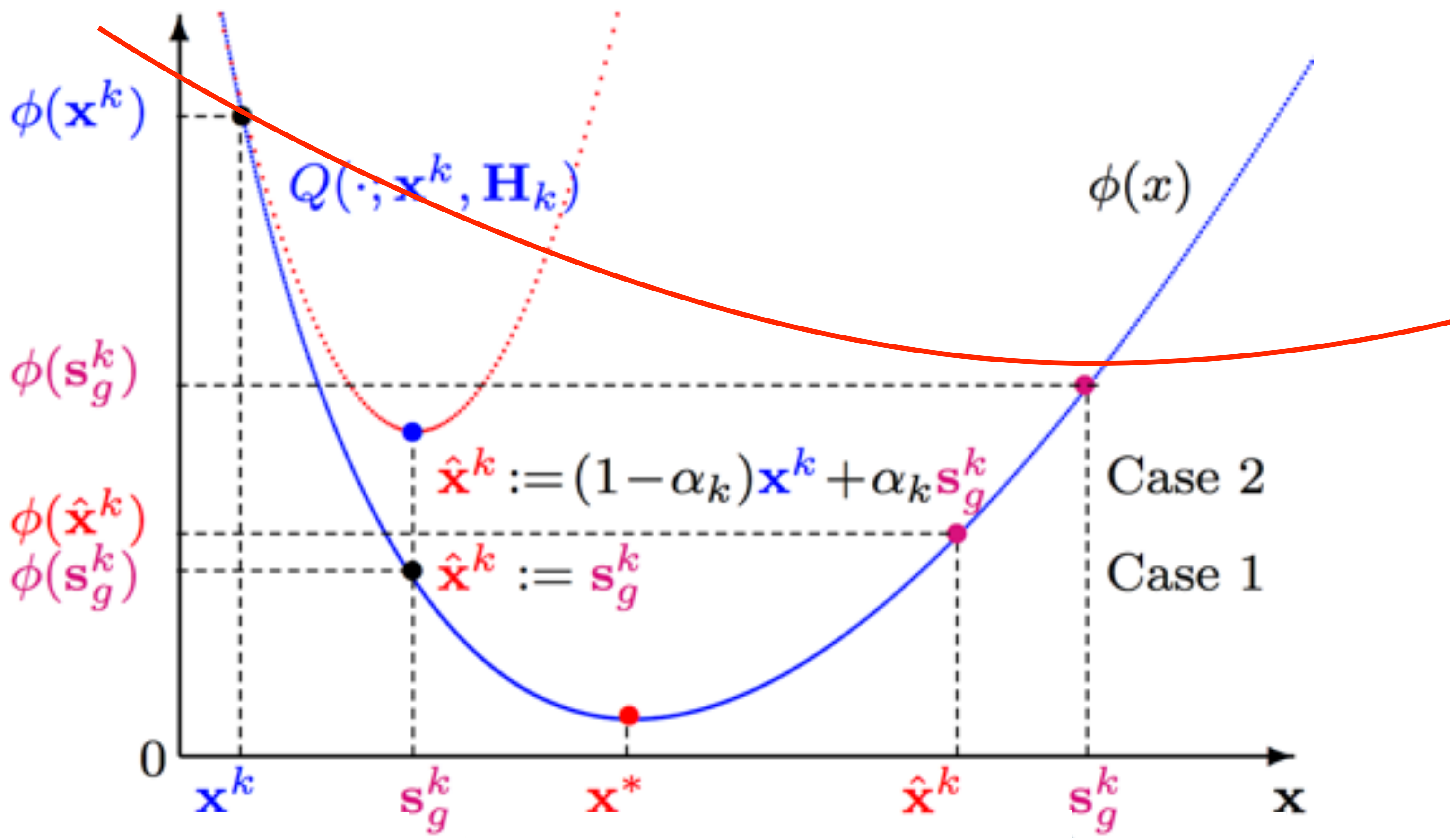


simple decision:

based on the function values $\phi(\mathbf{s}_g^k)$, $\phi(\hat{\mathbf{x}}^k)$ and $\phi(\mathbf{x}^k)$

cost:

practically none if implemented carefully

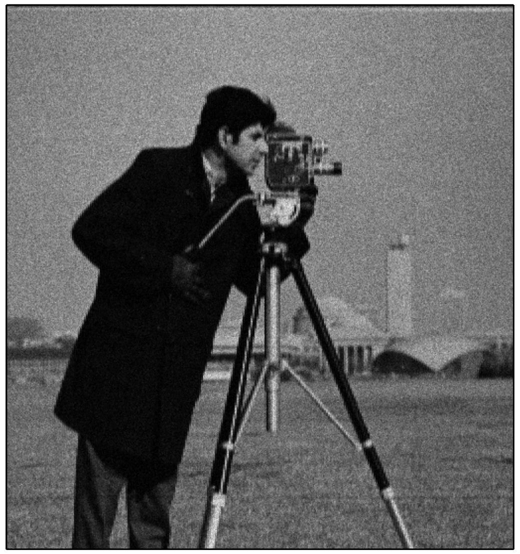
$$\hat{\mathbf{x}}^k := (1-\alpha_k)\mathbf{x}^k + \alpha_k\mathbf{s}_g^k$$

prox: $$\mathbf{s}_g^k := \arg\min_{\mathbf{x}\in\mathrm{dom}(F)} \{Q(\mathbf{x}; \mathbf{x}^k, \mathbf{D}_k) + g(\mathbf{x})\}$$

Poisson imaging reconstruction via TV

Our method vs SPIRAL-TAP [Harmany2012]

Original image



Poisson noise image

Reconstructed image (ProxGrad)

Reconstructed image (ProxGradNewton)

Reconstructed image (SPIRAL–TAP

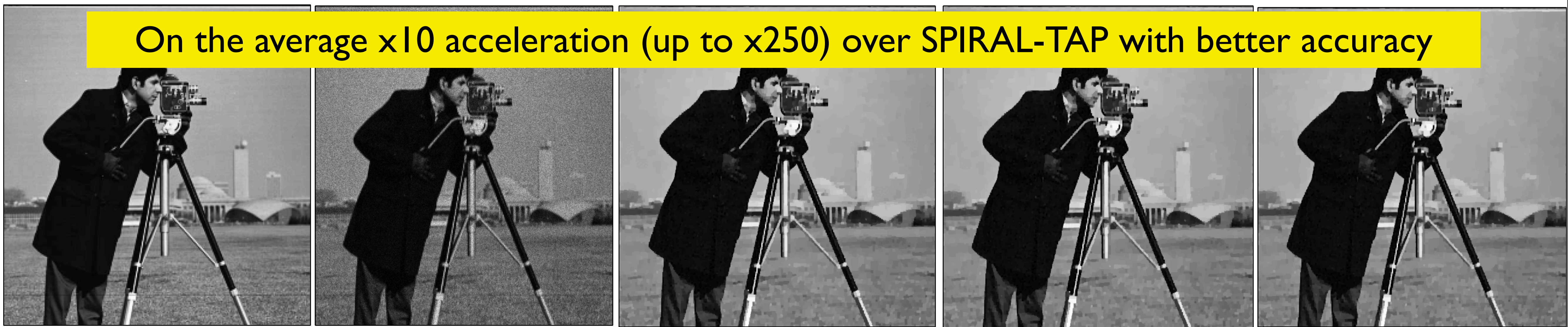
- Poisson imaging reconstruction via TV regularization

$$x^* \in \operatorname*{argmin}_x \Big\{ \underbrace{\sum_{i=1}^m a_i^T x - \sum_{i=1}^m y_i \log(a_i^T x + b_i)}_{f(x)} + g(x) \Big\}$$

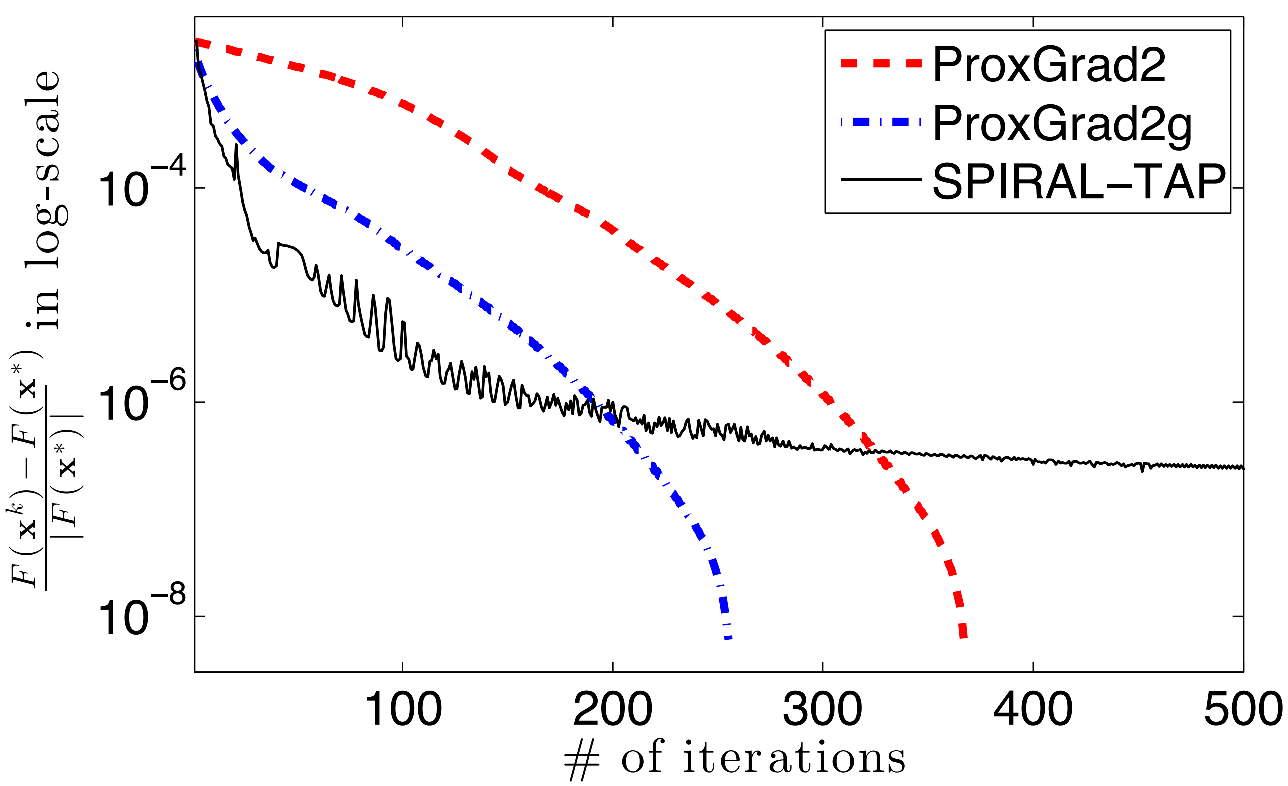
Poisson imaging reconstruction via TV

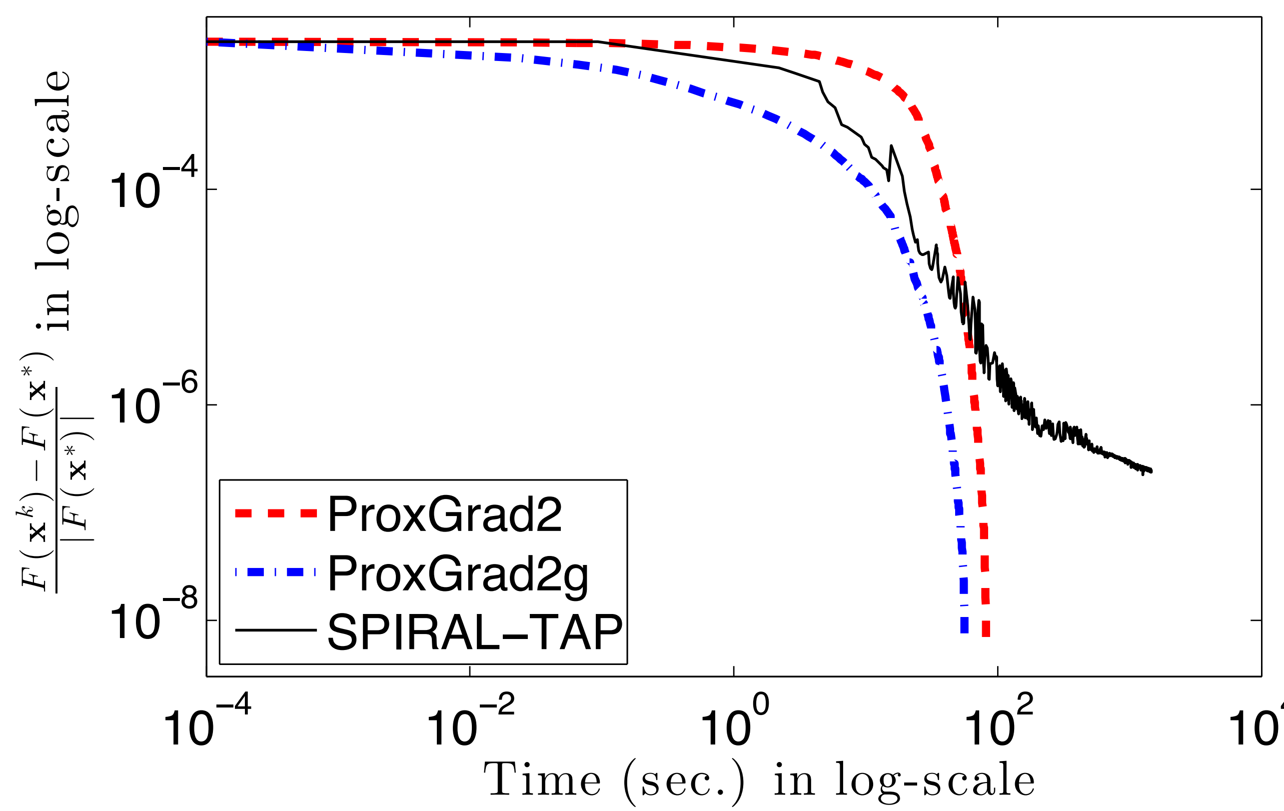
Our method vs SPIRAL-TAP [Harmany2012]

On the average x10 acceleration (up to x250) over SPIRAL-TAP with better accuracy



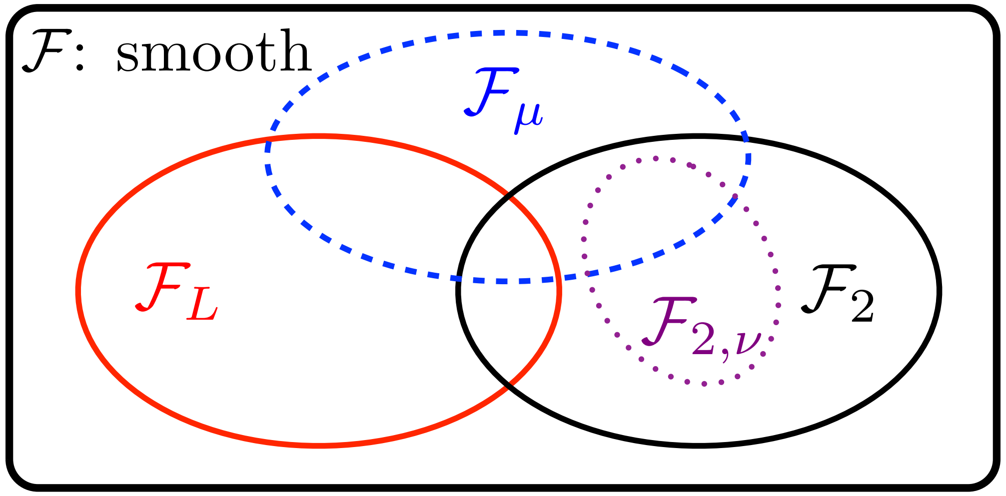
Original image | Poisson noise image | Reconstructed image (ProxGrad) | Reconstructed image (ProxGradNewton) | Reconstructed image (SPIRAL-TAP





Barrier extensions

Constrained convex problems



$$g^* := \min_{\mathbf{x}\in\Omega} g(\mathbf{x})$$

- Ω is endowed with a self-concordant barrier $f(x)$;

Self-concordant barrier

- Given **f** is smooth and convex on its domain. We define:

$$\varphi(t) := f(\mathbf{x} + t\mathbf{v}), \text{ where } \mathbf{x}, \mathbf{x} + t\mathbf{v} \in \mathrm{dom}(f), \mathbf{v} \in \mathbb{R}^n, t \in \mathbb{R}$$

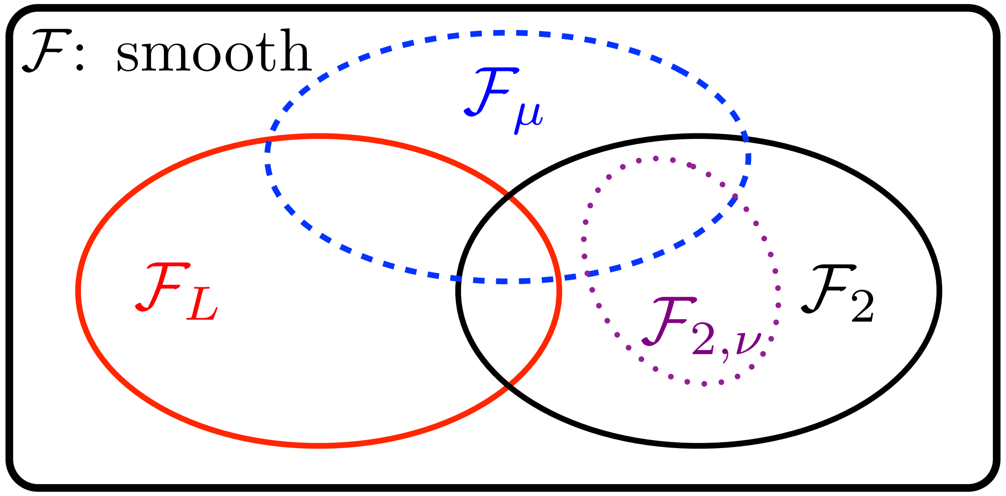
- ***Self-concordance***: **f** is standard self-concordant if

$$|\varphi'''(t)| \leq 2\varphi''(t)^{3/2}$$

- ***Self-concordant barrier***: **f** is a *self-concordant barrier* if there exists $\nu > 0$ such that:

$$\begin{cases} f \text{ is standard self-concordant,} \\ |\varphi'(t)| \leq \sqrt{\nu}\varphi''(t)^{1/2}, \\ f(\mathbf{x}) \to +\infty \text{ as } \mathbf{x} \to \partial\Omega \end{cases}$$

Constrained convex problems



$$g^* := \min_{\mathbf{x} \in \Omega} g(\mathbf{x})$$

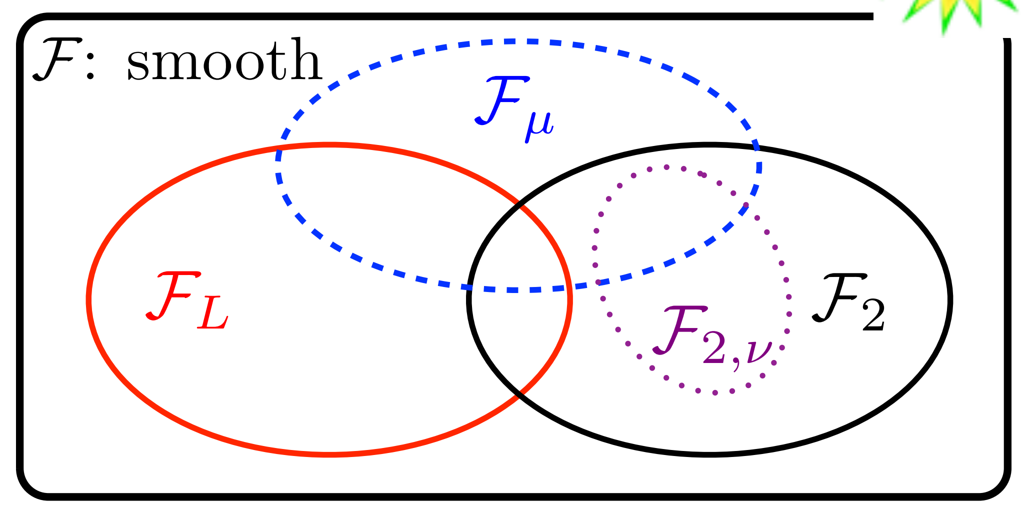
- Ω is endowed with a self-concordant barrier $f(x)$;

- Examples:

$$\begin{array}{lcl} \Omega : \mathbf{X} \succeq 0 & \Rightarrow & f_\Omega(\mathbf{X}) = -\log\det(\mathbf{X}) \\ \Omega : \mathbf{a}^T\mathbf{x} \geq 0 & \Rightarrow & f_\Omega(\mathbf{x}) = -\log(\mathbf{a}^T\mathbf{x}) \\ \Omega : \|\mathbf{A}\mathbf{x} - \mathbf{b}\| \leq \sigma & \Rightarrow & f_\Omega(\mathbf{x}) = -\log(\sigma^2 - \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2) \end{array}$$

f is a ν-self-concordant barrier if $\varphi(t) := f(\mathbf{x} + t\mathbf{d})$ satisfies $|\varphi'''(t)| \leq 2\varphi''(t)^{3/2}$ and $|\varphi'(t)| \leq \sqrt{\nu}\varphi''(t)^{1/2}$

Constrained convex problems



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- Main idea: ***solve a sequence of composite self-concordant problems***

$$\min_{\mathbf{x}\in\mathrm{int}(\Omega)} \left\{ F(\mathbf{x}, \tau) := f(\mathbf{x}) + \tau^{-1} g(\mathbf{x}) \right\}$$

Proximal Point + Interior Point

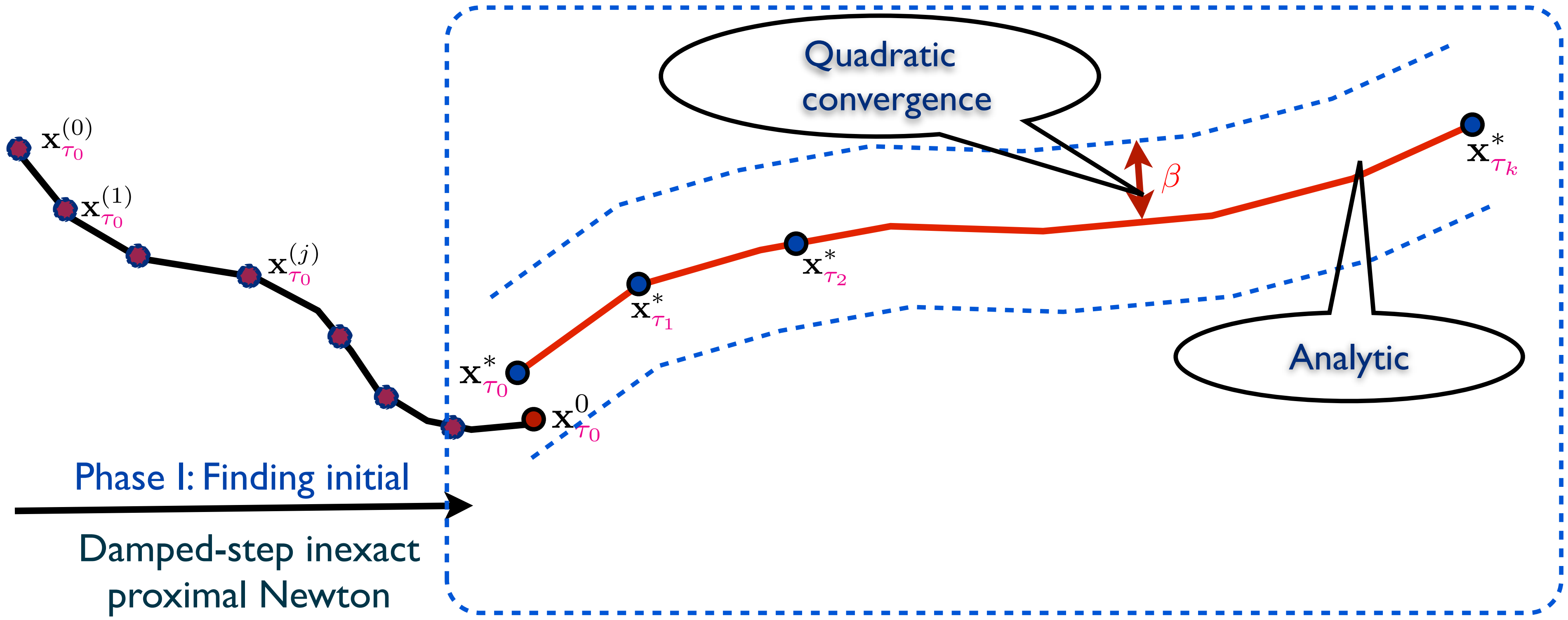
combine them together!!!

f is a ν-self-concordant barrier if $\varphi(t) := f(\mathbf{x} + t\mathbf{d})$ satisfies $|\varphi'''(t)| \leq 2\varphi''(t)^{3/2}$ and $|\varphi'(t)| \leq \sqrt{\nu}\varphi''(t)^{1/2}$

How does it work?

Main idea: solve a sequence of composite self-concordant problems as opposed to DCO

$$\min_{\mathbf{x}\in\mathrm{int}(\Omega)} \left\{ F(\mathbf{x},\tau) := f(\mathbf{x}) + \tau^{-1} g(\mathbf{x}) \right\}$$



One iteration **k** requires two updates **simultaneously**:

- Update the penalty parameter:

$$t_{k+1} := (1-\sigma_k)t_k, \quad \sigma_k \in [\underline{\sigma}, 1) \quad (\text{e.g.}, \underline{\sigma} = 0.0337/\sqrt{\nu}).$$

- Update the iterative vector (*can be solved approximately*):

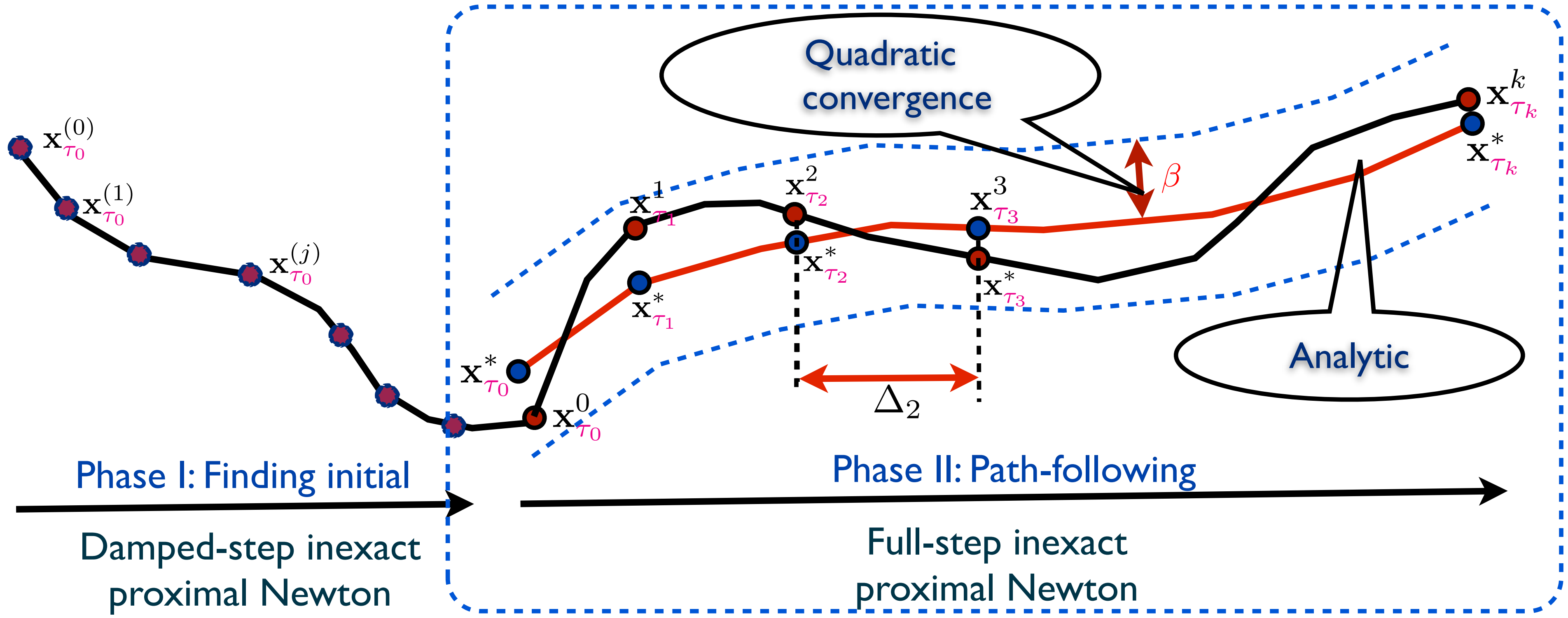
$$\mathbf{x}^{k+1} := \operatorname*{argmin}_{\mathbf{x}} \left\{ t_{k+1}\nabla f(\mathbf{x}^k)^T(\mathbf{x}-\mathbf{x}^k) + \frac{t_{k+1}}{2}(\mathbf{x}-\mathbf{x}^k)^T\nabla^2 f(\mathbf{x}^k)(\mathbf{x}-\mathbf{x}^k) + g(\mathbf{x}) \right\}$$

***Details:** "An Inexact Proximal Path-Following Algorithm for Constrained Convex Minimization," *optimization-online and lions.epfl.ch/publications.*

How does it work?

Main idea: solve a sequence of composite self-concordant problems as opposed to DCO

$$\min_{\mathbf{x}\in\mathrm{int}(\Omega)} \left\{ F(\mathbf{x},\tau) := f(\mathbf{x}) + \tau^{-1} g(\mathbf{x}) \right\}$$



One iteration **k** requires two updates **simultaneously**:

- Update the penalty parameter:

$$t_{k+1} := (1-\sigma_k)t_k, \quad \sigma_k \in [\underline{\sigma}, 1) \quad (\text{e.g.}, \underline{\sigma} = 0.0337/\sqrt{\nu}).$$

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How does it converge?

The algorithm consists of **two** PHASES:

- **Phase I**: Find a **starting point** $\mathbf{x}^0_{p_2}$ such that: $\|\mathbf{x}^0_{p_2} - \mathbf{x}^*(t_0)\|_{\mathbf{x}^*(t_0)} \leq 0.05$

- **Phase II**: Perform the ***path-following iterations***

Tracking properties on the penalty parameter and the iterative sequence in **Phase II**

- The penalty parameter t_k **decreases** at least with a factor $\tau := 1 - \frac{0.0337}{\sqrt{\nu}}$ at each iteration **k**

$$t_{k+1} = \tau t_k$$

- **Tracking error** of the iterative sequence:

$$\text{if } \boxed{\|\mathbf{x}^k - \mathbf{x}^*(t_k)\|_{\mathbf{x}^*(t_k)} \leq 0.05} \text{ then } \boxed{\|\mathbf{x}^{k+1} - \mathbf{x}^*(t_{k+1})\|_{\mathbf{x}^*(t_{k+1})} \leq 0.05}$$

Worst-case complexity:

- **Phase I**: Finding a starting point for **Phase II** requires at most

$$j_{\max} := \left\lfloor \frac{F(\mathbf{x}^0; t_0) - F(\mathbf{x}^*(t_0); t_0)}{0.0012} \right\rfloor + 1$$

- **Phase II**: The **worst-case complexity** to reach an ε**- solution** is at most:

$$\mathcal{O}\left(\sqrt{\nu}\log\left(\frac{Ct_0}{\varepsilon}\right)\right)$$

- **Note:** This worst-case complexity is as the **same** as in *standard path-following methods* [see Nesterov2004]

Proximal path-following **Upshot: no-heavy lifting!**

Proximal path following for conic programming with rigorous guarantees

$$g^* := \min_{\mathbf{x}\in\Omega} g(\mathbf{x})$$

- Ω is endowed with a self-concordant barrier $f(x)$;

Example: Low-rank SDP matrix approximation ...

$$\begin{array}{ll}\min\limits_{\mathbf{X}} & \rho\|\mathrm{vec}(\mathbf{X}-\mathbf{M})\|_1 + (1-\rho)\mathrm{tr}(\mathbf{X}) \\ \text{s.t.} & \mathbf{X}\succeq 0,\ \mathbf{L}_{ij}\leq \mathbf{X}_{ij}\leq \mathbf{U}_{ij},\ i,j=1,\ldots,n.\end{array}$$

ρ is a regularization parameter in (0, 1), $\mathbf{M}$ is the given input matrix.

	Solver\ n	80	100	120	140	160
Size	$[n_v; n_c]$	[16,200; 9,720]	[25,250; 15,150]	[36,300; 21,780]	[49,350; 29,610]	[64,400; 38,640]
Time (sec)	New Algorithm	15.738	24.046	24.817	25.326	36.531
	SDPT3	156.340	508.418	881.398	1742.502	2948.441
	SeDuMi	231.530	970.390	3820.828	9258.429	17096.580
Objective value	New Algorithm	306.9159	497.6706	635.4304	842.4626	1096.6516
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#variables
#constraints

ρ is a regularization parameter in (0, 1), **M** is the given input matrix.

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+ **Lift** the problem dimension (small to large-scale)

+ **Destroy** the sparsity due to **scaling-factors** in SDP (e.g., Nesterov-Todd scaling factor)

Proximal path-following **Upshot: desired scaling!**

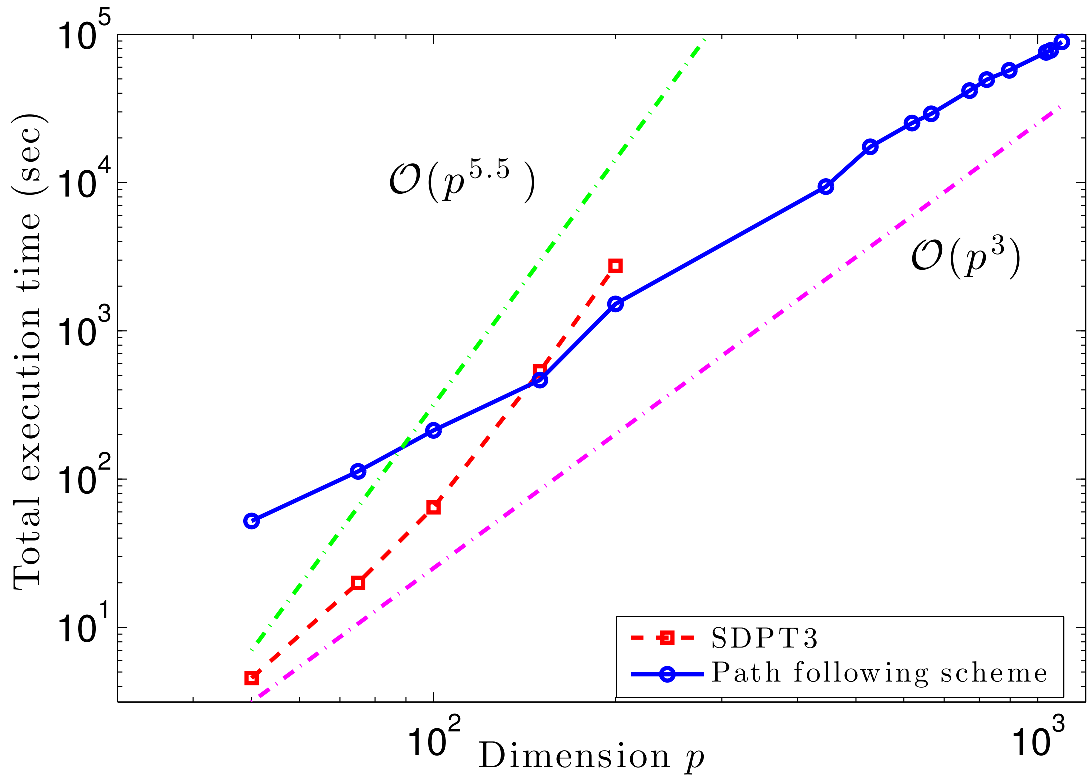
Proximal path following for conic programming with rigorous guarantees

$$g^* := \min_{\mathbf{x} \in \Omega} g(\mathbf{x})$$

- Ω is endowed with a self-concordant barrier $f(x)$;

Example: Max-norm clustering

$$\begin{array}{ll} \min\limits_{\mathbf{L},\mathbf{R},\mathbf{K}} & \|\mathrm{vec}\,(\mathbf{K}-\mathbf{A})\|_1 \\ \text{s.t.} & \begin{bmatrix} \mathbf{L} & \mathbf{K} \\ \mathbf{K}^T & \mathbf{R} \end{bmatrix} \succ 0,\ \mathbf{L}_{ii} \le 1,\ \mathbf{R}_{ii} \le 1,\ i = 1, \ldots, p. \end{array}$$



DCO:

	SDPT3		PF scheme
p	variables	constraints	variables
50	15.1	2.6	10
75	33.9	5.8	22.5
100	60.2	10.2	40
150	135.3	22.8	90
200	240.4	40.4	160

	p	50	75	100	150	200
Time (sec)	PF	62.450	109.426	202.600	416.044	1573.881
	SDPT3	4.396	21.282	64.939	522.021	2588.721
	splitting	102.217	236.366	354.444	778.904	1420.844
$g(\mathbf{K}^*)$	PF	549.1567	1293.6727	2232.5897	5396.0485	9809.6066
	SDPT3	549.1860	1293.7890	2233.0747	5396.7305	9809.6934
	splitting	597.8825	1387.1379	2496.6535	5583.8605	9958.0974

Proximal path-following **Upshot: auto. regularization!**

Proximal path following for conic programming with rigorous guarantees

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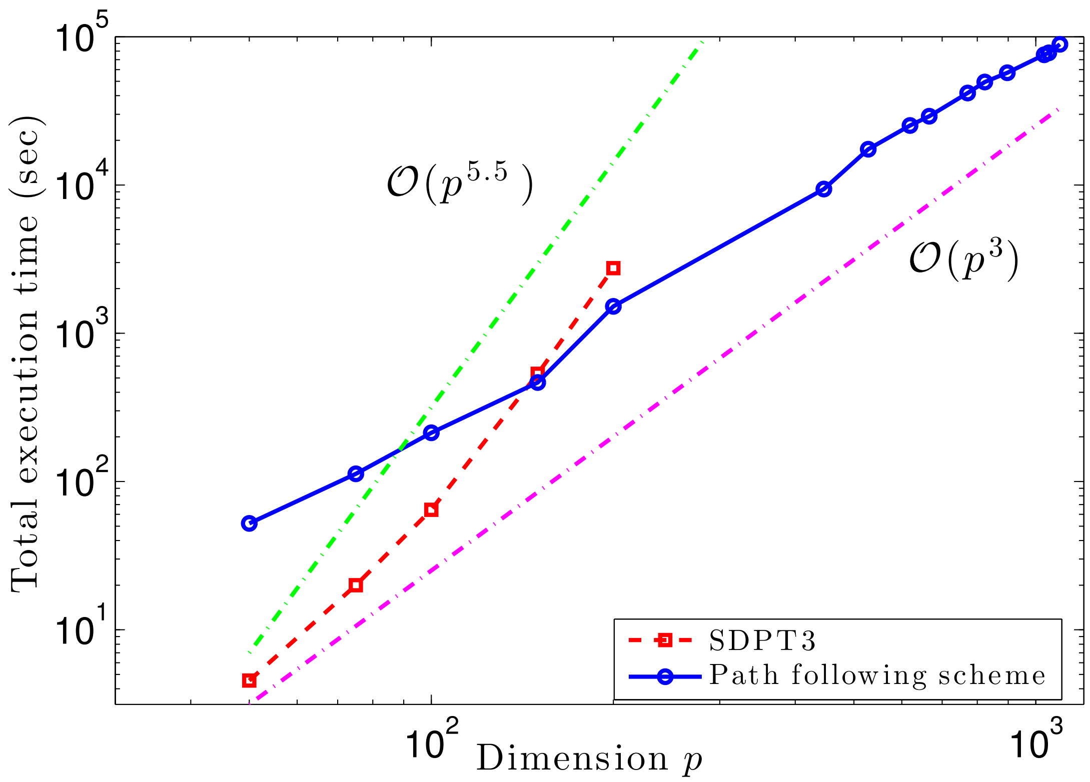
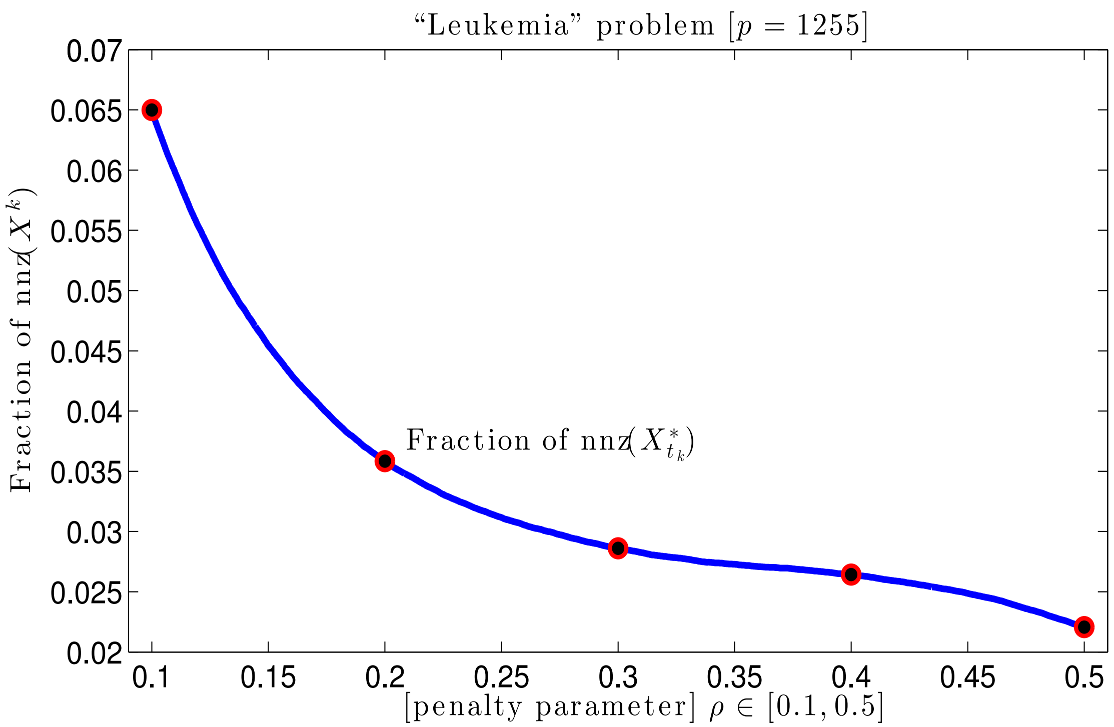


Example: Graph selection



Smoothing via self-concordant barrier

Composite convex minimization with nonsmooth-max structure **f**:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \{\phi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\}$$

- Nonsmooth function **f** is defined as:

$$f(\mathbf{x}) := \max_{\mathbf{u}\in\mathcal{U}} \{\langle \mathbf{A}^T\mathbf{x}, \mathbf{u}\rangle - G(\mathbf{u})\}$$

- $\mathcal{U}$: endowed with a **self-concordant barrier** $b_{\mathcal{U}}$, **G**: **convex**

Motivating examples

- *Nonsmooth convex function* induced by a **norm**:

$$f(\mathbf{x}) := \|\mathbf{D}\mathbf{x}\| = \max_{\mathbf{u}} \{\langle \mathbf{D}\mathbf{x}, \mathbf{u}\rangle \mid \mathbf{u} \in \mathcal{U} := \{\mathbf{v} \mid \|\mathbf{v}\|^* \leq 1\}\}$$

- Fenchel's ***dual function*** with compact domain:

$$f(\mathbf{x}) := \max_{\mathbf{u}\in\mathrm{dom} f^*} \{\langle \mathbf{x}, \mathbf{u}\rangle - f^*(\mathbf{u})\}$$

- ***Lagrange dual problem*** in *constrained convex optimization problem* $\max_{\mathbf{u}\in\mathcal{U}} \{-G(\mathbf{u}) \mid \mathbf{A}\mathbf{u} = b\}$

$$\boxed{\min_{\mathbf{x}} \left\{ \underbrace{\max_{\mathbf{u}\in\mathcal{U}} \{\mathbf{x}^T\mathbf{A}\mathbf{u} - G(\mathbf{u})\}}_{f(\mathbf{x})} \underbrace{-b^T\mathbf{x}}_{g(\mathbf{x})} \right\}}$$

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Self-concordant barrier smoother of f*

$$f_\sigma(\mathbf{x}) := \max_{\mathbf{u}\in\mathrm{int}(\mathcal{U})} \left\{\langle \mathbf{A}^T\mathbf{x}, \mathbf{u}\rangle - G(\mathbf{u}) - \sigma b_{\mathcal{U}}(\mathbf{u})\right\}$$

- $\sigma > 0$ is a *smoothness parameter*

- The solution $\mathbf{u}^*{}_\sigma(\mathrm{x})$ of the maximization problem satisfies the following optimality condition:

$$0 \in \mathbf{A}^T\mathbf{x} - \partial G(\mathbf{u}^*{}_\sigma(\mathbf{x})) - \sigma\nabla b_{\mathcal{U}}(\mathbf{u}^*{}_\sigma(\mathbf{x}))$$

- This is a **generalized equation**. *If **G** is **smooth** then it reduces to a system of nonlinear equations*

Often easier to solve than general convex problems

The **gradient** of f_σ is given as

$$\nabla f_\sigma(\mathbf{x}) = \mathbf{A}\mathbf{u}^*{}_\sigma(\mathbf{x})$$

Details: Manuscript under review for ICASSP'14.

Three key properties of the barrier smoother

1. Approximation property

$$f_\sigma(\mathbf{x}) \leq f(\mathbf{x}) \leq f_\sigma(\mathbf{x}) + \sigma C\nu.$$

- $\sigma > 0$ is the *smoothness parameter, C is a given constant and* ν *is the barrier parameter.*

2. Local Lipschitz-like property

$$\|\nabla f_\sigma(\mathbf{x}) - \nabla f_\sigma(\mathbf{y})\|_2 \leq \frac{c_A^2\|\mathbf{x}-\mathbf{y}\|_2}{\sigma - c_A\|\mathbf{x}-\mathbf{y}\|_2}, \quad \forall \mathbf{x},\mathbf{y} \;\; \|\mathbf{x}-\mathbf{y}\|_2 < \frac{\sigma}{c_A}$$

- Here: $c_A \equiv c_A(\mathbf{x}) = \left[\|\mathbf{A}\nabla^2 b_{\mathcal{U}}(\mathbf{u^*}_\sigma(\mathbf{x}))\mathbf{A}^T\|\right]_2^{1/2} \leq \bar{c}_A \;\; (\text{constant})$

3. Existence of the second order derivative

- If **G** is ***twice differentiable*** and A is ***full-row rank***, then

$$\nabla^2 f_\sigma(\mathbf{x}) = \mathbf{A}\left(\nabla^2 G(\mathbf{u^*}_\sigma(\mathbf{x})) + \sigma\nabla^2 b_{\mathcal{U}}(\mathbf{u^*}_\sigma(\mathbf{x}))\right)^{-1}\mathbf{A}^T \succ 0$$

- If, in addition, **G** is **self-concordant** with the parameter M_G , then f_σ is also **self-concordant** with

$$M_{f_\sigma} = \max\left\{M_G, \frac{2}{\sqrt{\sigma}}\right\}$$

First-order method for barrier smoothing

Barrier smoothed problem

$$\phi_\sigma^* = \min_{\mathbf{x}\in\mathbb{R}^n} \{\phi_\sigma(\mathbf{x}) := f_\sigma(\mathbf{x}) + g(\mathbf{x})\}$$

Here, f_σ is **smooth** and **g** is ***"tractably" proximal***

Proximal - gradient scheme

$$\begin{cases} \mathbf{s}^k & := \mathrm{prox}^g{}_{\gamma_k}\left(\mathbf{x}^k - \gamma_k \nabla f_\sigma(\mathbf{x}^k)\right), \\ \mathbf{d}^k & := \mathbf{s}^k - \mathbf{x}^k, \\ \mathbf{x}^{k+1} & := \mathbf{x}^k + \alpha_k \mathbf{d}^k. \end{cases}$$

- **Step-sizes:** $\gamma_k \geq \frac{\sigma}{\bar{c}_A^2}$ and $\alpha_k = \frac{\sigma}{c_A^k\left(c_A^k\gamma_k + \lambda_k\right)} \in (0, 1]$ with $\lambda_k = \|\mathbf{d}^k\|_2,\ c_A^k = c_A(\mathbf{x}^k)$

Convergence and convergence rate

- **Descent** property: $\boxed{\phi_\sigma(\mathbf{x}^{k+1}) \leq \phi_\sigma(\mathbf{x}^k) - \sigma\omega\left(\frac{\lambda_k}{c_A^k}\right),}$ $\omega(\tau) = \tau - \log(1+t).$

- **Convergence rate** (in the ergodic sense): ***If*** $\lambda_0 \leq M\bar{c}_A$ ***and*** $\bar{\mathbf{x}}^k = \left(\sum_{j=0}^k \alpha_j\right)^{-1} \sum_{j=0}^k \alpha_j \mathbf{x}^j$ ***then***

$$\boxed{\phi_\sigma(\bar{\mathbf{x}}^k) - \phi_\sigma(\mathbf{x}_\sigma^*) \leq \frac{(1+M)\bar{c}_A^2}{2\sigma k}\|\mathbf{x}^0 - \mathbf{x}_\sigma^*\|_2^2} \longleftarrow \boxed{\mathcal{O}\left(\frac{\bar{c}_A^2}{\sigma k}\|\mathbf{x}^0 - \mathbf{x}_\sigma{}^*\|_2^2\right)}$$

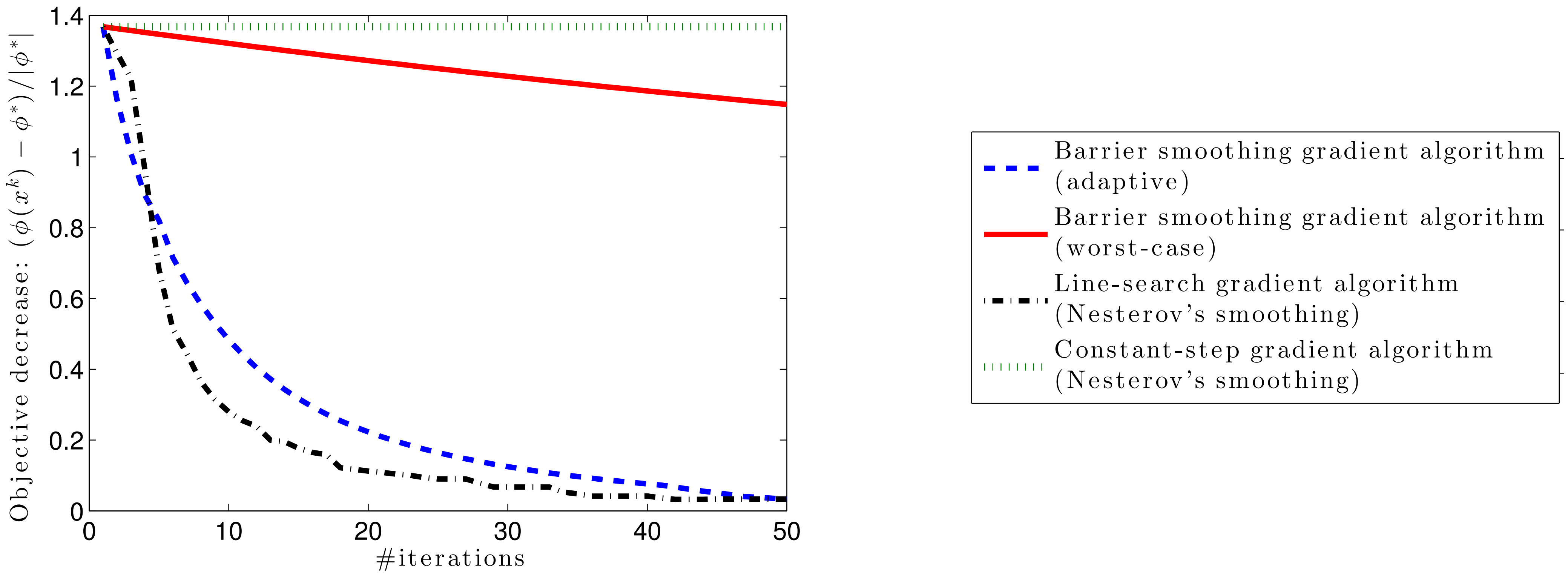
A stylized example

A simple quadratically constrained quadratic program

$$\phi^* = \min_{\mathbf{x}} \Big\{ \phi(\mathbf{x}) := \underbrace{\max_{\mathbf{u}^T\mathbf{B}\mathbf{u}\le 1} \Big\{ (\mathbf{A}\mathbf{x} - b)^T\mathbf{u} - \frac{1}{2}\mathbf{x}^T Q\mathbf{u} \Big\}}_{f(\mathbf{x})} + \underbrace{c^T\mathbf{x}}_{g(\mathbf{x})} \Big\}$$

- *Self-concordant barrier* for $\mathcal{U} := \{\mathbf{u} \in \mathbb{R}^m \mid \mathbf{u}^T\mathbf{B}\mathbf{u} \le 1\}$ is $b_{\mathcal{U}}(\mathbf{u}) := -\log(1 - \mathbf{u}^T\mathbf{B}\mathbf{u})$ with $\nu = 2$

Convergence of Nesterov's smoothing vs. Barrier smoothing (m = 200, n = 60) after **50** iterations



- Line-search ***gradient method*** *based on Nesterov's smoothing with proximity function* $p(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{B}\mathbf{x}$

- ***Gradient method*** *based on barrier smoothing with the* ***worst-case step size*** *using* $\bar{c}_A$
- ***Gradient method*** *based on barrier smoothing with the* ***adaptive step size*** *using* c_A^k

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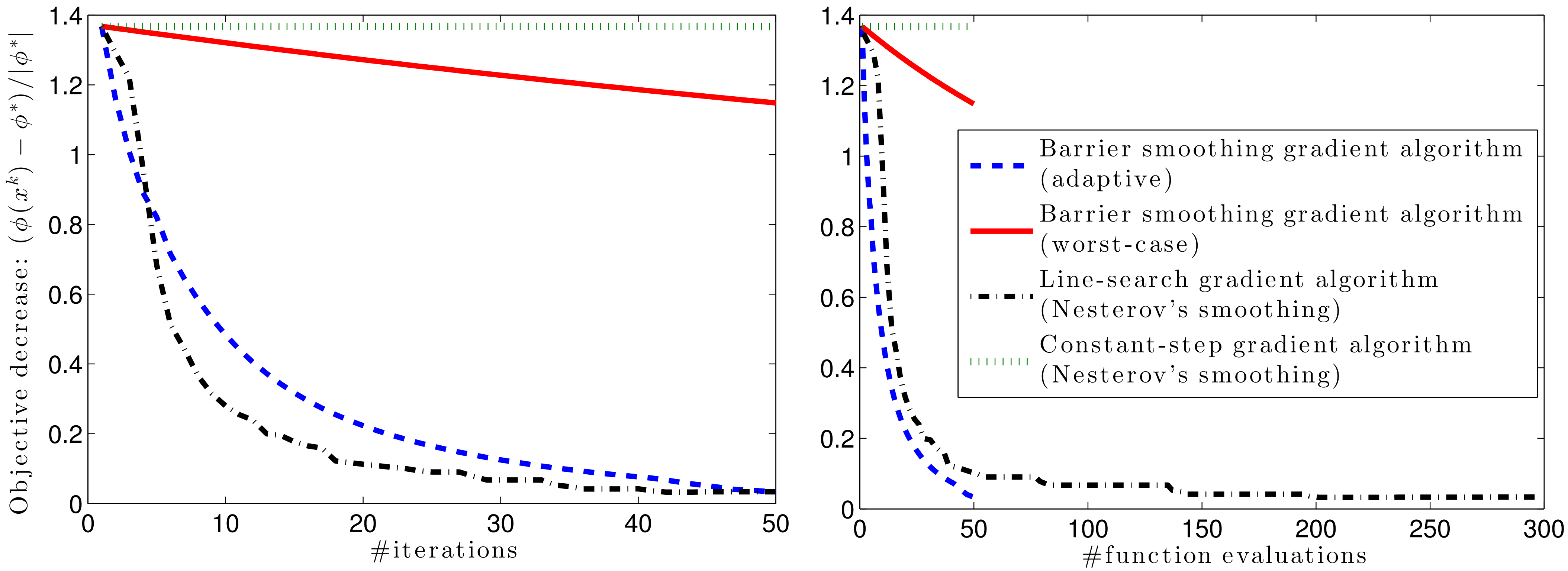

- Line-search ***gradient method*** *based on Nesterov's smoothing with proximity function* $p(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{B}\mathbf{x}$

  Line-search *requires additional function evaluations. The number of function evaluations is* ***296*** *after 50 iterations*

- ***Gradient method*** *based on barrier smoothing with the* ***worst-case step size*** *using* $\bar{c}_A$
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Second order method for barrier smoothing

When to apply proximal-Newton?

G is **self-concordant** and **A** is **full-row rank**

Proximal-Newton scheme

$$\begin{cases} \mathbf{s}^k & := \mathrm{argmin}_{\mathbf{x}} \left\{ Q(\mathbf{x}; \mathbf{x}^k) + g(\mathbf{x}) \right\}, \\ \mathbf{d}^k & := \mathbf{s}^k - \mathbf{x}^k, \\ \alpha_k & := (1 + \lambda_k)^{-1}, \ \ \lambda_k := \sqrt{\sigma^{-1}} \|\mathbf{d}^k\|_{\mathbf{x}^k}, \\ \mathbf{x}^{k+1} & := \mathbf{x}^k + \alpha_k \mathbf{d}^k. \end{cases}$$

- **Q** is the **quadratic surrogate** of f_σ around $\mathbf{x}^k$:

$$Q(\mathbf{x}; \mathbf{x}^k) := \nabla f_\sigma(\mathbf{x}^k)^T(\mathbf{x} - \mathbf{x}^k) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^k)^T \nabla^2 f_\sigma(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k)$$

Convergence

- Maintain ***local quadratic convergence*** as in *proximal-Newton method* for *self-concordant minimization*

- The worst-case complexity depends on the smoothness parameter:

$$\#\text{iterations} = \left\lfloor \frac{\phi_\sigma(\mathbf{x}^0) - \phi_\sigma{}^*}{0.017\sigma} \right\rfloor + \mathcal{O}\left(1.5 \ln\ln\left(\frac{0.28}{\sigma\varepsilon}\right)\right) + 2$$

- Possible to apply the **path-following proximal Newton scheme** for tuning $\sigma \downarrow 0^+$

Simultaneously update both σ and $\mathbf{x}^k$ at each iteration **k**

Conclusions

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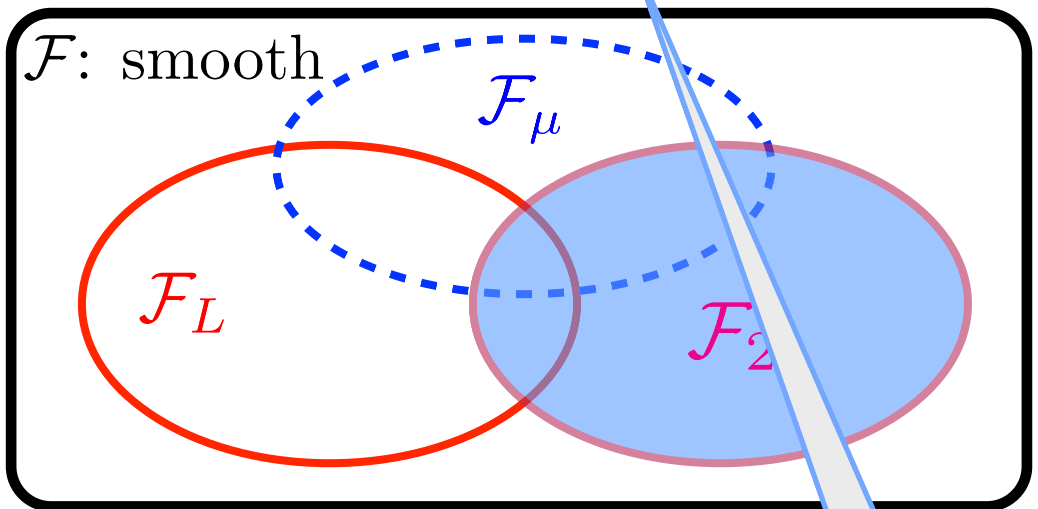
$$\min_{\mathbf{x}\in\mathbb{R}^n} \left\{\phi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\right\}$$



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- $\mathcal{F}_L$ - L-Lipschitz gradient
- $\mathcal{F}_2$ - self-concordant

A new variable metric proximal-point framework for composite self-concordant minimization
+
Extensions

Conclusions

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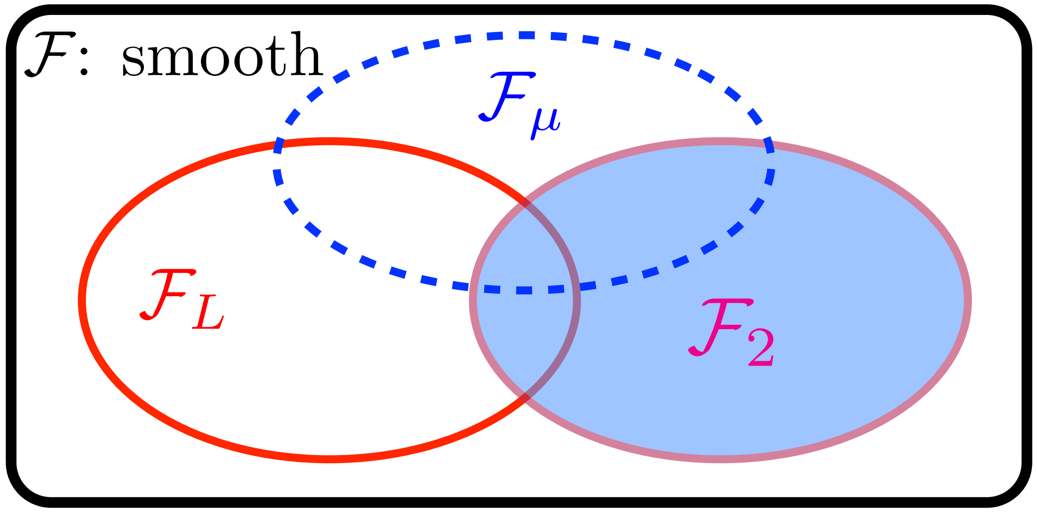

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- Highlights

- **Globalization**: a new strategy for finding step-size **explicitly**

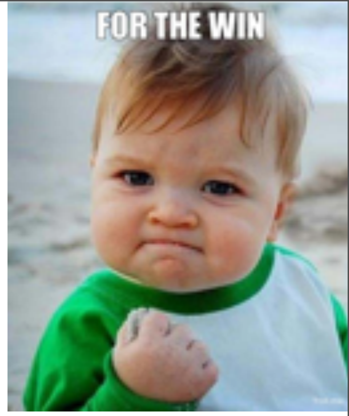
  motivate "*forward-looking*" line-search strategy

- **Search direction**: efficient (strongly convex program)

- **Local convergence**: quadratic convergence *without boundedness of the Hessian*

  analytic quadratic convergence region

Conclusions



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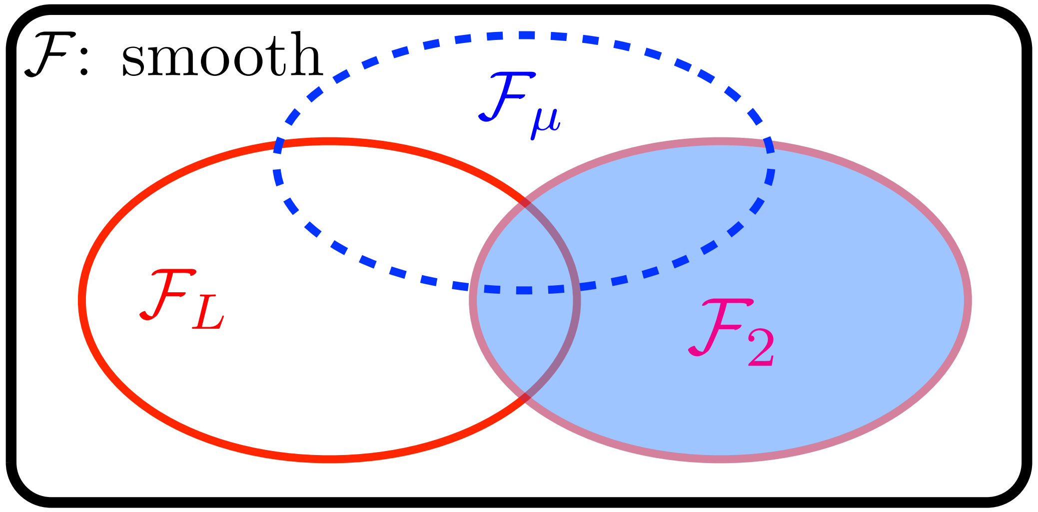

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- Practical contributions (this talk)
  - SCOPT package has quasi-Newton / first & second order methods @lions.epfl.ch/software
  - leverage fast proximal solvers for g(x) (structured norms etc.)
  - robust to subproblem solver accuracy

SCOPT FTW

- Postdoc & PhD positions @ LIONS / EPFL

contact: volkan.cevher@epfl.ch